1 A brief introduction to Volume conjecture

2 Linear Fractional Transformation and 2-dimensional hyperbolic geometry

3 Inversive geometry and hyperbolic geometry

3.1 Inversion(or reflection) and Möbius transformation

3.2 Möbius transformations as conformal maps

3.3 Möbius transformation as a cross-ratio preserving maps

3.4 Möbius transformation as a sphere preserving map

3.5 Poincare extension

Definition 3.5.1. Define $i : M(\widehat{\mathbb{R}^n}) \to M(\widehat{\mathbb{R}^{n+1}})$ by $\sigma = J_S \mapsto \tilde{\sigma} = J_{\tilde{S}}$ and $i : \phi = \sigma_1 \circ \cdots \circ \sigma_k \mapsto \tilde{\phi} = \tilde{\sigma_1} \circ \cdots \circ \tilde{\sigma_k}$, where $\tilde{S}$ is the sphere in $\mathbb{R}^{n+1}$ for which $\tilde{S} \cap \widehat{\mathbb{R}^n} = S$.

Check if i is uniquely well-defined and 1-1 :

Suppose $\tilde{\phi}_1, \tilde{\phi}_2$ are two extensions of $\phi \in M(\widehat{\mathbb{R}^n}) \Longrightarrow \tilde{\phi}_1 \circ \tilde{\phi}_2^{-1} = id$ on $\mathbb{R}^n$and preserves $\mathbb{H}^{n+1} \Longrightarrow \tilde{\phi_1} \circ \tilde{\phi_2}^{-1} = id$ by Proposition 3.5.2.

Theorem 3.5.1.

i) $\phi \in M(\mathbb{H}^{n+1}) := \{\phi \in M(\widehat{\mathbb{R}^n}) \,|\, \phi \text{ is an automorphism on } \mathbb{H}^{n+1}\} \Longrightarrow \phi|_{\widehat{\mathbb{R}^n}} \in M(\widehat{\mathbb{R}^n})$

ii) $i(M(\widehat{\mathbb{R}^n})) = M(\mathbb{H}^{n+1})$

iii) $\phi \in M(\mathbb{H}^{n+1}) \Leftrightarrow \phi = J_{S_1} \circ \cdots \circ J_{S_k}, S_i \perp \widehat{\mathbb{R}^n}$

Proof.

i) $\phi \in M(\mathbb{H}^{n+1}) \Longrightarrow \phi| : \widehat{\mathbb{R}^n} = \partial \mathbb{H}^{n+1} \circlearrowleft$. $\phi|$ preserves cross-ratio since ϕ does. $\Longrightarrow \phi| \in M(\widehat{\mathbb{R}^n})$

ii) $(\subset)$: clear. $(\supset)$: $\forall \phi \in M(\mathbb{H}^{n+1})$, consider $\phi|$. Then $\tilde{\phi|} \circ \phi^{-1} = id$ on $\widehat{\mathbb{R}^n}$ and $\mathbb{H}^{n+1} \circlearrowleft \Longrightarrow \tilde{\phi|} \circ \phi^{-1} = id \Longrightarrow \phi = \tilde{\phi|} = i(\phi|)$.

iii) $(\Longleftarrow)$: clear. $(\Longrightarrow)$: clear from ii). □

Note that $\forall \phi \in M(\mathbb{H}^{n+1})$, $\phi| \in M(\widehat{\mathbb{R}^n})$ as in the proof of i) and $\phi = \tilde{\phi|}$ by the proof of ii). Therefore if $\phi \in M(\widehat{\mathbb{R}^n})$ is a similarity, then $\tilde{\phi}$ is the unique similarity on $\mathbb{H}^{n+1}$ whose restriction is ϕ.

