III.1 Compact metric space

정의 1 A sequence in a space X is a function $x : \mathbb{N} \to X$ usually written as $(x_n)_{n=1}^{\infty}$ where $x_n = x(n)$.

정의 2 $x_0 \in X$ is a limit point of (x_n) if $\forall U$: open neighborhood of x_0, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \in U$

정의 3 x_0 is a cluster point of (x_n) if $\forall U$: open neighborhood of x_0 and $\forall N \in \mathbb{N}$, $\exists n > N$ such that $x_n \in U$.

정의 4 If $\mu : \mathbb{N} \to \mathbb{N}$ is a monotone increasing function (i.e., $n > m \Rightarrow \mu(n) > \mu(m)$), then $x \circ \mu : \mathbb{N} \to X$ is called a subsequence of (x_n)

Note Let X be a metric space. Then x_0 is a cluster point of (x_n) iff x_0 is a limit point of a subsequence.

(exercise)

Sequential compactness

정의 5 A metric space is sequentially compact if every sequence has a convergent subsequence.(i.e., every sequence has a cluster point)

정리 1 *If X is a metric space, X is sequentially compact $\Leftrightarrow$ X has BWP.*

증명
($\Rightarrow$) Let $A = \{x_1, x_2, \ldots\}$ be a countably infinite set in X. Then the sequence (x_n) has a cluster point x by sequential compactness and x is clearly an accumulation point of A.
($\Leftarrow$) Let (x_n) be a sequence. (i) If $\{x_n\}$ is an infinite set, then by BWP there exists an accumulation point x of $\{x_n\}$, and x is a cluster point of (x_n). (ii) If $\{x_n\}$ is a finite set, then infinitely many x_i's are identical points and such points constitute a convergent subsequence. □

Second countable metric space

정의 6 A space X is separable if it has a countable dense subset.

정리 2 *A metric space X is separable $\Leftrightarrow$ X is second countable.*

증명
($\Leftarrow$) Let $\{O_n | n = 1, 2, \ldots\}$ be a countable basis of X. Choose a point x_n from each O_n. Then $\{x_n\}$ is a countable dense subset of X.
($\Rightarrow$) Let S be a countable dense subset. Then $\{B_r(s) | s \in S, r \in \mathbb{Q}\}$ becomes a countable basis. □

Example

1. $\overline{\mathbb{Q}} = \mathbb{R}$, $\mathbb{Q}$: countable $\Rightarrow$ $\mathbb{R}$: separable.

Totally bounded metric space

정의 7 A subset A of a metric space X is called an ϵ-net if the following three conditions hold.
(i) A is a finite set.
(ii) $X = \bigcup_{a \in A} B_\epsilon(a)$.
(iii) $d(a, b) \geq \epsilon, \forall a, b \in A$ with $a \neq b$.

정의 8 A metric space X is said to be totally bounded if it has an ϵ-net for $\forall \epsilon > 0$.

remark X : totally bounded $\overset{\not\Leftarrow}{\Rightarrow}$ X : bounded

명제 3 *A metric space X is totally bounded $\Rightarrow$ X is separable.*

증명
Let A_n be a $\frac{1}{n}$-net. Then $A = \bigcup_{n=1}^{\infty} A_n$ is a countable dense subset of X. □

명제 4 *If a metric space X has BWP, then it is totally bounded.*

증명
Choose a point $a_1 \in X$. If $B_\epsilon(a_1)$ covers X, $\{a_1\}$ is a ϵ-net. If it does not, choose $a_2 \in X - B_\epsilon(a_1)$, and consider $\{B_\epsilon(a_1), B_\epsilon(a_2)\}$. If this covers X, then this is an ϵ-net. If it does not, choose $a_3 \in X - (B_\epsilon(a_1) \cup B_\epsilon(a_2))$, and consider $\{B_\epsilon(a_1), B_\epsilon(a_2), B_\epsilon(a_3)\}$,..., and so on.

(Claim) This process stops in a finite step to give an ϵ-net.

Suppose not. Then we obtain an infinite set $A = \{a_1, a_2, a_3, a_4, \ldots\}$with the property that $d(a_i, a_j) \geq \epsilon$ if $i \neq j$ by construction. Also, by BWP, A has an accumulation point $a \in X$. Now $B_{\epsilon/2}(a)$ contains infinitely many a_i's and this is a contradiction to the fact that $d(a_i, a_j) \geq \epsilon$ for $i \neq j$. □

Homework

1. Is there a separable metric space which is not totally bounded?
2. Is there a totally bounded metric space which does not satisfy BWP?

정리 5 *If X is a metric space, then X is countably compact* $\Leftrightarrow$ *X is compact.*

증명

($\Leftarrow$) Clear

($\Rightarrow$) Countably Compact $\Rightarrow$ BWP $\Rightarrow$ Totally bounded $\Rightarrow$ Separable $\Rightarrow$ 2nd countable. Let $\mathcal{O} = \{O_n\}$ be a countable basis and $\mathcal{U}$ be an open covering of X. Then $U \in \mathcal{U}$ is a union of basic open sets in $\mathcal{O}$. Since $\mathcal{O}$ is a countable basis of X, $\mathcal{U}$ has a countable subcovering and hence has a finite subcovering by contable compactness. □

remark (Summary for a metric space)

Compact $\Leftrightarrow$ Countably compact $\Leftrightarrow$ BWP $\Leftrightarrow$ Sequentially compact
$\Downarrow$
Totally Bounded
$\Downarrow$
Separable $\Leftrightarrow$ 2nd countable

정리 6 *(Lebesgue covering lemma)In a compact metric space X, every open covering* $\mathcal{U}$ *has a Lebesgue number. i.e.,* $\exists \epsilon > 0$ *depending only on* $\mathcal{U}$ *such that* $\forall x \in X, \exists U \in \mathcal{U}$ *with* $B_\epsilon(x) \subset U$

증명

$B_a \subset X$ is called a big ball if it is not contained in any $U \in \mathcal{U}$. Let $A = \{a \in \mathbb{R} | O_a(x)$ is a big ball for some $x \in X$ $\}$ and we'll show that $inf A > 0$.
Suppose $inf A = 0$, then $\forall n, \exists$a big ball $B_a(x_n)$ such that $0 < a < 1/n$. Since X is sequentially compact, (x_n) has a cluster point x. Hence (x_n) has a subsequence converging to x. Now that $\mathcal{U}$ is an open covering, $\exists U \in \mathcal{U}$ such that $x \in U$ and $B_\delta(x) \subset U$ for some $\delta > 0$. Choose sufficiently large n so that $x_n \in B_{\delta/2}(x)$ and $\frac{1}{n} < \frac{\delta}{2}$. Then $B_a(x_n) \subset B_\delta(x) \subset U$. We reach a contradiction. □

따름정리 7 *Let* $f : X \to Y$ *be a continuous function between metric spaces. If* X *is compact, then* f *is uniformly continuous.*

증명

Note that $\mathcal{U} = \{f^{-1}(B_\epsilon(y)) | y \in Y\}$ is an open covering of X. Apply the Lebesgue covering lemma. □