# Classification of Compact Surfaces

## 0. Introduction

By the term '*surface*', we mean a 2-dimensional manifold. Here are some examples of surfaces :

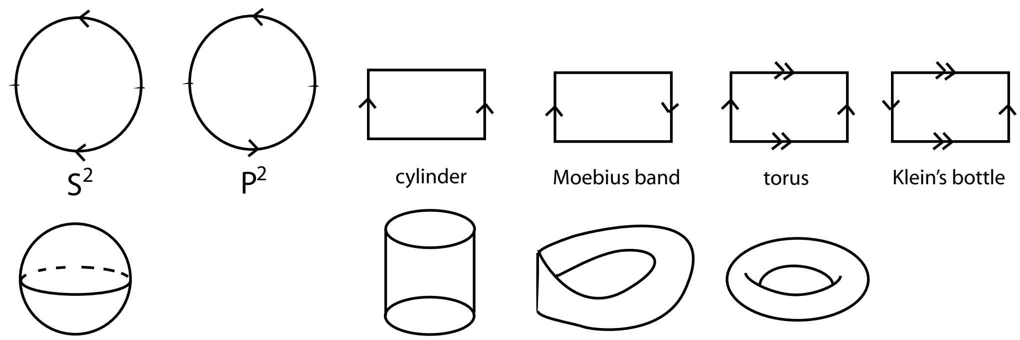


그림 1: **Examples of 2-manifolds.**

**Goal** :

1. Classify compact surfaces assuming the existence of triangulation on the surfaces.
2. Show each compact surface can be obtained by identifying the boundary edges of a polygon.
3. Discuss a complete set of topological invariants to distinguish surfaces.

## 1. Triangulation and Rado's theorem

**Definition 1** A **triangulation** of a compact surface $M$ consists of a finite set of triangles $K = \{T_1, \ldots, T_n\}$ and homeomorphisms $\{\phi_i : T_i \to T_i' \subset \mathbb{R}^2\}$, where $T_i' = \phi_i(T_i)$ is a triangle in $\mathbb{R}^2$ such that :

1. $\cup_i T_i = M$
2. For $i \neq j$, $T_i \cap T_j \neq \emptyset$ implies $T_i \cap T_j$ is either a single vertex or a single edge.

A vertex or an edge of $T_i$ is pre-image of a vertex or an edge of a triangle in $\mathbb{R}^2$ by $\phi_i$

**Theorem 1** *(Rado's theorem, 1925) There exists a triangulation on a compact surface.*

**Claim 1** *Each edge is an edge of exactly one or two triangles.*

**Claim 2** *Let $v$ be a vertex of a triangulation, which is not a boundary point of $M$. Then we can arrange triangles $T_1, \ldots, T_n$ around $v$ such that $T_i$ and $T_i + 1$ for $i = 1, \ldots, n-1$, and $T_n$ and $T_1$ have an edge in common cyclically.*

**Proof** 1 Suppose there is an edge which is an edge of three or more triangles. Take two among those triangles. There is a coordinate chart $\psi$ defined on a neighborhood $U$ of $p$ on the edge.

The space $S$ of two sheets of triangles is a topological disk. Consider a set $V \subset U \cap S$, which is relatively open in $S$, containing $p$, and homeomorphic to $\mathbb{R}^2$. Let $\psi' : V \to \mathbb{R}^2$ be the homeomorphism. Then $\psi'(V) \subset \mathbb{R}^2$ is open. Applying invariance of domain theorem to $\psi \circ \psi'^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$, we get $\psi(V) \subset \mathbb{R}^2$ is open.

Take a sequence $\{x_n\} \subset U - S$ converging to $p$. Continuity of $\psi$ implies that $\{\psi(x_n)\}$ converges to $\psi(p)$. Since $\psi(V)$ is a open neighborhood of $\psi(p)$, $\psi(V)$ must contain $\psi(x_i)$, for some $i$, which is contradictory to the injectiveness of $\psi$.

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**Proof** 2 This follows from the claim 1, and the definition of triangulation.

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