Now consider the ball model. Recall $\eta = J_{\widehat{\mathbb{R}^n}} \circ J_{S(e_{n+1}, \sqrt{2})} : \mathbb{B}^{n+1} \to \mathbb{H}^{n+1}$, $S^n = \partial \mathbb{B}^{n+1} \to \partial \mathbb{H}^{n+1} = \widehat{\mathbb{R}^n}$. Then

$$M(\mathbb{B}^{n+1}) = \eta^{-1} \circ M(\mathbb{H}^{n+1}) \circ \eta.$$

Proposition 3.5.1. Let $\phi \in M(\mathbb{B}^{n+1})$. Then the followings are equivalent.

i) $\phi(\infty) = \infty$

ii) $\phi(0) = 0$

iii) $\phi \in O(n+1)$

Proof. i) $\iff$ ii) since ϕ preserves the inversion $J_{S(0,1)}$. If i) holds, then ii) also holds and ϕ is a similarity:$x \mapsto \lambda A x$, where $A \in O(n)$. Now $\phi : \mathbb{B}^{n+1} \circlearrowleft \implies |\lambda| = 1$ and hence iii) follows. Now iii)$\implies$ii) is clear. □

3.6 Hyperbolic metric

3.6.1 $\mathbb{B}^n$ case

Let $\phi \in \mathcal{M}(\mathbb{B}^n)$ and $x^* = \sigma_1(x) = \frac{x}{|x|^2}$. Note that

$$|x^* - u^*|^2 = \sum_{i=1}^{n} \left(\frac{x_i}{|x|^2} - \frac{u_i}{|u|^2} \right) = \sum_{i=1}^{n} \frac{|x|^2 - 2x_i u_i + |u^2|}{|x|^2 |u^2|} = \left(\frac{|x-u|}{|x||u|} \right)^2.$$

This yields

$$[x, x^*, u, u^*] = \frac{|x-u||x^* - u^*|}{|x - \frac{x}{|x|^2}||u - \frac{u}{|u|^2}|} = \frac{|x-u|^2}{(1-|x|^2)(1-|u|^2)}.$$

Put $u = x + dx$, $y = \phi x$ and since the Möbius transformation ϕ preserves cross ratio, we conclude

$$\frac{2|dy|}{1-|y|^2} = \frac{2|dx|}{1-|x|^2}.$$

In other words, the Poincare matric is invariant under Möbius transformations.

3.6.2 $\mathbb{H}^{n+1}$ case

The inversive point is given as $x^* = (x_1, \cdots, x_{n-1}, -x_n)$ for any $x \in \mathbb{H}^{n+1}$. Then

$$[x, x^*, u, u^*] = \frac{|x-u||x^* - u^*|}{|x - x^*||u - u^*|} = \frac{|x-u|^2}{4x_{n+1} u_{n+1}},$$

and by letting $u = x + dx$, we see that

$$\frac{|dx|^2}{4x_{n+1}^2}$$

is an invariant metric and $\frac{|dx|}{x_{n+1}}$ is called the Poincare metric.

3.6.3 Canonical embedding

Fig3.1

Proposition 3.6.1. $M(\mathbb{H}^{n+1}) = Isom(\mathbb{H}^{n+1})$

Proof.

$(\subset)$: clear.

$(\supset)$: It suffices to show that $M(\mathbb{B}^{n+1}) = Isom(\mathbb{B}^{n+1})$. $M(\mathbb{B}^{n+1}) \subset Isom(\mathbb{B}^{n+1})$ and is already "full", *i.e.*, transitive and isotropy group $= O(n+1)$. Indeed $g \in Isom(M)$, M connected Riemannian, such that $g(x) = x$ and $dg(x) = id$, then $g = id$: Note $g = id$ on a neighborhood of x(since it fixes radial geodesics), and

$$\begin{aligned} A &= \{x \in M \mid g(x) = x \text{ and } dg(x) = id\} \\ &\Rightarrow A \text{ is open and closed} \\ &\Rightarrow A = M. \end{aligned}$$

□

3.7 Isometry types

3.7.1 $\mathbb{B}^{n+1}$ case

$\phi \in M(\mathbb{B}^{n+1}) \Longrightarrow \phi : \overline{\mathbb{B}^{n+1}} \circlearrowleft \Longrightarrow \phi$ has a fixed point in $\overline{\mathbb{B}^{n+1}}$ by Brouwer fixed point theorem.

Claim. ϕ has more than two fixed points on $S^n = \partial\mathbb{B}^{n+1} \Longrightarrow \phi$ has a fixed point in $\mathbb{B}^{n+1}$.

