1 A brief introduction to Volume conjecture

2 Linear Fractional Transformation and 2-dimensional hyperbolic geometry

2.1 Linear Fractional Transformation (LFT)

2.2 Geometry

2.3 Cross Ratio

2.4 Poincaré Upper Half Plane and Disk

2.5 2D Hyperbolic Geometry

Geodesic

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FIG.4

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Let γ be a C^1 curve parametrized by $z(t) = x(t) + iy(t)$ connecting two points on the imaginary axis, namely $P = ip$ and $Q = iq$. Then

$$\begin{aligned} l(\gamma) &= \int_a^b ds = \int_a^b \frac{|z'(t)|}{\text{Im}z} dt = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt \\ &\geq \int_a^b \frac{y'(t)}{y(t)} dt = \log \frac{q}{p} \end{aligned}$$

Here, the equality holds if and only if $x'(t) = 0$ and $y'(t) \geq 0$. Therefore,

$$\begin{aligned} d(P, Q) &= \inf\{\, l(\gamma) \mid \gamma \text{ is a curve connecting } P \text{ and } Q\,\} \\ &= \log \frac{q}{p} = \log[0, ip, iq, \infty] \end{aligned}$$

Note that $z(t) = ie^t$ is the geodesic with unit speed. ($\|z'(t)\| := \dfrac{z'(t)}{\text{Im}z} = \dfrac{e^t}{e^t} = 1$) Since Möbius transformations map circles to circles and preserve angles, any geodesic in $\mathbb{H}^2$ is a half circle perpendicular to the real axis.

On D, the distance s between 0 and $x \in \mathbb{R}$ is given by

$$s = \int_0^x ds = \int_0^x \frac{2|dz|}{1 - |z|^2} = \int_0^x \frac{2dx}{1 - x^2} = \log \frac{1 + x}{1 - x} = \log[-1, 0, x, 1]$$

and $x = \frac{e^t - 1}{e^t + 1} = \tanh\frac{t}{2}$ is the unit speed geodesic. Again, any geodesic is a circle perpendicular to the boundary of D.

Formula for distance

If we want to calculate the distance between arbitrary points z and w, first map them to the imaginary axis, say i and yi, by a Möbius transformation g. Then

$$d(z, w) = d(i, yi) = \log y = \log[0, i, yi, \infty] = \log[z', z, w, w']$$

where $z' = g^{-1}(0)$ and $w' = g^{-1}(\infty)$ are intersection points of the real axis and the half circle connecting z and w. In general, however, calculating z' and w' is cumbersome and we suggest to proceed as follows.

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. FIG.5 .

$$\begin{aligned} -\frac{|z-w|^2}{4\text{Im}z\text{Im}w} &= [z, \bar{z}, w, \bar{w}] = [i, -i, yi, -yi] = [1, -1, y, -y] \\ &= -\frac{(y-1)^2}{4y} = -\frac{1}{4}\left(y + \frac{1}{y} - 2\right) = -\frac{1}{2}\left(\frac{e^d + e^{-d}}{2} - 1\right) \\ &= -\frac{\cosh d - 1}{2} = -\sinh^2\frac{d}{2} \end{aligned} \tag{2.5.1}$$

Thus $\cosh d(z, w) = 1 + \frac{|z-w|^2}{2\text{Im}z\text{Im}w}$.

On the disk model, for $z, w \in D$, d = d(z,w) is given from (**??**) as follows.

$$\sinh^2\frac{d}{2} = -[z, z^*, w, w^*] = \frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)} \tag{2.5.2}$$

Here z^* is the reflection point of z with respect to the unit circle, i.e., $z^* = z/|z|^2$.

We will a "complete geodesic" (i.e., defined for $-\infty < t < \infty$) a line and we can easily check the following properties.

1. For a point p and a tangent vector v at p, there is a unique line l with initial data (p, v).
2. For given two lines l and l', there is a Möbius transformation g sending l

to l'. In fact, for any (p, v) and (p', v') with v and v' unit vectors, there is a unique Möbius transformation g sending (p, v) to (p', v').
3. For two non intersecting lines l and l', there exists a unique common perpendicular.
4. Compare with Euclidean 5th postulate: There are infinitely many parallel lines to a given line.

