I.7 Cantor set and Space filling curve

Definition 1 The set

$$C := \{\sum_{i=1}^{\infty} \frac{a_i}{3^i} \mid a_i = 0 \text{ or } 2\} \subset [0,1]$$

is called the Cantor set. Alternatively, let

$$\begin{aligned} M_0 &= [0,1] \\ M_1 &= M_0 - (\frac{1}{3}, \frac{2}{3}) \\ &\vdots \\ M_n &= M_{n-1} - \bigcup_{k=0}^{\infty} (\frac{1+3k}{3^n}, \frac{2+3k}{3^n}). \end{aligned}$$

Then

$$C = \bigcap_{n=1}^{\infty} M_n.$$

Proposition 1 *Let $A_i = \{0,2\}$ be a space with a discrete topology; let $X = \Pi_{i=1}^{\infty} A_i$. Define a function $f : X \to C \subset [0,1]$ as*

$$f((a_1, a_2, \ldots)) = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots = \sum_{i=1}^{\infty} a_i (\frac{1}{3^i}) \in C.$$

Then f is a homeomorphism.

Proof

1. f is bijective : Clear.

2. f is continuous : Given $\epsilon > 0$ and $c = \sum \frac{a_i}{3^i} \in C$, choose $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} \frac{2}{3^i} < \epsilon.$$

Let $\mathcal{U} := p_1^{-1}(a_1) \cap \ldots \cap p_N^{-1}(a_N) \subset X$. Note that $\mathcal{U}$ is open. Then for $a \in \mathcal{U}$,

$$|f(a) - c| \leq \sum_{i=N+1}^{\infty} \frac{2}{3^i} < \epsilon.$$

Thus

$$f(\mathcal{U}) \subset B_\epsilon(c).$$

Hence f is continuous.

3. f^{-1} is continuous : For a given $c \in C$, let $a = (a_i) = f^{-1}(c) \in X$. Then for a basic open set $U := p_{i_1}^{-1}(a_{i_1}) \cap \ldots \cap p_{i_n}^{-1}(a_{i_n}))$ containing a where $i_1 < \ldots < i_n = N$,

$$f^{-1}\big(B_{(\frac{1}{3})^{N+1}}(c)\big) \subset U \subset X.$$

Indeed if b is in $B_{(\frac{1}{3})^{N+1}}(c)$,

$$b_1 = a_1, \ldots, b_N = a_N.$$

From this $f^{-1}(b)$ is in U. Hence f^{-1} is continuous. □

Proposition 2 *Define* $g : \Pi A_i \to [0,1]$ *as* $(a_i) \mapsto \sum \frac{a_i/2}{2^i}$. *Then* g *is surjective and continuous.*

Proof

1. g is continuous : This follows using similar arguments as above.

2. g is surjective : Suppose x is in $[0,1]$. We know that there is a unique sequence $(b_i)_{i=1}^{\infty}$(the binary expansion of x) such that

$$x = \sum_{i=1}^{\infty} b_i(\frac{1}{2})^i.$$

Let $a_i = 2b_i$, then $g((a_i)) = x$.

[Figure] $C \to [0,1]$ □

Lemma 3 *Suppose* $A_i = \{0,2\}$ *is a space with a discrete topology. Then*

$$\Pi_{i=1}^{\infty} A_i \times \Pi_{i=1}^{\infty} A_i \cong \Pi_{i=1}^{\infty} A_i.$$

Corollary 4 *There exists a surjective continuous function* $h : \Pi A_i \to [0,1] \times [0,1] = I^2$.

Proof Let g be the function defined in the above proposition. Now consider the following diagram :

$$\begin{array}{ccc} \Pi A_i \times \Pi A_i & \xrightarrow{g \times g} & [0,1] \times [0,1] \\ \updownarrow \cong & \nearrow_{h} & \\ C \cong \Pi A_i & & \end{array}$$

[**Figure**] Space filling curve $\square$