Unique Path Lifting

정리 1 *(Unique path lifting property)*
Let $p : \widetilde{X} \to X$ be a covering map and let $\alpha : I \to X$ be a path with $\alpha(0) = x_0 \in X$ and $p(\widetilde{x_0}) = x_0$. Then α has a unique path lifting $\widetilde{\alpha} : I \to X$ with $\widetilde{\alpha}(0) = \widetilde{x_0}$ i.e., $p \circ \widetilde{\alpha}(t) = \alpha(t)$. $\forall\ t \in I$.

증명 *(Existence)*
For each t, $\alpha(t) \in X$ has an open neighborhood U_t which is evenly covered by $\coprod_{a\in A} V_{t,a}$. Since I=[0,1] is compact, we can choose a Lebesgue number $\epsilon > 0$ for a cover $\{\alpha^{-1}(U_t)|t \in I\}$ of I. Choose partition of I,
$0 = t_0 < t_1 < ... < t_{n+1} = 1$ so that $t_{i+1} - t_i < \epsilon$, $i = 1, ...n$.
Then note that $\alpha[t_i, t_{i+1}] \subset U_t$ for some t and we lift $\alpha|_{[t_i,t_{i+1}]}$ inductively :
Suppose $\alpha|_{[t_0,t_i]}$ is already lifted (note that the initial point x_0 is lifted to $\widetilde{x_0}$).
Then $\alpha[t_i, t_{i+1}] \subset U_t$ for some t and $p^{-1}(U_t) = \coprod_{a\in A} V_{t,a}$ and there exists a unique $a \in A$ such that $\tilde{\alpha}(t_i) \in V_{t,a}$.
And since $p|_{V_{t,a}} : V_{t,a} \to U_t$ is homeomorphism we can lift $\alpha|_{[t_i,t_{i+1}]}$ using $(p|_{V_{t,a}})^{-1}$ and the proof is completed.

(Uniqueness)
Suppose $p \circ \tilde{\alpha}_i = \alpha$ and $\tilde{\alpha}_i(0) = \widetilde{x_0}$ $i = 1, 2$. Then we show that $J = \{t \in I|\widetilde{\alpha_1}(t) = \widetilde{\alpha_2}(t)\}$ is open and closed non-empty set :
1 . (J is non-empty) : $x_0 \in I$.
2 . (J is open) : $t \in J$에 대해 $\alpha(t) \in X$는 evenly cover되는 U를 가지고 $p^{-1}(U) = \coprod V_a$ 에서 $\widetilde{\alpha}_i(t)$ 를 포함하는 V_a는 유일하다. 그리고 V_a 에서는 p가 homeomorphism이므로 $\widetilde{\alpha_1} = (p|_{V_a})^{-1} \circ \alpha = \widetilde{\alpha_2}$ on $(t - \epsilon, t + \epsilon)$ 이다. 따라서 $\exists\, \epsilon > 0$ *such that* $(t - \epsilon, t + \epsilon) \in J$.
3 . (J is closed) : $J^c = \{t \in I|\widetilde{\alpha_1}(t) \neq \widetilde{\alpha_2}(t)\}$이 open 임을 보이자.
$\widetilde{\alpha_1}(t) \neq \widetilde{\alpha_2}(t)$ for some $t \in I$ 라면, $\alpha(t)$ 에 대해 evenly cover되는 U 가 존재해서 $p^{-1}(U) = \coprod V_a$ 이다. 그리고 각 $\widetilde{\alpha}_i(t)$와 만나는 V_{a_1}, V_{a_2} 가 유일하게 존재하고, $V_{a_1} \neq V_{a_2}$ 이다. 즉 $\widetilde{\alpha_1}(t - \epsilon, t + \epsilon) \subset V_{a_1}$, $\widetilde{\alpha_2}(t - \epsilon, t + \epsilon) \subset V_{a_2}$ 를 만족하는 $\epsilon > 0$ 이 존재한다. 따라서 J^c 는 open이다.
I는 connected 이므로 위 1,2,3, 에 의해 J = I 이다. 따라서 I 내부 전체에서 $\widetilde{\alpha_1} = \widetilde{\alpha_2}$ 이다. □

Remark. *(Uniqueness of lifting)* Y가 connected 이고 $f : (Y, y_0) \to (X, x_0)$ 가 lifting $\tilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x_0})$를 가지면, 이는 unique하다.
(증명) Y의 connectedness 를 이용, 위 정리의 증명에서 I 대신 Y 를 써서 똑

같이 하면 된다.