Distance and Angle between Lines

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. FIG.6 .

$$[z_1, z_2, w_1, w_2] = [-1, 1, a, -a] = \frac{(1+a)^2}{4a} = \frac{1}{4}\left(a + \frac{1}{a} + 2\right)$$
$$= \frac{1}{2}\left(\frac{e^d + e^{-d}}{2} + 1\right) = \frac{1}{2}(\cosh d + 1) = \cosh^2 \frac{d}{2} \qquad (2.5.3)$$

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FIG.7 .

$$[z_1, z_2, w_1, w_2] = [-1, 1, e^{i\theta}, -e^{i\theta}] = \frac{(1+e^{i\theta})^2}{4e^{i\theta}} = \frac{1}{2}\left(\frac{e^{i\theta} + e^{-i\theta}}{2} + 1\right)$$
$$= \frac{1}{2}(\cos\theta + 1) = \cos^2 \frac{\theta}{2} \qquad (2.5.4)$$

Exercise. .
FIG.8 .
Calculate the distance between w and the line connecting z_1 and z_2. (Hint: Consider w^* and show $[z_1, z_2, w_1, w_2] = \frac{i}{2}\sinh d + \frac{1}{2}$.)

Circumference and Area of a Ball

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FIG.9

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Let C be a circle centered at the origin of hyperbolic radius ρ. Since the metric is rotationally symmetric, C looks like an ordinary circle. Let r be the Euclidean radius of C. Then $\sinh^2 \frac{\rho}{2} = \frac{r^2}{1-r^2}$, $\cosh^2 \frac{\rho}{2} = \frac{1}{1-r^2}$, and

$\sinh\rho = 2\sinh\frac{\rho}{2}\cosh\frac{\rho}{2} = \frac{2r}{1-r^2}$. Consequently, the circumference of C is given by

$$\int_C ds = \int_C \frac{2|dz|}{1-r^2} = \int_0^{2\pi} \frac{2rd\theta}{1-r^2} = \frac{4\pi r}{1-r^2} = 2\pi\sinh\rho \tag{2.5.5}$$

For the area of $B(0;\rho)$, note that the volume form on D is given by $dvol = \left((\frac{2}{1-|z|^2})^2\right) dx\wedge dy$. Thus

$$\begin{aligned} vol(B(0;\rho)) &= \int_{B(0;\rho)} \left(\frac{2}{1-|z|^2}\right)^2 dx\wedge dy = \int_0^{2\pi}\int_0^r \frac{4rdrd\theta}{(1-r^2)^2} \\ &= \int_0^{2\pi}\left[\frac{2}{1-r^2}\right]_0^r d\theta = \frac{4\pi r^2}{1-r^2} = \pi\left(2\sinh\frac{\rho}{2}\right)^2 \end{aligned} \tag{2.5.6}$$

Here, $2\sinh\frac{\rho}{2}$ is the "horospherical distance".

Determination of a Triangle

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FIG.10

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Proposition 2.5.1. For a triangle in D with (hyperbolic) side lengths a, b, c and the opposite (hyperbolic) angles A, B, C (see FIG.10), the follwing rules hold.

$$\text{The Sine Rule: } \frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c} \tag{2.5.7}$$

$$\text{The Cosine Rule I: } \cosh c = \cosh a\cosh b - \sinh a\sinh b\cos C$$

$$\text{or } \cos C = \frac{\cosh a\cosh b - \cosh c}{\sinh a\sinh b} \tag{2.5.8}$$

$$\text{The Cosine Rule II: } \cos C = -\cos A\cos B + \sin A\sin B\cosh c$$

$$\text{or } \cosh c = \frac{\cos A\cos B + \cos C}{\sin A\sin B} \tag{2.5.9}$$

Note that a triangle is completely determined by its three angles unlike in the Euclidean case.

Proof. .

FIG.11

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Once we prove the Cosine Rule I, the rest follows easily. First, move the vertex C to the origin O and let $\hat{a}, \hat{b}, \hat{c}$ be the Euclidean lengths of $\overline{OB}, \overline{OA}, \overline{AB}$ respectively. By (2.5.2) and the cosine rule in the Euclidean plane (which is valid since $\overline{OA}$ and $\overline{OB}$ are straight lines),

$$\cosh c = 2\sinh^2 \frac{c}{2} + 1 = \frac{2\hat{c}^2}{(1-\hat{a}^2)(1-\hat{b}^2)} + 1 = \frac{2(\hat{a}^2 + \hat{b}^2 - 2\hat{a}\hat{b}\cos C)}{(1-\hat{a}^2)(1-\hat{b}^2)} + 1$$
$$= \frac{(1+\hat{a}^2)(1+\hat{b}^2) - 4\hat{a}\hat{b}\cos C}{(1-\hat{a}^2)(1-\hat{b}^2)} = \cosh a \cosh b - \sinh a \sinh b \cos C$$

□

Exercise. Derive the sine rule and the cosine rule II from the cosine rule I. Notice that the square of each of the terms appeared in the Sine Rule has a common expression symmetric with respect to a, b, c.

