# II.2 Compactness

**Definition 1** Let $X$ be a topological space and $S$ be a subset of $X$. A collection $\mathcal{U}$ is a covering of $S$ if $S \subset \bigcup_{U \in \mathcal{U}} U$. If each $U \in \mathcal{U}$ is an open set, $\mathcal{U}$ is called an open covering of $S$.

**Definition 2** A topological space $X$ is compact if every open covering of X has a finite subcovering, i.e., for each open covering $\mathcal{U}$ there exists a finite number of sets $\{U_1, U_2, \ldots, U_n\} \subset \mathcal{U}$ such that $X = \bigcup U_i$.

Note that a subset $A$ of $X$ is compact iff an open covering of $A$ in $X$ has a finite subcovering.

**Example** 1. $X$ has a indiscrete topology. $\Rightarrow X$ is compact.
2. $X$ is a finite set. $\Rightarrow X$ is compact.
3. A finite union of compact sets is compact.
4. $\mathbb{R}$ is not compact.
5. $(a, b]$ is not compact. Consider $(0, 1]$ and its open covering

$$\{(1/n, 1] | n = 1, 2, 3, \ldots\}.$$

6. (Heine-Borel property) Every closed and bounded interval $[a, b]$ in $\mathbb{R}$ is compact.
**Proof** Let $\mathcal{U}$ be an open covering of $I = [a, b]$.
Let $S := \{x \in I | [a, x] \textit{ can be covered by finitely many open sets in } \mathcal{U}\}$. Then $S \neq \emptyset$ since $a \in S$. Let $c := \sup S$. Since $\mathcal{U}$ is a covering, $\exists U \in \mathcal{U}$ such that $c \in U$. If $c \neq b$, choose $\epsilon > 0$ so that $(c - 2\epsilon, c + 2\epsilon) \subset U$. Since $[a, c - \epsilon]$ is covered by finitely many open sets in $\mathcal{U}$, say $U_1, \cdots U_n$, we have $[a, c + \epsilon] \subset (\bigcup_{i=1}^{n} U_i) \cup U$, i.e., $[a, c + \epsilon]$ is covered by finitely open sets, and hence $c + \epsilon \in S$. This is a contradiction to the fact $c = \sup S$, and we conclude that $c = b$. But in this case, by the same argument as above, $c = b \in S$. □

**Proposition 1** *$X$ is compact. $\Leftrightarrow$ If $\mathcal{C}$ is a family of closed subsets of $X$ with finite intersection property, i.e., $C_1 \bigcap C_2 \bigcap \ldots \bigcap C_n \neq \emptyset$ for any finite subcollection $\{C_1, \ldots, C_n\} \subset \mathcal{C}$, then $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.*

**Proof** Every collection of open subsets whose union is $X$ contains a finite subcollection whose union is X. $\Leftrightarrow$ Every collection of closed subsets whose intersection is empty contains a finite subcollection whose intersection is empty.
□

**Example** $\{(-\infty, n] \mid n \in \mathbb{Z}\}$ satisfies FIP but $\bigcap\{(-\infty, n] \mid n \in \mathbb{Z}\} = \emptyset$. $\Rightarrow \mathbb{R}$ is not compact.

**Proposition 2** *A closed subset of a compact space is compact.*

**Proof** Suppose $\mathcal{U}$ is an open covering of closed subset A. Then $\mathcal{U} \cup \{A^c\}$ is an open covering of X and hence $\exists$ a finite open subcovering $\mathcal{V} \subset \mathcal{U}$ of $X$. Now $\mathcal{V} - \{A^c\}$. is the desired finite subcovering for $A$. $\square$

**Lemma 3** *Suppose $X$ is Hausdorff. If $C$ is a compact subset of $X$ and $x$ is a point disjoint from $C$, then there exist two disjoint open neighborhoods $U$ and $V$ of $C$ and $x$ respectively. In other words, a compact set and a point can be separated by open sets in a Hausdorff space.*