정리 2 *(Lifting of homotopy of paths.)*
Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be a covering space. And $\alpha : I \to X$ with $\alpha(0) = x_0$ and $F : \alpha \simeq \beta$ be a homotopy between α and β. Then $\exists!\ \widetilde{F} : I \times I \to \widetilde{X}$ such that $p \circ \widetilde{F} = F$ and $\widetilde{F}(0,0) = \widetilde{x_0}$.
In particular, $\widetilde{F}$ gives a homotopy between $\widetilde{\alpha} = F_0$ and $\widetilde{\beta} = F_1$.
Furthermore (1)if F keeps initial point x_0 fixed, i.e., $F(0,u) = x_0\ \forall u \in I$, then $\widetilde{F}$ keeps initial point $\widetilde{x_0}$ fixed,
and (2)if F keeps end points $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$ fixed, then $\widetilde{F}$ keeps end points $\widetilde{\alpha(0)} = \widetilde{\beta(0)}$ and $\widetilde{\alpha(1)} = \widetilde{\beta(1)}$ fixed.

증명 이 증명 역시 존재성만 보이면, 유일성은 I^2의 connectedness에 의해 보장된다.
For each (t,u), $F(t,u)$ has an open neighborhood $U_{(t,u)}$ which is evenly covered by $p^{-1}(U_{(t,u)}) = \coprod_{a \in A} V_{(t,u),\,a}$. Choose a Lebesgue number $\epsilon > 0$ for a cover $\{\, F^{-1}(U_{(t,u)}) \mid (t,u) \in I \times I \,\}$ for compact $I \times I$. Choose a partition
$0 = t_0 < t_1 < \cdots < t_{n+1} = 1$ with $t_{i+1} - t_i < \frac{\epsilon}{2}$
$0 = u_0 < u_1 < \cdots < u_{n+1} = 1$ with $u_{i+1} - u_i < \frac{\epsilon}{2}$ so that
each $[t_i, t_{i+1}] \times [u_i, u_{i+1}] \subset F^{-1}(U_{(t,u)})$ for some (t,u).
As in Theorem 1, F is defined inductively starting from $[t_0,t_1] \times [u_0,u_1]$ so that $F(0,0) = \widetilde{x_0} \in V_{(0,0),\,\alpha}$ using the homeomorphism $p|_{V_{(0.0),\,a}} : V_{(0,0),\,a} \to U_{(0,0)}$. Then lift $F|_{[t_1,t_2]\times[u_0,u_1]}$, $\cdots$, $F|_{[t_n,t_{n+1}]\times[u_0,u_1]}$ successively as before to obtain a lifting of $F|_{[0,1]\times[u_0,u_1]}$.
Now lift $F|_{[0,1]\times[u_1,u_2]}$ using the already lifted portion as above, and lift $F|_{I\times[u_2,u_3]}$, $\cdots$ etc., finally to get a lifting $\widetilde{F} : [0,1] \times [0,1] \to \widetilde{X}$.
위 정리의 (1)과 (2)는 lifting의 uniqueness에 의해서 constant map의 lifting은 constant map일 수 밖에 없으므로 성립한다. □

Exercise. $\widetilde{F}$ 가 연속임을 보여라.
(Hint) $X = A \cup B$, A and B both closed(or open) in X 라면
$f : X \to Y$ 에서 $f|_A$ *and* $f|_B$ 가 연속이면 f는 연속임을 보인후 이를 이용하라.

따름정리 3 $\alpha \sim \beta \Rightarrow \widetilde{\alpha} \sim \widetilde{\beta}$.
여기서 α, β는 X의 path들이고 $\widetilde{\alpha}$,$\widetilde{\beta}$는 같은 initial point를 가지는 lifting 들이다.