IV.Normal Space

1. Normal space

Definition 1 A topological space X is **normal(T_4)** if it is Haussdorff and for any two disjoint closed sets of X, there are separating open neighborhoods, i.e., for any disjoint closed subsets C_1 and C_2 of X, there exist open sets O_1 and O_2 s.t. $C_1 \subset O_1$ and $C_2 \subset O_2$ and $O_1 \cap O_2 = \phi$.

Recall X is regular if there are separating open neighborhoods for a point and a disjoint closed set.

Example A compact Hausdorff space is normal.

proof. A closed subset of compact space is compact. We can separate two disjoint compact subsets of a Hausdorff space. □

Example A metric space X is normal.

proof. For given disjoint closed subsets C_1 and C_2 in X, let
$O_1 := \{x \in X | d(x, C_1) < d(x, C_2)\}$ and $O_2 := \{x \in X | d(x, C_1) > d(x, C_2)\}$.
Then clearly $O_1 \cap O_2 = \phi$ and $C_1 \subset O_1, C_2 \subset O_2$. Now let's show that O_1 and O_2 are open. Since $g(x) = d(x, C_1) - d(x, C_2)$ is a continuous function and $O_1 = g^{-1}(-\infty, 0)$ and $O_2 = g^{-1}(0, \infty)$. Hence a metric space is normal. □

Proposition 1 *X is normal.*
$\Leftrightarrow \forall$ *closed $A \subset X$ and open $U \supset A$, there exists open V s.t. $A \subset V \subset \overline{V} \subset U$.*

proof. $(\Rightarrow) U^c$ is closed.
U^c and A are two disjoint closed sets.
$\Rightarrow \exists$ disjoint open sets V and W s.t. $A \subset V$ and $U^c \subset W$.
$A \subset V \subset W^c$ and W^c is closed
$\Rightarrow \overline{V} \subset W^c$
$\Rightarrow \overline{V} \cap W = \phi$.
Since $U^c \subset W$, $\overline{V} \cap U^c = \phi$ and then $\overline{V} \subset U$.
$(\Leftarrow)$Let C and D be two disjoint closed sets. Then $U := D^c$ is an open set containing C. Hence there exists open V s.t. $A \subset V \subset \overline{V} \subset U$. Also $V \cap \overline{V}^c = \phi$. Then V and $\overline{V}^c$ are two seperating open neighborhoods of C and D. □

Note that similar statement also holds for a regular space.

Homework. Prove the followings.
(a) A subspace of a regular space is regular.
(b) A product of regular spaces is regular.

Remark A product of normal spaces need not be normal: It is not difficult to show that $\mathbb{R}_l$ is normal (and hence regular). But $\mathbb{R}_l \times \mathbb{R}_l$(=Sorgenfrey plane) is not normal. (See Munkres p198.) But notice that $\mathbb{R}_l \times \mathbb{R}_l$ is regular.
(regular + 2nd countable $\Rightarrow$ normal. See p200.)

Remark If J is uncountable, the product space $\mathbb{R}^J$ is not normal. (A difficult exercise: p206)

This example serves three purposes. Firstly, it shows that a regular space $\mathbb{R}^J$ need not be normal. Secondly, it shows that a subspace of a normal space need not be normal, for $\mathbb{R}^J \cong (0,1)^J \subset [0,1]^J$. We can easily see that $[0,1]^J$ is normal since it is compact by Tychonoff theorem and Hausdorff. (It is easy to show that a closed subset of normal space is normal.) Lastly, it shows that a product of normal space need not be normal.