Compact surface from polygon

정리 1 *M is a connected closed(=compact and without boundary) surface. Then M is homeomorphic to a finite polygon with edge identified in pairs.*

정리증명
1^{st}*Step* : We can number triangles as $T_1, T_2, \cdots, T_n$ so that each T_i has an edge e_i in common with at least one of the triangles $T_1, \cdots, T_{i-1}$.
$\because$ Choose inductively. Suppose $A = \{T_1, \cdots, T_k\}$ are chosen and no triangles of $B = \{T_{k+1}, \cdots, T_n\}$ has an edge in common with a triangles in A. Then $T_1 \bigcup \cdots \bigcup T_k$ and $T_{k+1} \bigcup \cdots \bigcup T_n$ are disjoint closed sets in M. This is contradiction to connectedness of M.
2^{nd}*Step* : We can arranged $T_1, \cdots, T_n$ and $e_2, \cdots, e_n$(e_i is T_i's edge for all $i = 1, \cdots, n$) in the order as in the 1^{st} Step. Then we may assume that $\phi_i(T_i) = T_i'$ are all disjoint in $\mathbb{R}^2$ for some homeomorphism ϕ_i.
Claim $P = \coprod_{i=1}^n T_i'/\sim$ is topologically a disk, where $\sim$ is defined by $\phi_i(p) \sim \phi_{i-1}(p)$, $\forall p \in e_i < T_i$, $i = 2, \cdot, n$. This claim is clearly true by induction.

Now
$$\begin{array}{ccc} P = \coprod_{i=1}^n T_i'/\sim & \xrightarrow{\psi = \bigcup \phi_i^{-1}} & M. \\ \searrow & & \nearrow \overline{\psi} \\ & P/\sim & \end{array}$$

Here another '$\sim$' defined on P is a side pairing. Since P is compact, $P/\sim$ is also compact. Therefore $\overline{\psi}$ is a continuous bijective map from compact space to Hausdorff space. Hence $\overline{\psi}$ is a homeomorphism. □

정리 2 *A closed surface M is homeomorphic to a surface obtained from S^2 from which finite number of discs are removed and for which either a "handle"(=torus with a hole) or a "cross-cap"(=Möbius band) is attached along the boundary of the disc, i.e., M is a connected sum of tori and projective planes.*

정리증명
Step 1 By theorem 1, there is a polygon with edges identified in pairs homeomorphic to M. We can make sequence of edges like $\cdots abca \cdots$. The rule of naming edges is that edges that have same name are identified and that if clockwise edge's name is 'a', counterclockwise edge's name is 'a^{-1}'. Edges like 'aa^{-1}' can be deleted.

Step 2 All the vertices of the polygon D may be assumed to be identified to a single point in M.
$\because$ Suppose that $A \neq B$ in M.

그림

$[A]_D = \{$vertices in D that are identified to A$\}$
$\Rightarrow \sharp[A]_{D'} = \sharp[A]_D - 1$ and $\sharp[B]_{D'} = \sharp[B]_D + 1$
Therefore, by induction, we can make $\sharp[A] = 0$ in a certain modified polygon D.
(A가 한개 남았을 때는 Step1으로 돌아간다.)

Step 3 For the type of edges of the form $XaYa$, we can cancel a and introduce dd by cutting and pasting, i.e., make of the form $XY^{-1}dd$.

그림

Step 4 Suppose we have $\cdots a \cdots a^{-1} \cdots$. Then there exists b such that $\cdots a \cdots b \cdots a^{-1} \cdots b^{-1} \cdots$. There is no edges of the form $\cdots a \cdots b \cdots a^{-1} \cdots b \cdots$ by step 3. Suppose not.

그림

Therefore we must identify edges in A with edges in A. Then we can not identify all the vertices to a single point. This is a contradiction to step 2.

Step 5 We can change $\cdots a \cdots b \cdots a^{-1} \cdots b^{-1} \cdots$ to $\cdots cdc^{-1}d^{-1} \cdots$ by the following sequence of cut and paste.

그림

Note that through step 5, $\cdots aa \cdots$ or $\cdots aba^{-1}b^{-1} \cdots$ is not destroyed.

Conclusion We have only successive edges of the form $\cdots aa \cdots$ or $\cdots bcb^{-1}c^{-1}\cdots$ in the clockwise sequencing. □

HW. Identify $abacb^{-1}c$.

Note M, N : connected closed surface

1. $M\sharp N$ is well-defined.

   $\because$ Let $x, y \in M$ with small open disc neighborhood U and V respectively. Then it can be shown that there exists a homeomorphism ϕ on M such that $\phi(x) = y$ with $\phi(U) = V$. Then $M\sharp N$ obtained by deleting U and $M\sharp N$ obtained by deleting V are homeomorphic. (why?)

   사실 위 homeomorphism ϕ 가 존재한다는 것을 증명하기 위해서는 다음 사실들이 필요하다.

   (1) (Schönflies Thm): Let C be a simple closed curve in $\mathbb{R}^2$. Then $\exists$ a homeomorphism $h : \mathbb{R}^2 \longrightarrow R^2$ such that h carries C onto the unit circle. (Ref: Moise, "Geometric topology in dim 2 and 3")

   (2) (Annulus Thm): Let A be an annulus region in $\mathbb{R}^2$ bounded by two simple closed curve. Then A is homeomorphic to the standard annulus:
   $$\{x = (x_1, x_2) \in \mathbb{R}^2 \,|\, 1 \leq ||x|| \leq 2\}.$$

   OHW.

   (1) Prove Annulus Thm from Schönflies Thm.

   (2) Improve Annulus Thm so that the homeomorphism can be chosen as an ambient homeomorphism on $\mathbb{R}^2$.

   (3) Improve Schönflies Thm so that the homeomorphism h can be chosen identity outside a big disk.

   OHW. connected sum은 유일하게 결정된다. 이 사실을 보다 엄밀하게 논증하려면 위에서 언급한 것 이외에 어떤 것들을 보여야 하나?
   (1) connected sum은 gluing homeomorphism의 isotopy class 에만 depend한다.
   (2) gluing homeomorphism의 orientation에 무관하다.

2. $M\sharp N = N\sharp M$

3. $(M\sharp N)\sharp L = M\sharp(N\sharp L)$

정리 3 *M is a closed surface and M' is a closed surface obtained from M by replacing 3 cross-caps with a handle and a cross-cap. Then, $M \cong M'$, i.e., $P^2\sharp P^2\sharp P^2 \cong T^2\sharp P^2$.*

정리증명 Recall that $P^2\sharp P^2 = K$.
Therefore, if we can show $K\sharp P^2 = T^2\sharp P^2$, we can prove this theorem.
$K\sharp P^2 = T^2\sharp P^2$ is true by following figure.

그림

□

따름정리 4 *If M has a cross-cap, we can replace all the handle by cross-caps, i.e., $M = P^2\sharp M^1 \Rightarrow M = P^2\sharp\ldots\sharp P^2$. In this case, one handle is replaced by two cross-caps.*

따름정리 5 *M : closed surface*

$$\Rightarrow M = \begin{cases} T^2\sharp\ldots\sharp T^2 = \sharp^g T^2 & (= \Sigma_g) \\ P^2\sharp\ldots\sharp P^2 = \sharp^k P^2 & (= N_k) \end{cases}$$