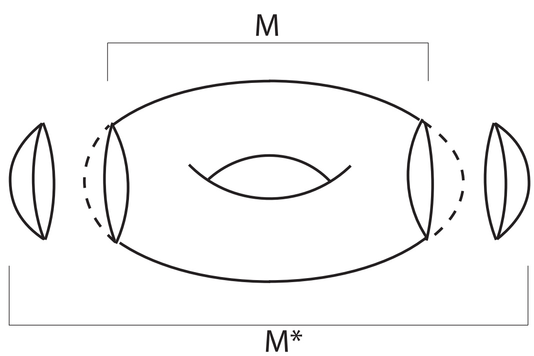
Compact Surfaces with Boundary

Let M be a compact surface with boundary, then a closed surface M^* is obtained from M by *capping off* the boundaries of M, i.e. gluing each boundary component of M with a boundary of a disk.

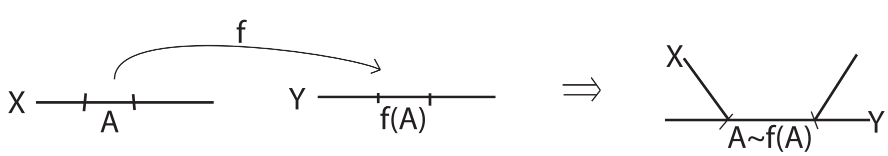
$$M^* = M \bigcup_{\partial} \big(\coprod_i D_i \big)$$



In general, given Y and $A \subset X$, let $f : A \to Y$ be an imbedding of A. Then,

$$X \bigcup_f Y := X \bigcup Y / \sim$$

where the relation is to identify $a \sim y = f(a)$.



Let $\phi : \amalg_i S_i^1 \to \partial M$ be a homeomorphism, then

$$M^* = M \bigcup_{\phi} \big(\coprod D_i \big)$$

Theorem 1 *Let M, N be compact surfaces with boundary. Then $M \cong N$ if and only if $M^* \cong N^*$ and the number of boundary components are same.*

Proof

($\Rightarrow$) The theorem of the invariance of domain implies that an interior point is mapped to an interior point by the homeomorphism between M and N. Also it maps a boundary point to a boundary point. It follows that the number of boundary components are the same.

($\Leftarrow$) (1) Let x and y be different points in M, a compact surface. Then

$$M - B_\epsilon^\circ(x) \cong M - B_\epsilon^\circ(y)$$

(2) Let $p_i \in M^*$, $i = 1, 2, \ldots, b$ where b is the number of the boundary components of M. From (1),

$$M \cong M^* - \coprod_{i=1}^{b} B_\epsilon(p_i)$$

□

Corollary 2 *The topological type of compact surface with boundary is determined by the orientability, the Euler Characteristic, and the number of boudary components.*

Note $\chi(M) = \chi(M^*) - b$

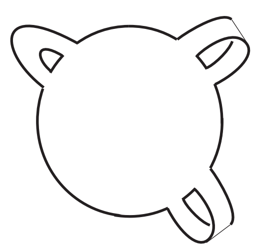
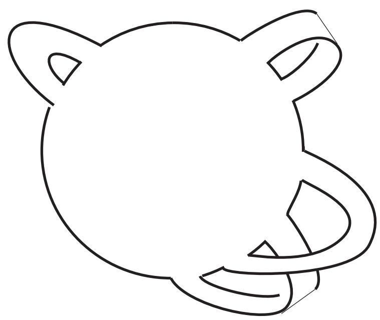
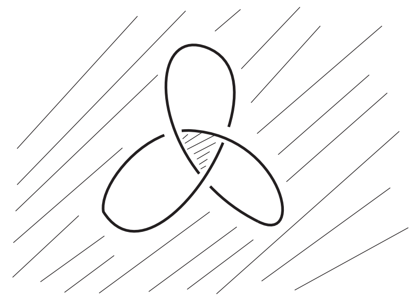


Figure 1: An example of compact surface with boundary. $\chi = -2$ and $b = 2$, and hence $\chi^* = 0$, and non-orientable. Therefore it is a Klein bottle minus two discs.

Exercises Determine the topological type of the following surface.





with one-point-compactification