

# Euler Characteristic

**Definition 1** Given a triangulation  $K$  on a compact surface  $M$ , we define the **Euler characteristic** of  $M$  by

$$\chi_K(M) := V - E + F$$

where  $V$  is the number of vertices,  $E$  is the number of edges, and  $F$  is the number of faces (triangles).

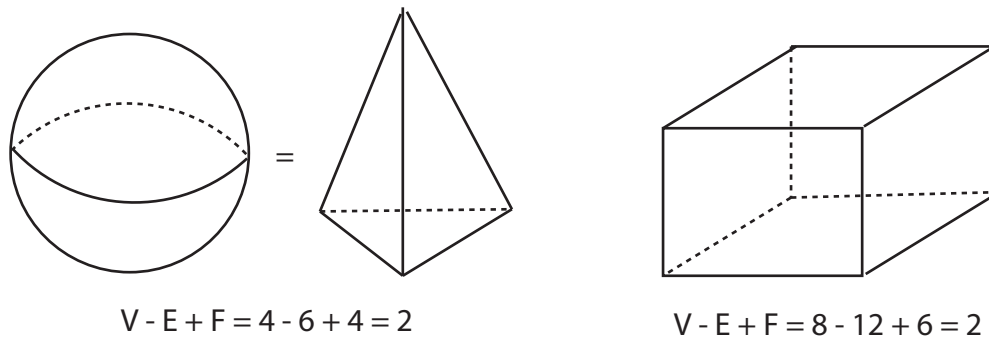


Figure 1: **Euler Characteristic**

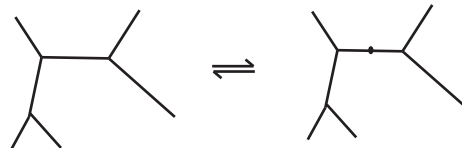
**Theorem 1**  $\chi(M)$  does not depend on the choice of the triangulation on  $M$ .

**Proof**(sketch proof)

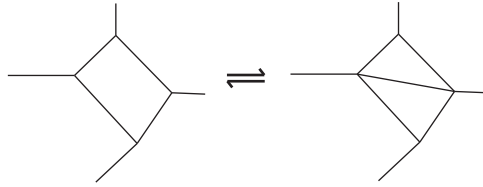
*Step 1* Similarly one can define  $\chi$  by the same formula for a *cell-division* of  $M$ , say, using polygons instead of triangles.

*Step 2*  $\chi$  is invariant under the following change in a cell-division:

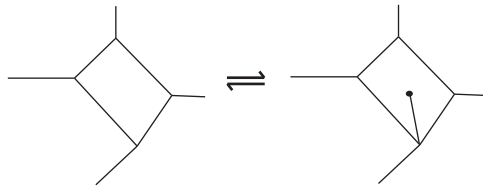
(a) introducing a new vertex on an edge or deleting.



(b) introducing a new edge connecting vertices or deleting.

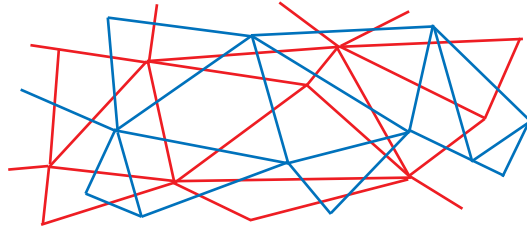


(c)\* introducing a new vertex and a new edge inside a face or deleting.



Now note that the step 2 implies that  $\chi$  is invariant under *subdivision* of a cell complex. Subdivision means a procedure through any sequence of (a), (b), or (c) in any order.

*Step 3* Given two triangulations  $K$  and  $K'$ , by superposing the two triangulations *transversely* we obtain a common subdivision as a cell-complex  $K''$  on  $M$ . The term ‘transversely’ means no two edges meet tangentially.



Then,

$$\chi_K(M) = \chi_{K''}(M) = \chi_{K'}(M)$$

□

**Note**  $\chi(M)$  is a topological invariant.

Further by the step 2, one can compute  $\chi(M^2)$  by counting the number of vertices, edges, and faces in a polygon identification. Consider a triangulation on the surface as one on its standard polygon identification as in Figure 2. By the step 2, one can delete all the edges and vertices inside the polygon. The resulting cell division has only one face. See Figure 2.

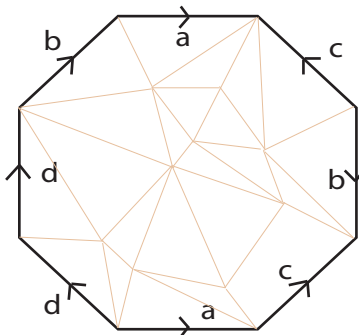


Figure 2: An example of computing Euler characteristic from polygon identification. Since all vertices are identified to one point in the figure above, there is one vertex. Clearly, there is only one face. To count the number of edges, we just count the number of different letters. Therefore, the Euler characteristic is  $1 - 4 + 1 = -2$ .

Given a polygon identification of  $(\#^k P^2) \# (\#^g T^2)$ , all vertices are identified to a single point, and there is only one face. Observe that there are one edge in a projective plane and two edges in a torus. Therefore, we get the following equality.

$$\chi((\#^k P^2) \# (\#^g T^2)) = 1 - (k + 2g) + 1 = 2 - k - 2g$$

**Theorem 2**  $\chi(M^2 \# N^2) = \chi(M^2) + \chi(N^2) - 2$

**Proof** Consider a connected sum of two surfaces. Clearly,  $\chi(M^2 \amalg N^2) = \chi(M^2) + \chi(N^2)$ . Eliminating a disk from each surface is equivalent to eliminating a triangle. Deleting one triangle's face decreases the Euler characteristic by 1 from each surface. Pasting two boundaries of triangles of two surfaces decreases the number of edges and vertices both by 3, giving no change in Euler characteristic. Hence the total  $V - E + F$  is decreased by 2.

□

**Corollary 3**

$$\begin{aligned} \chi(\#^k P^2) = 2 - k & \quad \text{or} \quad k = 2 - \chi \\ \chi(\#^g T^2) = 2 - 2g & \quad \text{or} \quad g = (2 - \chi)/2 \end{aligned}$$

**Theorem 4** *Let  $M$  and  $N$  be closed surfaces ( i.e., compact without boundary ).  $M$  is homeomorphic to  $N$  if and only if  $M$  and  $N$  are both orientable or both non-orientable and have the same Euler characteristic.*

**Note** { orientability,  $\chi$  } is a complete set of invariants.