III.2 Complete Metric Space

Definition 1 A metric space is complete if every Cauchy sequence is convergent.

Note A sequence (x_n) is a Cauchy sequence if $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $n, m > N \Rightarrow d(x_n, x_m) < \epsilon$

Theorem 1 *A metric space X is compact if and only if it is complete and totally bounded.*

proof
$(\Rightarrow)$ Since X is compact, it is totally bounded. And it is also sequentially compact. Thus every sequence(therefore every Cauchy sequence) has a convergent subsequence. Then a cluster point becomes a limit point of the sequence by the Cauchy condition. Hence X is complete.
$(\Leftarrow)$ Since X is a metric space, it suffices to show that X is sequentially compact. Let (x_n) be a sequence in X. We want to find a subsequence which is Cauchy, then this is convergent by completeness of X. Now we apply totally boundedness to X: Let A_1 be a 1-net of $X = \bigcup_{i=1}^{n_1} O_1(a_i^1)$. Since A_1 is finite, there is a subsequence $x^1 = (x_1^1, x_2^1, \ldots, x_n^1, \ldots)$ of $x = (x_n)$ such that $\{x_i^1 | i = 1, 2, \ldots\}$ is contained in one of $O_1(a_i^1)$'s. For a given $\frac{1}{2}$-net A_2 of $X = \bigcup_{i=1}^{n_2} O_{\frac{1}{2}}(a_i^2)$, there is a subsequence $x^2 = (x_1^2, x_2^2, \ldots, x_n^2, \ldots)$ of x^1 such that $\{x_i^2 | i = 1, 2, \ldots\}$ is contained in one of $O_{\frac{1}{2}}(a_i^2)$'s. In this way we can construct subsequences x^n, $n = 1, 2, \ldots$ and consider a sequence $(x_1^1, x_2^2, x_3^3, \ldots)$. Then this is obviously a Cauchy sequence. □

Homework 1 A subset of a complete metric space X is complete if and only if it is closed.

Proposition 2 $\mathbb{R}^n$ *is complete.*

proof Let (x_n) be a Cauchy sequence.
(1^{st} *step*) (x_n) is bounded: $\exists N$ such that $n, m \geq N \Rightarrow |x_n - x_m| < \varepsilon = 1$.
Let $M = max\{|x_1|, \cdots |x_N|, |x_N| + 1\}$. Then $|x_n| \leq M, \forall n$.
(2^{nd} *step*) Since $B(0, M)$ (= closed ball of radius M) is a compact metric space, it is complete and hence a Cauchy sequence (x_n) converges. □

Recall Let X be a topological space. Let $\mathcal{B}(X, \mathbb{R}^n)$ be the vector space of all bounded functions of X into $\mathbb{R}^n$, and let $\mathcal{C}(X, \mathbb{R}^n)$ be the vector space of all bounded continuous functions of X into $\mathbb{R}^n$. The norm defined by $\|f\| := sup_{x\in X}|f(x)|$ gives a normed linear space structure on these spaces.

Proposition 3 *(1)* $\mathcal{B}(X, \mathbb{R}^n)$ *is complete.*
(2) $\mathcal{C}(X, \mathbb{R}^n)$ *is a closed subset of* $\mathcal{B}(X, \mathbb{R}^n)$ *and hence is complete.*

proof
Let (f_n) be a Cauchy sequence in $\mathcal{B}(X, \mathbb{R}^n)$, i.e., $\forall \varepsilon > 0, \exists N$ such that $m, n > N \Rightarrow \|f_m - f_n\| < \varepsilon$. Then for each $x \in X$, $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}^n$, and hence converges to a point, say $f(x)$.
Claim f is a uniform limit of (f_n), i.e., $\forall \epsilon > 0, \exists N$ such that $n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon$, $\forall x \in X$.
proof of claim Since (f_n) be a Cauchy sequence, $\forall \epsilon > 0, \exists N$ such that $n \geq N \Rightarrow |f_n(x) - f_N(x)| < \epsilon/2$, $\forall x \in X$, which implies that $|f(x) - f_N(x)| \leq \frac{\epsilon}{2}, \forall x \in X$. Now $|f(x)-f_n(x)| \leq |f(x)-f_N(x)|+|f_N(x)-f_n(x)| < \epsilon, \forall x \in X$. This completes the proof of Claim.
Now f is clearly bounded since $f_n \to f$ uniformly and f_n is bounded. This shows that $f \in \mathcal{B}(X, \mathbb{R}^n)$.
Uniform limit of continuous functions is continuous
Suppose that $f_n \to f$ with $f_n \in \mathcal{C}(X, \mathbb{R}^n)$. We want to show that f is continuous. Fix any $x_0 \in X$ and $\epsilon > 0$.
Since $f_n \to f$ uniformly, $\exists N$ such that $n \geq N \Rightarrow |f_n(x)-f(x)| < \epsilon/3, \forall x \in X$.
For a such f_n with $n \geq N$, since f_n is continuous, $\exists U(x_0)$: open nbhd of x_0 such that $x \in U(x_0) \Rightarrow |f_n(x) - f_n(x_0)| < \epsilon/3$.
Now $x \in U(x_0) \Rightarrow |f(x)-f(x_0)| \leq |f(x)-f_n(x)|+|f_n(x)-f_n(x_0)|+|f_n(x_0)-f(x_0)| < \epsilon$. □

[Diagram: Summary for metric spaces]

Homework 2 Find the examples of metric spaces which fit into the 'blanks' of the above diagram.

Homework 3 Let D be a dense subset of a metric space X. If $f : D \to Y$ is a uniformly continuous function which is defined on D, where Y is a complete metric space, then f has a uniformly continuous extension $\overline{f}$ on X to Y and the extension is unique.