Exercise. .
FIG.12

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Derive the following analogous results in the spherical case.

$$\text{The Sine Rule: } \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$
$$\text{The Cosine Rule I: } \cos C = -\frac{\cos a \cos b - \cos c}{\sin a \sin b}$$
$$\text{The Cosine Rule II: } \cos c = \frac{\cos A \cos B + \cos C}{\sin A \sin B}$$

Remark Note that if we replace a, b, c by ia, ib, ic in the above formulae, then we have the hyperbolic sine and cosine Rule.
Special Case for a right triangle: If $C = \frac{\pi}{2}$, then by the hyperbolic cosine rule I, we obtain Pythagoras' theorem as follows.

$$\cosh c = \cosh a \cosh b.$$

From this it is easy to see that for a fixed point and a line the distance function from the point to a point on a line is a convex function with its minimum attained for a perpendicular drop. Also, by the cosine rule II and the sine rule, we have

$$
\begin{aligned}
\cosh c &= \cot A \cot B \\
\cosh a &= \frac{\cos A}{\sin B} \\
\sin A &= \frac{\sinh a}{\sinh c}.
\end{aligned}
$$

It follows from these that $\cos B = \dfrac{\tanh a}{\tanh c}$.

Gauss-Bonnet Theorem

Theorem 2.5.1. The hyperbolic area of a tringle with internal angles α, β, γ is $\pi - \alpha - \beta - \gamma$.

Proof. We may assume that one vertex is at ∞ since the general case easily follows from the following observation as in the picture.

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FIG.13

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$$\pi - (\alpha + \gamma + \delta) - (\pi - (\delta + \pi - \beta)) = \pi - (\alpha + \beta + \gamma)$$

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FIG.14

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It suffices to show that the area of Δ is $\pi - \alpha - \beta$. On $\mathbb{H}^2$, the volume form is given by $dvol = \dfrac{dx \wedge dy}{y^2}$. Hence

$$
\begin{aligned}
\int_{\Delta} \frac{dx \wedge dy}{y^2} &= \int_{\cos(\pi-\alpha)}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} dy dx = \int_{\cos(\pi-\alpha)}^{\cos\beta} \frac{1}{\sqrt{1-x^2}} dx \\
&= -\arccos x \Big|_{\cos(\pi-\alpha)}^{\cos\beta} = \pi - \alpha - \beta
\end{aligned}
$$

Or applying Stokes' theorem,

$$
\begin{aligned}
\int_{\Delta} \frac{dx \wedge dy}{y^2} &= \int_{\Delta} d\left(\frac{dx}{y}\right) = \int_{\partial\Delta} \frac{dx}{y} \quad [x = \cos\theta,\ y = \sin\theta] \\
&= -\int_{\beta}^{\pi-\alpha} \frac{-\sin\theta}{\sin\theta} d\theta = \pi - \alpha - \beta
\end{aligned}
$$

□

Corollary 2.5.1. The area of n-gon with angles $\alpha_1, ...\alpha_n$ is given by $(n-2)\pi - (\alpha_1 + \cdots \alpha_n)$.

Proposition 2.5.2. Let $(\theta_1, \cdots, \theta_n)$ be an ordered n-tuple with $0 \le \theta_j < \pi$. Then $\exists$ a convex polygon P with interior angles $(\theta_1, \cdots, \theta_n)$ in its order $\iff$ $(n-2)\pi - \sum \theta_j > 0$.

Proof. $\Rightarrow$): Follows from Cor 2.5.1
$\Leftarrow$): Given θ, consider the following right triangle.

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FIG.15

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Then, $\sin\alpha = \dfrac{\cos\left(\frac{\theta}{2}\right)}{\cosh d}$ and $\alpha = \arcsin\left(\dfrac{\cos\left(\frac{\theta}{2}\right)}{\cosh d}\right)$

Consider

$$g(t) := \sum_{i=1}^{n} \arcsin\left(\frac{\cos\left(\frac{\theta_i}{2}\right)}{\cosh t}\right)$$

We want to find d such that $g(d) = \pi$.
$g(0) = \sum \arcsin(\cos\frac{\theta_i}{2}) = \sum(\frac{\pi}{2} - \frac{\theta_i}{2}) = \frac{1}{2}(n\pi - \sum\theta_i) > \pi$. Also, as $t \to \infty$, $g(t) \to 0$.
$\therefore \exists d > 0$ s.t $g(d) = \pi$
For such d we have a desired polygon P. □

Note. Therefore $\exists P$ with $\theta_1 = \cdots = \theta_n = \frac{\pi}{2}$ iff $n \ge 5$