IV.2 Basic topological properties

3. Tychonoff Theorem

Theorem 1 (Tychonoff Theorem) *The product of compact spaces is compact.*

Definition 1 Let $\mathcal{B}$ be a basis for X. Then $\mathcal{B}' \subset \mathcal{B}$ is called a basic open covering if $\bigcup \mathcal{B}' = X$. Also, let $\mathcal{S}$ be a subbasis for X. Then $\mathcal{S}' \subset \mathcal{S}$ is called a subbasic open covering if $\bigcup \mathcal{S}' = X$.

Lemma 2 *A topological space X is compact.*
$\Leftrightarrow$ *For a given basis, any basic open covering has a finite subcovering.*

proof. ($\Rightarrow$)Trivial.
($\Leftarrow$)Let $\mathcal{U} = \{U\}$ be an open covering of X. Since each $U \in \mathcal{U}$ is a union of basic open sets, the collection of such basic open sets is a covering of X and hence there exists a finite subcovering $\{B_1, B_2, \cdots, B_k\}$
Since each B_i is contained in some $U_i \in U$, $\{U_1, U_2, \cdots, U_n\}$ gives a finite subcovering of X.
□

Lemma 3 *A topological space X is compact.*
$\Leftrightarrow$ *For a given subbasis, any subbasic open covering has a finite subcovering (i.e., every collection of subbasic closed set = complement of subbasic open set) of finite intersection property has a nonempty intersection.*

proof.($\Rightarrow$)Trivial.
($\Leftarrow$)Let $\mathcal{S}$ be a given subbasis and $\mathcal{B} = \mathcal{B}(\mathcal{S})$ be the induced basis. Let $\mathcal{C}$ be a collection of basic closed sets with finite intersection property.
$\mathfrak{A} = \{\mathfrak{D}$: collection of basic closed sets with finite intersection property $\mid \mathcal{C} \subset \mathfrak{D}\}$ is a partially ordered set with respect to inclusion and each chain has an upper bound(the union of each chain has a finite intersection property and contains $\mathcal{C}$). By Zorn's lemma, $\mathfrak{A}$ has a maximal element $\mathcal{M} \in \mathfrak{A}$(collection of basic closed sets with finite intersection property).
Now it suffices to show that $\phi \neq \bigcap\{M \in \mathcal{M}\}(\subset \bigcap\{C \in \mathcal{C}\})$.
Each $M \in \mathcal{M}$ is basic closed and this is a finite union of subbasic closed sets S_i^c, i.e., $M = (\cap S_i)^c = S_1^c \cup S_2^c \cup \cdots \cup S_k^c$.

Claim 1 *At least one ${S_i}^c \in \mathcal{M}$.*

proof of claim Suppose not. Then $\mathcal{M} \cup \{S_1^c\} \supset \mathcal{C}$ does not have finite intersection property by maximality.
Thus, $S_1^c \bigcap (M_1^1 \cap \cdots \cap M_{l_1}^1) = \phi$ for some M_i^1's in $\mathcal{M}$
Similarly, $S_j^c \bigcap (M_1^j \cap \cdots \cap M_{l_j}^j) = \phi$ for $j = 1, 2, \cdots, k$
Therefore $(S_1^c \cup \cdots \cup S_k^c) \bigcap (\bigcap_{i=1}^{l_1} M_i^1) \bigcap \cdots \bigcap (\bigcap_{i=1}^{l_k} M_i^k) = \phi$
which is a contradiction to the fact that $\mathcal{M}$ has finite intersection property. $\square$

Let's denote this subbasic closed set S_i^c obtained in the above claim by $S^c(M)$ so that $S^c(M)$ is a subbasic closed set, s.t. $S^c(M) \subset M$ and $S^c(M) \in \mathcal{M}$.
Let $\mathcal{F} := \{S^c(M) \in \mathcal{M} | M \in \mathcal{M}\}$
Then $\mathcal{F}$ has a finite intersection property since $\mathcal{F} \subset \mathcal{M}$ and by hypothesis, $\phi \neq \bigcap_{M \in \mathcal{M}} S^c(M) \subset \cap_{M \in \mathcal{M}} M$ ($\because S^c(M) \subset M$). $\square$

proof of Tychonoff theorem Let $X = \prod_{i \in I} X_i$ where X_i are compact. We want to show that X is compact. Let $\mathcal{S}$ be the subbasis defining the product topology of X and $\mathcal{F}$ be a collection of subbasic closed sets with finite intersection property.
Show $\bigcap \mathcal{F} \neq \phi$:
Let $F \in \mathcal{F}$. Then $p_i(F)$ is a closed set in X_i or equal to X_i
$\{p_i(F) | F \in \mathcal{F}\}$ has a finite intersection property since $\mathcal{F}$ has a finite intersection property. Therefore $\bigcap_{F \in \mathcal{F}} p_i(F) \neq \phi$ by compactness of X_i
$\Rightarrow \exists x_i \in \bigcap_{F \in \mathcal{F}} p_i(F)$ for all i
$\Rightarrow x = (x_i) \in \bigcap_{F \in \mathcal{F}} F$
($x \in \forall F$ since $x_i \in p_i(F)$ for all i and F is a subbasic closed set) $\square$