Compact surface from polygon

정리 1 *M is a connected closed(=compact and without boundary) surface. Then M is homeomorphic to a finite polygon with edge identified in pairs.*

정리증명

1^{st} : We can number triangles as $T_1, T_2, \cdots, T_n$ so that each T_i has an edge e_i in common with at least one of the triangles $T_1, \cdots, T_{i-1}$.

$\because$ Choose inductively. Suppose $A = \{T_1, \cdots, T_k\}$ are chosen and no triangles of $B = \{T_{k+1}, \cdots, T_n\}$ has an edge in common with a triangles in A. Then $T_1 \bigcup \cdots \bigcup T_k$ and $T_{k+1} \bigcup \cdots \bigcup T_n$ are disjoint closed sets in M. This is contradiction to connectedness of M.

2^{nd} : We can arranged $T_1, \cdots, T_n$ in order with $e_2, \cdots, e_n$(e_i is T_i's edge for all $i = 1, \cdots, n$) as 1^{st} step. Then we may assume that $\phi_i(T_i) = T_i'$ are all disjoint in $\mathbb{R}^2$ for some homeomorphism ϕ_i.

Claim $\coprod_{i=1}^n T_i'/\sim$ is topologically a disk. '$\sim$' is defined by $\phi_i(p) \sim \phi_{i-1}(p)$, $\forall p \in e_i < T_i$, $i = 2, \cdot, n$. This claim is clearly true by induction. Now,

$$\begin{array}{ccccc} P = \coprod_{i=1}^n T_i'/\sim & & \overset{\psi=\bigcup \phi_i^{-1}}{\longrightarrow} & & M \\ & \searrow & & \nearrow \overline{\psi} & \\ & & P/\sim & & \end{array}$$

'$\sim$' is side pairing. Since P is compact, $P/\sim$ is also compact. So, $\overline{\psi}$ is a continuous bijective map from compact space to Hausdorff space. Hence $\overline{\psi}$ is homeomorphism. □

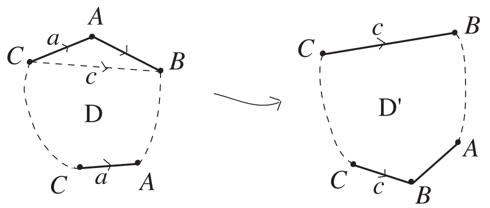
정리 2 *A closed surface M is homeomorphic to S^2 , from which finite number of discs are removed and for which either a "handle"(=torus with a hole) or a "cross-cap"(=Möbius band) is attached along the boundary of the disc, i.e., M is a connected sum of tori and projective planes.*

정리증명

Step 1 By theorem 1, there is a polygon with edges identified in pairs homeomorphic to M. We can make sequence of edges like $\cdots abca \cdots$. The rule of naming edges is that edges that have same name are identified and if clockwise edge's name is 'a', counterclockwise edge's name is 'a^{-1}'. Edges like 'aa^{-1}' can be deleted.

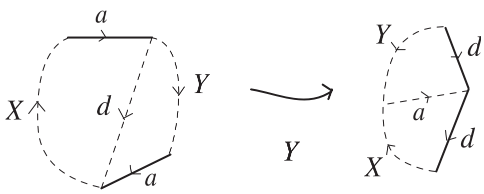
Step 2 All the vertices of the polygon D may be assumed to be identified to a single point in M.

$\because$ Suppose that $A \neq B$ in M.

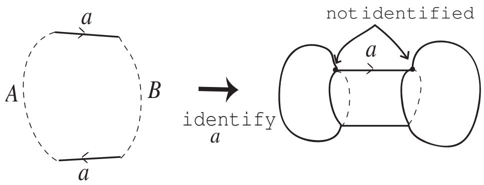


$[A]_D = \{\text{vertices in D that are identified to A}\}$
$\Rightarrow \sharp[A]_{D'} = \sharp[A]_D - 1$ and $\sharp[B]_{D'} = \sharp[B]_D + 1$
Therefore, by induction, we can make $\sharp[A] = 0$ in a certain modified polygon D.

Step 3 For the type of edges of the form $XaYa$, we can cancel a and introduce dd by cutting and pasting, i.e., make of the form $XY^{-1}dd$.