**Proof** For each $y \in C$, the Hausdorff property implies that there are disjoint open neighborhoods $U_y \ni y$ and $V_y \ni x$. Since C is compact, $C \subset U_{y_1} \bigcap U_{y_2} \bigcap \ldots \bigcap U_{y_n} = U$. Set $V = V_{y_1} \bigcap V_{y_2} \bigcap \ldots \bigcap V_{y_n}$. $\square$

**Proposition 4** *A compact subset of a Hausdorff space is closed.*

**Proof** $C^c$ is open since, for any point $x \in C^c$, there exists an open neighborhood $O_x$ disjoint with $C$. $\square$

**Proposition 5** *Two disjoint compact subsets of a Hausdorff space have disjoint open neighborhoods.*

**Proof** Suppose C and D are compact subsets. For each $y \in D$, there exist disjoint open neighborhoods $U_y, V_y$ of $C$ and $y$ respectively. By compactness, $D$ is covered by finitely many $V_y$'s. Also $C$ is contained in the intersection of the corresponding finitely many open sets $U_y$'s. $\square$

**Theorem 6** *The product of finitely many compact spaces is compact.*

**Proof** It is sufficient to show that the product of two compact spaces is compact. Let $X$ and $Y$ be two compact spaces. Suppose $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ is an open covering of $X \times Y$. Fix $x \in X$. Then for each $y \in Y$, there is an open set $U \in \mathcal{U}$ which contains $(x, y)$, and a basic open neighborhood of $(x, y)$, $V \times W \subset \mathcal{U}$. The collection of all such basic open sets covers the compact set $\{x\} \times Y$ and hence has a finite subcovering $\{V_1 \times W_1, \cdots, V_n \times W_n\}$ where each $V_i \times W_i \subset U_i$ for some $U_i \in \mathcal{U}$.
Let $V_x = \bigcap_{i=1}^n V_i$, then $V_x \times Y \subset \bigcup_{i=1}^n (V_i \times W_i) \subset \bigcup_{i=1}^n U_i$ is an open set containing $\{x\} \times Y$. Now $\mathcal{V} = \{V_x | x \in X\}$ is an open covering of $X$. Since $X$

is compact, there is a finite subcovering $\{V_{x_1}, \ldots V_{x_n}\}$ of $\mathcal{V}$. Also each $V_{x_i} \times Y$ is covered by a finite number of elements in $\mathcal{U}$. Thus $X \times Y$ is covered by finitely many open sets in $\mathcal{U}$. $\square$

**Theorem 7** *(Generalized Heine-Borel) Suppose $A$ is a subset of $\mathbb{R}^n$, then $A$ is compact if and only if $A$ is closed and bounded.*

**Proof** Suppose that $A$ is compact, then $A$ is closed since $\mathbb{R}^n$ is Hausdorff. Consturct an open covering $\mathcal{B}$ of $A$ as

$$\mathcal{B} = \{B_n(0) \mid n = 1, 2, \ldots\}.$$

Since $A$ is compact, $\mathcal{B}$ has a finite subcovering and hence $A$ is contained in $B_n(0)$ for some $n \in \mathbb{N}$.
Conversely suppose that $A$ is closed and bounded, then there exists $r \in \mathbb{R}$ such that $A$ is contained in $B_r(0)$. Note that $A \subset B_r(0) \subset [-r, r]^n$. Thus $A$ is a closed subset of the compact space $[-r, r]^n$. Hence $A$ is compact. $\square$

**Example** 1. Closed balls and spheres are compact.
2. Cantor set is compact.
3. $I \bigcap \mathbb{Q}$ is not compact.
4. $\mathbb{R} \subset \mathbb{R}^2$ is not compact.

**Remark** Let $X$ be a metric space. If $A \subset X$ is compact, then $A$ is closed and bounded. But not conversely.

**Example** Consider a metric space $X$ with a metric

$$d(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the topology induced by $d$ is discrete. Thus, if $X$ is an infinite set, $X$ cannot be compact. However $X$ is bounded.

