Definition and Examples

Definition 1 A topological space M is an **n-dimensional manifold** or (**n-manifold**) if

1. M is Hausdorff,
2. M is locally Euclidean, i.e., $\forall x \in M$, $\exists$ a neighborhood U of x such that U is homeomorphic to an open set in $\mathbb{R}^n$.

여기서 U가 $\mathbb{R}^n$과 homeomorphic한 neighborhood 라고 요구하여도 위 정의와 equivalent 하다.

For an open set U of a manifold, let $\phi : U \to \phi(U) \subset \mathbb{R}^n$ be a homeomorphism. We call (U, ϕ) a **coordinate chart**.

Definition 2 Let M be an n-manifold. M is a **differentiable manifold** if there is a system of open coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ covering M such that

$$g_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta), \forall \alpha, \beta$$

is differentiable. $g_{\beta\alpha}$ is called a **coordinate transition map**.

일반적으로 coordinate transition map에 어떤 구조를 주느냐에 따라서 M의 구조가 결정된다. 예를 들어 coordinate transition map들이 $\mathcal{C}^k$ differentiable 이면 M은 $\mathcal{C}^k$ differentiable manifold라 부르고, real analytic이 되게 coordinate system을 잡아 줄 수 있으면 M은 real analytic manifold가 되고 transition이 Euclidean rigid motion 이 되게 잡을 수 있으면 M은 Euclidean manifold가 된다. 만일 coordinate chart들을 $\mathbb{R}^n$ 대신 $\mathbb{C}^n$ 로 잡고 transition을 holomorphic map 으로 잡을 수 있으면 complex manifold가 된다. 특히 1차원 complex manifold를 Riemann surface라고 부른다.

Example 1 Manifolds

1. $\mathbb{R}^n$ itself.
2. A space with discrete topology, in which every set is open, is 0-manifold.
3. An open set in $\mathbb{R}^n$.
4. $S^n \subset \mathbb{R}^{n+1}$.
5. A smooth surface in $\mathbb{R}^3$: $\{(x, y, z) | f(x, y, z) = 0\}$ when $\nabla f \neq 0$.
6. An open set of n-manifold.

7. non-Hausdorff manifold: Consider the real line $\mathbb{R}$ with standard topology. Add one more point $0'$ to $\mathbb{R}$ set-theoretically, and give a topology as follows: (1)The open sets of the original real line are open. (2)For any open set U containing 0, the set $(U \cup \{0'\}) - \{0\}$ is open. Then this space $\mathbb{R} \cup \{0'\}$ is locally homeomorphic to $\mathbb{R}$ but not Hausdorff since any two neighborhoods U of 0 and V of $0'$ intersect.

8. ∞(Figure 8) is not a manifold.

HW1. Show that S^n is a $\mathcal{C}^\infty$ manifold.