Further examples

1. Projective space

(a) Define an equivalence relation $\sim$ on $\mathbb{R}^{n+1}\backslash\{0\}$ by

$$x = (x_1, \ldots, x_{n+1}) \sim y = (y_1, \ldots, y_{n+1})$$

if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}\backslash\{0\}$.
Now let $\mathbb{R}P^n := \mathbb{R}^{n+1}/\sim$. Denote equivalence class of x by $[x] = [x_1, x_2, \ldots, x_{n+1}]$.
This is a manifold, in fact C^∞-manifold.
Consider $U_1 = \{[x] = [x_1, \ldots, x_{n+1}] : x_1 \neq 0\}$ and define $\phi_1 : U_1 \to \mathbb{R}^n$ by

$$[x_1, \ldots, x_{n+1}] \mapsto \left(\frac{x_2}{x_1}, \ldots, \frac{x_{n+1}}{x_1}\right).$$

We claim that ϕ_1 is a homeomorphism:
Let $V_1 = \{x = (x_1, \ldots, x_{n+1}) : x_1 \neq 0\} \subset \mathbb{R}^{n+1}\backslash\{0\}$, and define $\psi_1 : V_1 \to \mathbb{R}^n$ by

$$(x_1, \ldots, x_{n+1}) \mapsto \left(\frac{x_2}{x_1}, \cdots, \frac{x_{n+1}}{x_1}\right).$$

Since U_1 is a quotient space of V_1 and ϕ_1 is induced by ψ_1, ϕ_1 is a continuous map. Let $q : V_1 \to U_1$ be a quotient map and let $\sigma_1 : \mathbb{R}^n \to V_1$ be given by $(y_1, \ldots, y_n) \mapsto (1, y_1, \ldots, y_n)$. then $\phi_1 \circ (q \circ \sigma_1) = id_{\mathbb{R}^n}$ and $(q \circ \sigma_1) \circ \phi_1 = id_{\mathcal{U}_1}$.
Hence ϕ_1 is a homeomorphism.

HW Show that $\mathbb{R}P^n$ is a C^∞-manifold.

(b)

$$\begin{array}{ccc} & i & \\ S^n & \hookrightarrow & \mathbb{R}^{n+1}\backslash\{0\} \\ \downarrow q & & \downarrow q \\ S^n/\sim & \longrightarrow & \mathbb{R}\mathbf{P}^n \\ & \bar{i} & \end{array}$$

$\sim$: antipodal identification, where antipodal map $A : S^n \to S^n$ is defined by $x \mapsto -x$. Equivalence relation on S^n is given by $x \sim -x = Ax$.
Then there exists a well-defined bijective continuous map $\bar{i}$, because i is an embedding and q is a quotient map. Since S^n is compact and $\mathbb{R}\mathbf{P}^n$ is Hausdorff, $\bar{i}$ is a homeomorphism.

(dim 1) $\mathbb{R}\mathbf{P}^1 = S^1/\sim = S^1$
The quotient map may be given by $z \mapsto z^2$.

(dim 2)
$\mathbb{R}\mathbf{P}^2$ with one point deleted is homeomorphic to open Möbius band.

2. Product manifold

M^m, N^n : manifolds $\Rightarrow M \times N$ is an $(m+n)$-manifold.
$\because \forall (p,q) \in M \times N$, $p \in M$ has a coordinate neighborhood (U, ϕ) homeomorphic to an open subset of $\mathbb{R}^m$ and $q \in N$ has a coordinate neighborhood (V, ψ) homeomorphic to an open subset of $\mathbb{R}^n$. It follows that (p,q) has a coordinate neighborhood $(U \times V, \phi \times \psi)$ homeomorphic to an open subset of $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$.

Example 1 $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$, $T^2 = S^1 \times S^1$, $T^3 = S^1 \times S^1 \times S^1$, ...

3. Other constructions

Connected sum, boundary identification, ...

Example 2 S^3 can be obtained by identifying the boundaries of two solid tori as follows:
$D^2 \times D^2 = D^4$
$\Rightarrow \partial(D^2 \times D^2) = \partial(D^4) = S^3$
$\Rightarrow (\partial D^2 \times D^2) \bigcup (D^2 \times \partial D^2) = (S^1 \times D^2) \bigcup (D^2 \times S^1) = S^3$

The picture shows the decomposition of S^3 as a union of two solid tori.

HW Can you decompose S^3 as a union of two handle bodies of genus 2?

4. Lie group

정의 1 A topological space X is a **topological group** if
1. X is a group.
2. $\mu : X \times X \to X$ given by $(x, y) \mapsto xy^{-1}$ is continuous.

Example 3 Topological group

1. Any group G with discrete topology.
2. $\mathbb{R}^n$: additive group.
   $\because (x,y) \mapsto x-y$ is continuous.
3. $S^1 \subset \mathbb{C}$ is a multiplicative group.
   $\because S^1 \times S^1 \to S^1$ by $(z,w) \mapsto \frac{z}{w}$ is continuous.
4. General linear group $Gl(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) : detA \neq 0\} \subset \mathbb{R}^{n^2}$
   $\because$ The map given by $(A,B) \mapsto AB^{-1}$ is continuous.

정의 2 A ***Lie group*** is a topological group X which is a smooth n-manifold such that μ is C^∞

Above examples are all *Lie groups*.

5. Manifold with boundary

정의 3 A Hausdorff space M is an n-manifold with boundary if $\forall p \in M, \exists$ a coordinate chart (U,ϕ) of p which is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n$, where $\mathbb{H}^n = \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n : x_n \geq 0\}$, with $\phi(p) = 0$. $\partial M = \{x \in M : x$ has a coordinate neighborhood homeomorphic to $\mathbb{H}^n\}$ is called a boundary of M.

The notion of boundary point is well-defined by the following theorem.

정리 1 ***(Invariance of domain)*** *Let $U \subset \mathbb{R}^n$ be an open set and $h : U \to \mathbb{R}^n$ be 1-1 and continuous map. The $h(U)$ is open in $\mathbb{R}^n$.*

Invariance of domain implies that the image of an interior point by 1-1 and continuous map is also an interior point and the image of a boundary point is also a boundary point.

If M is an n-manifold, then ∂M is an $(n-1)$-manifold.