I.4 Properties of Continuous Functions

Definition 1 Let X and Y be topological spaces. Then $f : X \to Y$ is continuous if $f^{-1}(V)$ is open for all V open in Y.

Proposition 1 *Let X and Y be topological spaces and f be a function from X to Y. Then the followings are equivalent.*

1. *f is continuous, i.e., f^{-1}(open)=open,*
2. *$\forall x \in X$, f is continuous at x, i.e., for $\forall$ open neighborhood V of $f(x)$, $\exists$ open nbd U_x of x such that $f(U_x) \subset V$,*
3. *f^{-1}(basic open)=open,*
4. *f^{-1}(subbasic open)=open,*
5. *f^{-1}(close)=closed,*
6. *$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for all $B \subset Y$,*
7. *$f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$.*

Proof ($1 \Rightarrow 2$) Suppose f is continuous, and let $V \subset Y$ be an open set containing $f(x)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is open. Hence f is continuous at x.
($1 \Leftarrow 2$) Suppose f is continuous at every point $x \in X$, and let $V \subset Y$ be open. For every $x \in f^{-1}(V)$, there exists an open set $U_x \subset X$ such that $x \in U_x \subset f^{-1}(V)$. Hence $f^{-1}(V)$ is open and f is continuous.
It is easy to see that 1 and 2 are equivalent.
($1 \Rightarrow 3$) Trivial.
($1 \Leftarrow 3$) Let B_α be a basic open set of Y. Then $f^{-1}(\cup_\alpha B_\alpha) = \cup_\alpha f^{-1}(B_\alpha) = \cup_\alpha Open = Open$.
($3 \Rightarrow 4$) Trivial.
($3 \Leftarrow 4$) Let S_i be a subbasic open set of Y. Then $f^{-1}(\cap_{i=1}^k S_i) = \cap_{i=1}^k f^{-1}(S_i) = \cap_{i=1}^k Open = Open$.
($1 \Leftrightarrow 5$) Let B be an open set in Y. Then it is clear that $f^{-1}(B^c) = f^{-1}(B)^c$.
($5 \Rightarrow 6$) Let B be a set in Y. Clearly $f^{-1}(\overline{B})$ is closed by 5. Since $f^{-1}(B) \subset f^{-1}(\overline{B})$, $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.
($5 \Leftarrow 6$) Let B be a closed set in Y. Then $f^{-1}(B) = f^{-1}(\overline{B}) \supset \overline{f^{-1}(B)}$ by 5. At the same time $f^{-1}(B) \subset \overline{f^{-1}(B)}$. Thus $f^{-1}(B) = \overline{f^{-1}(B)}$ and hence $f^{-1}(B)$ is closed.

(6 $\Rightarrow$ 7) Let $A \subset X$ and $B := f(A) \subset Y$. Since $B = f(A)$, $A \subset f^{-1}(B)$. Thus $\overline{A} \subset \overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$. Consequently $f(\overline{A}) \subset f(f^{-1}(\overline{B})) \subset \overline{B} = \overline{f(A)}$.
(6 $\Leftarrow$ 7) Let $B \subset Y$ and $A := f^{-1}(B) \subset X$. Then $f(A) = f(f^{-1}(B)) \subset B$. Thus $f(\overline{A}) \subset \overline{f(A)} \subset \overline{B}$ and hence $\overline{A} = \overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$. $\square$

Proposition 2 *Let X, Y, and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.*

Proof Let U be open in Z. Then $g^{-1}(U)$ is open in Y, and $f^{-1}(g^{-1}(U))$ is open in X. But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ and hence $g \circ f$ is continuous. $\square$

Homework Determine whether the following statements are equivalent to any of the statements of the proposition 3 or not.

1. $f^{-1}(B)^i \supset f^{-1}(B^i)$ for all $B \subset Y$.
2. $f(A^i) \supset f(A)^i$ for all $A \subset X$.