Proof. Work on $\mathbb{H}^{n+1}$ and suppose $\#|Fix| \geq 3$ on $\widehat{\mathbb{R}^n} = \partial\mathbb{H}^{n+1}$. We may assume $\phi(\infty) = \infty$, $\phi(0) = 0$, $\phi(e_1) = e_1$.

$\Longrightarrow \phi(x) = Ax$, $A \in O(n+1)$
$\Longrightarrow \phi(x)$ fixes x_{n+1} axis since it is perpendicular to $\mathbb{R}^n$, *i.e.*, fixes points in $\mathbb{H}^{n+1}$ □

Therefore we have the following trichotomy for a conjugacy class of ϕ :

(1) ϕ fixes a point in $\mathbb{B}^{n+1}$: elliptic

(2) ϕ fixes exactly one point on $\partial\mathbb{B}^{n+1} = S^n$: parabolic

(3) ϕ fixes exactly two points on $\partial\mathbb{B}^{n+1} = S^n$: loxodormic or hyperbolic

3.7.2 $\mathbb{H}^2$ and $\mathbb{H}^3$ case

$$g(z) = \frac{az+b}{cz+d}, \quad A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Fixed point : $z = g(z) \Longrightarrow cz^2 + (d-a)z - b = 0 \Longrightarrow D = (d-a)^2 + 4bc = (a+d)^2 - 4(ad-bc) = \mathrm{tr}^2(A) - 4\det(A)$. Then we define

$$\mathrm{tr}(g) := \frac{(\mathrm{tr}A)^2}{\det(A)},$$

which is an invariant of a projective transformation. Note that we have the following trichotomy for $G = PSL_2(\mathbb{R})$

(1) g is elliptic $\Longleftrightarrow D < 0 \Longleftrightarrow \mathrm{tr}^2 < 4$

(2) g is parabolic $\Longleftrightarrow D = 0 \Longleftrightarrow \mathrm{tr}^2 = 4$

(3) g is hyperbolic $\Longleftrightarrow D > 0 \Longleftrightarrow \mathrm{tr}^2 > 4$

Proposition 3.7.1. Let $G = PSL_2(\mathbb{C})$ or $PSL_2(\mathbb{R})$. Then $\forall f, g \in G$ we have $f \sim g$(*i.e.*, f is conjugate to g) $\Longleftrightarrow \mathrm{tr}^2(f)$=$\mathrm{tr}^2(g)$

Proof.

$\Longrightarrow$) : Clear.

$\Longleftarrow$) :

(1) $\mathrm{tr}^2(g) = 4 \Longrightarrow$ there exists a unique fixed point, say ∞.
$\Longrightarrow g(z) = az + b$ and $a = 1$ ($\exists$ another fixed point otherwise)
$\Longrightarrow g \sim f : z \mapsto z + 1$ ($\because f = h^{-1} \circ g \circ h$ with $h(z) = bz$)

(2) $\mathrm{tr}^2(g) \neq 4 \Longrightarrow$ there are two fixed points, say $0, \infty$
$\Longrightarrow g \sim f(z) = az$ $(a \neq 1\ a \neq 0) \Longrightarrow \mathrm{tr}^2 = a + \frac{1}{a} + 2$

(i) $|a| = 1 \Longrightarrow g$ is elliptic
(ii) $|a| \neq 1 \Longrightarrow g$ is loxodromic (hyperbolic if a is real)

Notice that, tr^2 determines $a, \frac{1}{a}$ and $g(z) = az \sim f(z) = \frac{1}{a}z$ via $h(z) = -1/z$, and this proves the proposition. ☐

In the above proof, we notice

$$(2)i) \Longrightarrow \mathrm{tr}^2 g = a + \frac{1}{a} + 2 = a + \overline{a} + 2 = 2\cos\theta + 2 \in [0, 4).$$

Conversely, if $\mathrm{tr}^2 g \in [0, 4)$, then by the above dichotomy $g \sim f(z) = az$ with $a = e^{i\theta}$, *i.e.*, elliptic. Hence we have the following map,

$$\mathrm{tr}^2 : (G \setminus \{id\})/\sim \quad \longrightarrow \quad \mathbb{C}$$

such that tr^2(elliptic)$= [0, 4)$, tr^2(parabolic)$= 4$, and loxodromic otherwise.