# Compact Surfaces with Boundary

Let $M$ be a compact surface with boundary, then a closed surface $M^*$ is obtained from $M$ by *capping off* the boundaries of $M$, i.e. gluing each boundary component of $M$ with a boundary of a disk. (Note that the number of boundary components is finite.)

$$M^* = M \bigcup_{\partial} (\coprod_i D_i)$$

In general, given $Y$ and $A \subset X$, let $f : A \to Y$ be an imbedding of A. Then,

$$X \bigcup_f Y := X \bigcup Y / \sim$$

where the relation is to identify $a \sim y = f(a)$.

Let $\phi : \amalg_i S_i^1 \to \partial M$ be a homeomorphism, then

$$M^* = M \bigcup_{\phi} (\coprod D_i)$$

**Theorem 1** *Let $M$, $N$ be compact connected surfaces with boundary. Then $M \cong N$ if and only if $M^* \cong N^*$ and the number of boundary components are same.*

**Proof**

( $\Rightarrow$ ) The theorem of the invariance of domain implies that an interior point is mapped to an interior point by the homeomorphism between $M$ and $N$. Also it maps a boundary point to a boundary point. It follows that the number of boundary components are the same.

( $\Leftarrow$ ) (1) Let $x$ and $y$ be different points in $M$, a compact connected surface. Then

$$M - \text{“}B_\epsilon^\circ(x)\text{”} \cong M - \text{“}B_\epsilon^\circ(y)\text{”}$$

(2) Let $p_i \in M^*$, $i = 1, 2, \ldots, b$ where $b$ is the number of the boundary components of $M$. From (1),

$$M \cong M^* - \coprod_{i=1}^{b} B_\epsilon(p_i)$$

Use collar neighborhood theorem to prove this. (CN Thm can be shown easily using triangulation.)

□

**Corollary 2** *The topological type of compact surface with boundary is determined by the orientability, the Euler Characteristic, and the number of boudary components.*

**Note** $\chi(M) = \chi(M^*) - b$

**Exercises** Determine the topological type of the following surface.