**Theorem 8** *Let $f : X \to Y$ be a continuous function from a space $X$ to a space $Y$. Then the followings hold.*
*1. If $X$ is compact, $f(X)$ is compact.*
*2. If $X$ is compact and $Y$ is Hausdorff, $f$ is a closed map.*
*3. If $X$ is compact, $Y$ is Hausdorff and $f$ is bijective, then $f$ is a homeomorphism.*

**Proof**
1. Let $\mathcal{U}$ be an open covering of $f(X)$, then $\mathcal{V} = \{f^{-1}(U) \mid U \in \mathcal{U}\}$ is an open covering of $X$ and has a finite subcovering $\{f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_n)\}$. Since $X \subset f^{-1}(U_1) \bigcup f^{-1}(U_2) \bigcup \ldots \bigcup f^{-1}(U_n) = f^{-1}(U_1 \bigcup \ldots \bigcup U_n)$, $f(X)$ is covered by a finite subcovering $\{U_1, \ldots, U_n\}$.
2. Suppose $A$ is a closed subset of $X$. Since $X$ is compact, $A$ is compact and $f(A)$ is compact. Now $f(A)$ is a compact subset of a Hausdorff space $Y$. Hence $f(A)$ is closed.
3. It is clear from the above result. □
**(exercise)** Let $f : X \to \mathbb{R}$ be a continuous map from a compact space $X$ to the real line $\mathbb{R}$. Show that $f$ attains its maximum and minimum.

**Corollary 9** *Let $f : X \to Y$ be a continuous function from a compact space $X$ to a Hausdorff space $Y$. If $f$ is continuous and injective, $f$ is an imbedding.*

**Example** 1. The function $f : \prod A_i \to [0,1]$, given by $f((a_i)) = \sum \frac{a_i}{3^i}$, is an imbedding.
2. A wild arc in $\mathbb{R}^3$ is an imbedding of an unit interval $[0,1]$ into $\mathbb{R}^3$.

**Definition 3** $X$ is countably compact if every countable covering has a finite subcovering.

**Definition 4** $X$ has the Bolzano Weierstrass Property(BWP) if every infinite subset of $X$ has an accumulation point.

**Theorem 10** *If $X$ is countably compact, the $X$ has BWP.*

**Proof** It suffices to show that every countably infinite subset has an accumulation point. Suppose that a countably infinite set $A = \{a_1, a_2, \ldots\}$ does not have an accumulation point. Then

(i) $A$ is closed since $\bar{A} = A \bigcup A' = A$,

(ii) each $a_i$ is an isolated point of $A$. Thus, for each $a_i$, there is an open neighborhood $O_i$ such that $O_i \bigcap A = \{a_i\}$.

Thus $\{O_i\} \bigcup \{X - A\}$ is an countable open covering of $X$. By the countable compactness there is a finite subcovering $\{O_{i_1}, \ldots, O_{i_n}\} \bigcup \{X - A\}$. Therefore some $O_{i_k}$ must contain infinitely many $a_i$'s in $A$. This is a contradiction. □

**Theorem 11** *If a space $X$ is Hausdorff and has BWP, then $X$ is countably compact.*

**Proof** Let $\mathcal{U} = \{U_1, \dots\}$ be a countable open covering of $X$. We may assume that $\mathcal{U}$ is not redundant, i.e., $U_{n+1}$ is not contained in $U_1 \bigcup \dots \bigcup U_n$ for each $n$. Suppose that $\mathcal{U}$ does not have a finite subcovering. Then there is a set $A = \{x_n \in X \mid n = 1, 2, \dots\}$ such that each $x_n$ is in $U_n - (U_1 \bigcup \dots \bigcup U_{n-1})$. $A$ is an infinite set since $x_i \neq x_j$ if $i \neq j$. By BWP, $A$ has an accumulation point $x \in A$. There is $U_n \in \mathcal{U}$ which contains $x$. From the Hausdorff condition, $U_n$ should contain infinitely many $x_i$'s. This is a contradiction. $\square$

**Remark** Compact $\Rightarrow$ Countably compact $\Rightarrow$ BWP