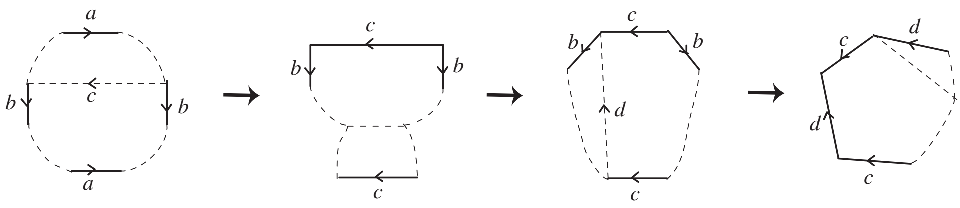


Step 4 Suppose we have $\cdots a \cdots a^{-1} \cdots$. Then there exists b such that $\cdots a \cdots b \cdots a^{-1} \cdots b^{-1} \cdots$. Suppose not. There is no edges of the form $\cdots a \cdots b \cdots a^{-1} \cdots b \cdots$ by step 3.



Therefore we must identify in A with edge in A. Then, we can not identified all the vertices to a single point. This is a contradiction by step 2.

Step 5 We can change $\cdots a \cdots b \cdots a^{-1} \cdots b^{-1} \cdots$ to $\cdots cdc^{-1}d^{-1} \cdots$ by the following sequence of cut and paste.



Note that through step 5, $\cdots aa \cdots$ or $\cdots aba^{-1}b^{-1} \cdots$ is not destroyed.

Conclusion We have only successive edges of the form $\cdots aa \cdots$ or $\cdots bcb^{-1}c^{-1} \cdots$ in the clockwise sequencing. □

Note M, N : connected closed surface

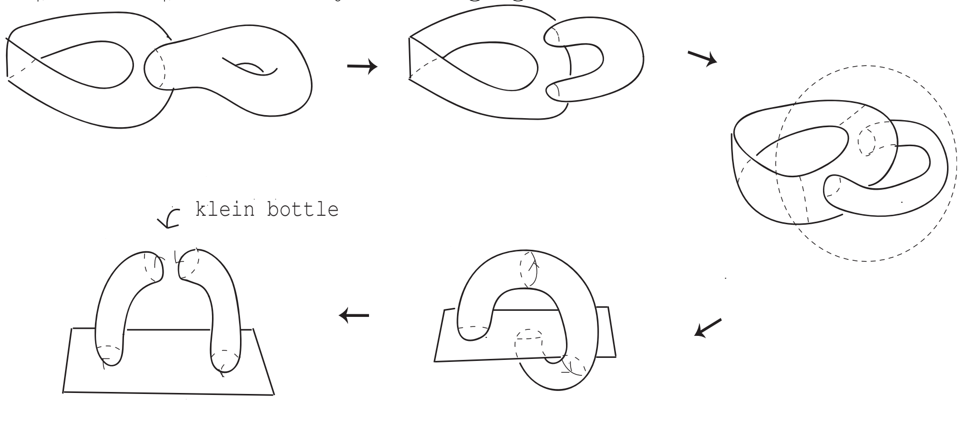
1. $M \sharp N$ is well-defined.
   $\because$ Let $x, y \in M$ with open disc neighborhood U and V respectively. Then it is known that there exists a homeomorphism ϕ such that $\phi(x) = y$, $\phi(U) = V$. So, $M \sharp N$ obtained by deleting U and $M \sharp N$ obtained by deleting V are homeomorphic.

2. $M \sharp N = N \sharp M$

3. $(M \sharp N) \sharp L = M \sharp (N \sharp N)$

정리 3 *M is a closed surface and M' is a closed surface obtained from M by replacing 3 cross-caps with a handle and a cross-cap. Then, $M \cong M'$, i.e., $P^2 \sharp P^2 \sharp P^2 \cong T^2 \sharp P^2$.*

정리증명 Recall that $P^2 \sharp P^2 = K$.
Therefore, if we can show $K \sharp P^2 = T^2 \sharp P^2$, we can prove this theorem.
$K \sharp P^2 = T^2 \sharp P^2$ is true by following figure.



□

따름정리 4 *If M has a cross-cap, we can replace all the handle by cross-caps, i.e., $M = P^2 \sharp M^1 \Rightarrow M = P^2 \sharp \ldots \sharp P^2$. In this case, one handle is replaced by two cross-caps.*

따름정리 5 *M : closed surface*

$$\Rightarrow M = \begin{cases} T^2 \sharp \ldots \sharp T^2 = \sharp^g T^2 & (= \Sigma_g) \\ P^2 \sharp \ldots \sharp P^2 = \sharp^k P^2 & (= N_k) \end{cases}$$