I.3 Interior and Closure

Definition 1 Let X be a topological space. $C \subset X$ is closed if the complement of C, C^c is open.

Note 1. $\emptyset$ and X are closed and open at the same time.

2. An arbitrary intersection of closed sets is closed.

3. A finite union of closed sets is closed.

Definition 2 (Trichotomy) Let X be a topological space and A be a subset of X.

1. A point $x \in X$ is called an interior point of A if there is an open set U containing x such that $U \subset A$. (The set of interior points of A is called the interior of A and denoted by A^i.)

2. $x \in X$ is called a boundary point of A if for each open set U containing x, $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$. (The set of boundary points of A is called the boundary of A and denoted by A^b.)

3. $x \in X$ is called an exterior of A if there is an open set V containing x such that $V \subset A^c$. (The set of exterior points of A is called the exterior of A and denoted by A^e.)

Note Given a subset A of a topological space X, the interior point of A is the exterior point of A^c, and the interior point of A^c is the exterior point of A.

Definition 3 The closure of A is defined as the union of the interior of A and the boundary of A, i.e., $\overline{A} = A^i \cup A^b$.

Proposition 1 *1.* $(A^c)^i = A^e$, $(A^c)^e = A^i$, $(A^c)^b = A^b$.

2. $(A^i)^c = \overline{A^c}$, $(A^e)^c = \overline{A}$.

Proposition 2 *1.* $A^i = \cup\{O : open|O \subset A\}$ = *the largest open set contained in* A.

2. $\overline{A} = \cap\{C : closed|A \subset C\}$ = *the smallest closed set containing* A.

Proof

1. ($\subset$) Let $x \in A^i$. Then there is an open set O such that $x \in O \subset A$. Thus $A^i \subset \cup\{O : open | O \subset A\}$.
   ($\supset$) Choose an open set O contained in A. Then clearly $O \subset A^i$.

2. Let $B = A^c$. Then $B^i = \cup\{O : open | O \subset B\}$ by 1. Therefore $\overline{A} = \overline{B^c} = (B^i)^c = \cap\{O^c : closed | O^c \supset B^c\}$.

$\square$

Proposition 3 *1. A is open $\Leftrightarrow A = A^i$, and A is closed $\Leftrightarrow A = \overline{A}$.*

2. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$ and $A^i \subset B^i$.

3. $\overline{\overline{A}} = \overline{A}$ and $(A^i)^i = A^i$.

4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

5. $(A \cap B)^i = A^i \cap B^i$ and $(A \cup B)^i \supset A^i \cup B^i$

Definition 4 Given a set A of a topological space X, $x \in X$ is an accumulation point of A if for each open neighborhood U of x, $(U - \{x\}) \cap A \neq \emptyset$.
The derived set of A, denoted by A', is the set of all accumulation points of A.

Proposition 4 *The closure of A is the union of A and the derived set of A, i.e., $\overline{A} = A \cup A'$.*

Proof ($\subset$) Let $x \in \overline{A} = A^i \cup A^b$. If $x \in A^i$, then $x \in A$ and hence $x \in A \cup A'$. If $x \in A^b$ and $x \notin A$, then $x \in A'$ and hence $\overline{A} \subset A \cup A^b$.
($\supset$) Since $A \subset \overline{A}$, it suffices to show that $A' \subset \overline{A}$. Suppose $x \notin \overline{A}$. Then $x \in A^e$ and hence $x \notin A'$. $\square$

Corollary 5 *A is closed if and only if $A' \subset A$.*

Proof A is closed $\Leftrightarrow A = \overline{A} = A \cup A' \Leftrightarrow A' \subset A$. $\square$

Definition 5 $x \in A - A'$ is called an isolated point of A.

Example 1. If $A = \mathbb{Q}$, then there is no isolated point.

2. If $A = \{\frac{1}{n} | n \in \mathbb{N}\}$, then each point in A is an isolated point.

3. If $A = \mathbb{Z}$, then each point in A is an isolated point.

4. If $A = (0, 1) \subset \mathbb{R}$, then there is no isolated point.

Homework 4 p.34 #6, p.43 #1, #5 in Kahn's book.