Classification of Compact Surfaces

0. Introduction

By the term '*surface*', we mean a 2-dimensional manifold. Here are some examples of surfaces :

Goal :

1. Classify compact surfaces assuming the existence of triangulation on the surfaces.

2. Show each compact surface can be obtained by identifying the boundary edges of a polygon.

3. Discuss a complete set of topological invariants to distinguish surfaces.

1. Triangulation and Rado's theorem

Definition 1 A **triangulation** of a compact surface M consists of a finite set of triangles $K = \{T_1, \ldots, T_n\}$ and homeomorphisms $\{\phi_i : T_i \to T_i' \subset \mathbb{R}^2\}$, where $T_i' = \phi_i(T_i)$ is a triangle in $\mathbb{R}^2$ such that :

1. $\cup_i T_i = M$

2. For $i \neq j$, $T_i \cap T_j \neq \emptyset$ implies $T_i \cap T_j$ is either a single vertex or a single edge.

3. The transition map on a common edge is a linear homeomorphism, i.e., for $e = T_i \cap T_j$, $\phi_j \circ \phi_i^{-1} : \phi_i(e) \subset T_i' \to \phi_j(e) \subset T_j'$ is a linear homeomorphism.

A vertex or an edge of T_i is pre-image of a vertex or an edge of a triangle in $\mathbb{R}^2$ by ϕ_i

Theorem 1 *(Rado's theorem, 1925) There exists a triangulation on a compact surface.*

Claim 1 *Each edge is an edge of exactly one or two triangles.*

Claim 2 *Let v be a vertex of a triangulation, which is not a boundary point of M. Then we can arrange triangles $T_1, \ldots, T_n$ around v such that T_i and T_i+1 for $i = 1, \ldots, n-1$, and T_n and T_1 have an edge in common cyclically.*

Proof 1 Suppose there is an edge which is an edge of three or more triangles. Take two among those triangles. There is a coordinate chart ψ defined on a neighborhood U of p on the edge.

The space S of two sheets of triangles is a topological disk. Consider a set $V \subset U \cap S$, which is relatively open in S, containing p, and homeomorphic to $\mathbb{R}^2$. Let $\psi' : V \to \mathbb{R}^2$ be the homeomorphism. Then $\psi'(V) \subset \mathbb{R}^2$ is open. Applying invariance of domain theorem to $\psi \circ \psi'^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$, we get $\psi(V) \subset \mathbb{R}^2$ is open.

Take a sequence $\{x_n\} \subset U - S$ converging to p. Continuity of ψ implies that $\{\psi(x_n)\}$ converges to $\psi(p)$. Since $\psi(V)$ is a open neighborhood of $\psi(p)$, $\psi(V)$ must contain $\psi(x_i)$, for some i, which is contradictory to the injectiveness of ψ.

□

Proof 2 This follows from the claim 1, and the definition of triangulation.

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