# Basic topological property

**1**. $M$ : n-manifold.
⇒ M is Hausdorff, locally compact, locally connected, 1st countable.

**2**. $M$ is connected ⇔ M is path-connected
**증명** At first, path-connected space is connected clearly. Conversely, since $M$ is locally homeomorphic to Euclidean space, every point in $M$ has a path-connected neighborhood. Recall that $X$ is path-connected if and only if $X$ is connected and every point in $X$ has a path-connected neighborhood. (Proof: Each path-component of $X$ is open because each point in path-component has a path-connected neighborhood inside its component by maximality. A path-component is also closed since the compliment is open. By connectedness, open and closed set is only $X$ itself.) □

**정리 1** *Let $M$ be a compact n-manifold.*
*1) Let $\mathcal{U}$ be an open cover of $M$. Then there exists a **partition of unity** subordinate to $\mathcal{U}$ .*
*2) There exists $\phi : M \hookrightarrow \mathbb{R}^N$, an **embedding** into a Euclidean space.*

**증명** 1) We can choose a finite coordinate refinement of $\mathcal{U}$ because $M$ is compact. In fact, we can choose $\{V_1, V_2, \ldots, V_k\}$, an open covering of $M$ such that $V_j \subset \overline{V_j} \subset U_j$ for a coordinate chart $\{U_j, \phi_j\}$, where $\{U_1, \ldots, U_k\}$ is an open refinement of $\mathcal{U}$. $M$ is a normal space since it is compact Hausdorff.
Using Urysohn lemma, we can construct $g_j : M \to [0,1]$ by

$$g_j(x) = \begin{cases} 1, & x \in \overline{V_j} \\ 0, & x \in U_j^c \end{cases} .$$

Define

$$f_j(x) := \frac{g_j(x)}{\sum_{j=1}^{k} g_j(x)} .$$

Then we can easily check $\{f_j\}$ is a partition of unity.

2) For each $i = 1, \ldots, k$, define $\psi_i : M \to \mathbb{R}^{n+1}$ by $\psi_i(x) = (g_i(x)\phi_i(x), g_i(x))$. then $\psi_i$ is well-defined on $M$. Let $\psi : M \to \mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1} = \mathbb{R}^{k(n+1)} = \mathbb{R}^N$ be given by $\psi = (\psi_1, \ldots, \psi_k)$. We claim $\psi$ is one-to-one. Suppose that $\psi(x) = \psi(y)$, then $\psi_i(x) = \psi_i(y)$ and $g_i(x) = g_i(y)$ for every $i = 1, \ldots, k$. And if $x \in V_j$ then $g_j(y) = g_j(x) = 1$ and hence $y \in U_j$. Now $\phi_j(x) = \phi_j(y)$ implies $x = y$ as $\phi_j$ is a homeomorphism. Hence $\psi$ is an embedding since $M$ is compact. □

**정리 2** *If $M$ is $2^{nd}$ countable, then $M$ is paracompact.*
*In fact, each open cover has a countable locally finite refinement consisting of open sets with compact closures.*

**증명** At first, we show that there exists a countable basis $\mathcal{A}$ consisting of relatively compact open sets. Since $M$ is $2^{nd}$ countable, there exists a countable basis $\mathcal{B}$. Let $\mathcal{A} = \{B \in \mathcal{B} : \overline{B} \text{ is compact}\}$.
$M$ 은 locally compact 이기때문에 $x$를 포함하는 임의의 neighborhood $U$에 대하여 relatively compact open set $V$를 선택할 수 있고, $V$또한 $x$를 포함하는 open set이므로 $x \in B \subset V$ 인 basis element $B$를 찾을수 있다. $\overline{B} \in \overline{V}$ 즉 compact set 안의 닫힌 집합이므로 $B$는 relatively compact 이다. 따라서 $\mathcal{A}$ is also a countable basis and denote $\mathcal{A} = \{A_1, A_2, \ldots\}$.

Secondly, there exists a compact exhaustion, i.e., $\{G_i : i = 1, 2, \ldots\}$ such that

$$G_1 \subset \overline{G_1} \subset G_2 \subset \overline{G_2} \subset G_3 \subset \ldots$$

and $M = \bigcup_{i=1}^{\infty} G_i$, $G_i$ is open and $\overline{G_i}$ is compact. Indeed define $G_k$ inductively with $G_1 = A_1$ and $G_k = A_1 \bigcup \cdots \bigcup A_{j_k}$. Then $\overline{G_k} \subset A_1 \bigcup \cdots \bigcup A_{j_k} \bigcup A_{j_k+1} \bigcup \cdots \bigcup A_{j_{k+1}}$ for some $j_{k+1} > j_k$ since $\overline{G_k}$ is compact. Now let $G_{k+1} = A_1 \bigcup \cdots \bigcup A_{j_{k+1}}$.
Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{I}\}$ be a given cover of $M$. For each fixed index $i$, let $V_\alpha = U_\alpha \bigcap (G_{i+2} - \overline{G_{i-1}})$. Then $\{V_\alpha : \alpha \in \mathcal{I}\}$ is an open cover of a compact set $\overline{G_{i+1}} - G_i$ and hence there exists a finite subcover $\mathcal{V}_i$. Now $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ is the desired refinement. □

**따름정리 3** *If $M$ is $2^{nd}$ countable, then there exists a partition of unity subordinate to an arbitrarily given open cover.*

* 일반적으로 manifold $M$ 은 $2^{nd}$ countable 을 가정한다.

HW2. (1) 1학기때 배운 필요한 topology를 review할것.(제출할 필요없음.)
(2) smooth bump function을 construct할 수 있나?