II.4 Existence of covering spaces

1. Existence of a covering space assuming the existence of a universal covering space.

$$\boxed{\begin{array}{l} (\widetilde{X}, \widetilde{x}) \\ \downarrow p : \text{ a universal covering with a deck transformation group } G \overset{\theta(\cong)}{\longleftarrow} \pi_1(X,x) \\ (X,x) \end{array}}$$

정리 1 *If $H < G$, then*

$$\begin{tikzcd} (\widetilde{X}, \widetilde{x}) \arrow[dd, "p"'] \arrow[dr, "q"] & \\ & (\widetilde{X}_H, \bar{x}) \arrow[dl, "r"] \\ (X,x) & \end{tikzcd}$$

where $\widetilde{X}_H = H \setminus \widetilde{X}$ with $r_\sharp \pi_1(\widetilde{X}_H, \bar{x}) = \theta^{-1}(H)$ and the quotient map q and the induced map r become covering maps.

증명

왼쪽 그림에서와 같이 p에 의해 evenly cover되는 U를 잡으면 H는 $p^{-1}(U)$에 permutation으로 작용하므로 H action에 대한 quotient map q가 covering map이 되는 것은 분명하고 r을 q에 의해 p로 부터 induced되는 $\bar{p}$로 정의

하면 r은 당연히 onto이고, U가 역시 r에 의해서 evenly covered 되므로 r은 covering 이 된다. 그리고

$\alpha \in \theta^{-1}(H) \subset \pi_1(X, x) \quad \Leftrightarrow \quad \theta(\alpha) \in H \quad \Leftrightarrow \quad \tilde{x} \cdot \alpha \in H \cdot \tilde{x}$
$\Leftrightarrow \quad \bar{x} := q(\tilde{x}) = q(\tilde{x} \cdot \alpha) = \bar{x} \cdot \alpha \quad \Leftrightarrow \quad \alpha \in \Pi_{\bar{x}} = r_\sharp \pi_1(\widetilde{X}_H, \bar{x})$ 가 성립하므로,
$r_\sharp \pi_1(\widetilde{X}_H, \bar{x}) = \theta^{-1}(H)$ 이다.

$\square$

2. Existence of a universal covering space.

Idea : the set of homotopy classes of paths from x to y $\underset{1-1}{\leftrightarrow} p^{-1}(y) \subset \widetilde{X}$, universal covering , via $[\alpha] \leftrightarrow \tilde{\alpha}(1)$

정의 1 A space X is said to be **semilocally simply connected** if for each $x \in X$, there is a neighborhood U of x such that the homomorphism

$$i_\sharp : \pi_1(U, x) \to \pi_1(X, x)$$

induced by inclusion is trivial.

Assume that X is path-connected, locally path-connected and semilocally simply connected.
Let $(P, x) = \{\alpha : I \to X \mid \alpha(0) = x\}$
Define $\widetilde{X} = (P, x)/\sim$ (recall $\alpha \sim \beta \Leftrightarrow \alpha \simeq \beta$ rel ∂) and $p : \widetilde{X} \to X$ by $p([\alpha]) = \alpha(1)$

Topology of $\widetilde{X}$
For $\alpha \in (P, x)$ and open set U with $\alpha(1) \in U$, let $(\alpha, U) := \{\alpha * \alpha' | \alpha' : I \to U$ with $\alpha'(0) = \alpha(1)\}/\sim \ \subset \widetilde{X}$.
이때 주어진 $y = \alpha(1) \in X$에 대해, U를 path-connected , semilocally simply connected neighborhood of $y(= \alpha(1))$ 라고 하면 $p| : (\alpha, U) \to U$는 onto 이고 one-to-one 이다.

이제 p가 evenly covered 임을 보이자. 즉, $p^{-1}(U) = \coprod_{\alpha(1)=y} (\alpha, U)$:
$(\alpha, U) \cap (\beta, U) \neq \varnothing$ 라고 하자.
$\gamma \in (\alpha, U) \cap (\beta, U)$에 대하여 α'와 β'가 존재해서
$\alpha * \alpha' \sim \gamma \sim \beta * \beta'$
$\Rightarrow \quad \alpha * \alpha' * \bar{\alpha'} \sim \beta * \alpha' * \bar{\alpha'}(\because U$가 semilocally simply connected이므로 $\beta * \beta' \sim \beta * \alpha')$

$\Rightarrow \quad \alpha \sim \beta$

$\therefore (\alpha, U) \cap (\beta, U) \neq \varnothing \quad \Rightarrow \quad (\alpha, U) = (\beta, U)$

Take $\{(\alpha, U) | U :$ open neighborhood of $\alpha(1)$, $\alpha \in (P, x)\}$ as a base for a topology of $\widetilde{X}$.

Check

1. $\bigcup(\alpha, U) = \widetilde{X}$(obvious)
2. $\gamma \in (\alpha, U) \cap (\beta, V)$
   $\Rightarrow \exists W \subset U \cap V$ such that $(\gamma, W) \subset (\alpha, U) \cap (\beta, V)$ 여기서 W는 path-connected neighborhood of $\gamma(1)$이고 U가 semilocally simply connected 이므로 subset W도 마찬가지(obvious)

$\widetilde{X}$가 path-connected 임을 보이자.
임의의 $[\alpha] \in \widetilde{X}$에 대해서 $\alpha_s(t) := \alpha(st)$라고 하면 α_s는 α와 x(constant path)를 잇는 path이다. 여기서 $\tilde{\alpha}(s) := [\alpha_s]$라고 정의하면 $\tilde{\alpha}(0) = [x], \tilde{\alpha}(1) = [\alpha]$가 되어 $[\alpha]$와 $[x]$는 path 로 연결된다. (**exercise** : $\tilde{\alpha}$ is continuous)

마지막으로 $\widetilde{X}$가 simply connected임을 보이자.
Let τ be a loop in $(\widetilde{X}, \widetilde{x})$, where $\tilde{x} = [x]$
 $\Rightarrow \alpha := p \circ \tau$ is a loop in X and $\tilde{\alpha} = \tau$
 $\Rightarrow [\alpha] = \tilde{\alpha}(1) = \tau(1) = [x] \quad \Rightarrow \quad \alpha \sim x \quad \Rightarrow \quad \tau \sim \tilde{x} \quad \blacksquare$

따름정리 2 $\forall H < \pi_1(X, x), \exists$ *a covering space* $(\widetilde{X}, \widetilde{x})$ *corresponding to* H, *i.e.* $p_\sharp \pi_1(\widetilde{X}, \widetilde{x}) = H$

Remark

1. Universal covering is "universal", i.e. it covers every other covering by lifting theorem and universal covering is clearly unique up to isomorphism.

2. X has a universal covering.
 $\Rightarrow X$ is semilocally simply connected.

$$\begin{array}{ccc} \pi_1(\tilde{U}, \tilde{x}) & \xrightarrow{i_\sharp} & \pi_1(\widetilde{X}, \widetilde{x}) = 0 \\ \cong\downarrow p_\sharp & \circlearrowleft & \downarrow p_\sharp \\ \pi_1(U, x) & \xrightarrow{i_\sharp} & \pi_1(X, x) \end{array}$$

로 부터 당연하다.

숙제 8

Let (G, e) be a topological group and $p : (\tilde{G}, \tilde{e}) \to (G, e)$ be a covering. Then we can lift the group structure of G to $\tilde{G}$ so that p becomes a homomorphism unique up to the choice of identity $\tilde{e} \in p^{-1}(e)$.