5. 1-dimensional manifold

We will classify 1-dimensional manifold. To do this, first we need a concept of triangulation.

Definition 1 A collection $\{T_i\}$ of subsets of 1-manifold is called a **triangulation** of M if

1. $\forall i, T_i \approx [0, 1]$
2. $M = \cup_i T_i$
3. $T_i \cap T_j = \emptyset$ or "a single vertex" if $i \neq j$.
4. (local finiteness) $\forall p \in M$, $\exists$ a neighborhood U of p s.t. U meets only finitely many T_i's.

cf) In general, a triangulation is a certain well matched collection of n-*simplexes* which are defined as the closed convex hull of $n + 1$ affinely independent points. A 1-simplex is a finite line segment. A 2-simplex is a triangle and a 3-simplex is a tetrahedron.

Theorem 1 *Let M be a 2nd countable and connected 1-manifold with boundary. Then*

1. *$\exists$ a countable triangulation on M.*
2. *M is homeomorphic to $\mathbb{R}^1$,S^1, $[0, 1]$, or $(0, 1](\approx [0, 1))$.*

Proof (Existence of triangulation:)
Consider a system of coordinate charts $\{(U_x, \phi_x : U_x \to \mathbb{R} \text{ (or } \mathbb{H}))\}_{x \in M}$, and let $U'_x = \phi_x^{-1}((-1, 1))$. We may cover M by $\{U'_x\}$. Since M is 2nd countable, we may choose a countable subcover$\{U'_1, U'_2, U'_3, \ldots\}$.

Let $K_1 = \{\phi_1^{-1}([-1, 1])\}$. Inductively, define $K_{n+1} := K_n \cup$ {closures of components of $\{\phi_{n+1}^{-1}([-1, 1]) - |K_n|\}\}$ where $|K_n| = \bigcup_{A \in K_n} A$. Note that

$|K_n| \supset U'_n$. Then each K_n is a finite triangulation on a subset of M containing $\cup_{i=1}^n U'_i$, and $K_1 \subset K_2 \subset \ldots$. Now $K = \cup_{i=1}^\infty K_i$ is the desired triangulation of M.

(Proof of classification:)

M has a countable triangulation K. Since each element of K is homeomorphic to $[-1, 1]$, we may consider K as a collection of line segments, the end point of which should be identified with another end point or with nothing.

A vertex is the end point of a line segment or the intersection of two line segments. It is not possible for three or more line segments to meet at one vertex because of the manifold condition. Suppose it is possible, and consider the homeomorphic image of the three or more line segments meeting at a vertex into $\mathbb{R}$. Eliminating the vertex gives three or more components, while eliminating the image of the vertex in $\mathbb{R}$ gives two components, which is a contradiction. Therefore, each end point must be identified with at most one another point.

Take a line segment of K, pick another segment which has common point with the previous segment, and paste it. We must be able to do this for all elements of K. Otherwise, we get two or more components, which is contradictory to the connectedness of M. Let l_n be the resulting space after $n - 1$ pasting. Clearly $l := \cup_n l_n \approx M$.

In case of K finite, l is homeomorphic to S^1, or to a closed interval. In case of K infinite, l is homeomorphic to $\mathbb{R}$, or to a half-open interval.

$\square$