ON THE VOLUME OF A SPHERICAL OCTAHEDRON WITH SYMMETRIES

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Dedicated to Ernest Borisovich Vinberg
on the occasion of his 70-th anniversary

Abstract. In the present paper closed integral formulae for the volumes of spherical octahedron and hexahedron having non-trivial symmetries are established. Trigonometrical identities involving lengths of edges and dihedral angles (Sine-Tangent Rules) are obtained. This gives a possibility to express the lengths in terms of angles. Then the Schläfli formula is applied to find the volume of polyhedra in terms of dihedral angles explicitly. These results and the canonical duality between octahedron and hexahedron in the spherical space allowed to express the volume in terms of lengths of edges as well.

Keywords: *spherical polyhedron, volume, Schläfli formula, Sine-Tangent Rule, symmetric octahedron, symmetric hexahedron.*

1. Introduction and Preliminaries

The calculation of volume of polyhedron is very old and difficult problem. Probably, the first result in this direction belongs to Tartaglia (1499–1557) who found the volume of an Euclidean tetrahedron. Nowadays this formula is more known as Caley-Menger determinant. A few years ago it was shown by I. Kh. Sabitov [Sb] that the volume of any Euclidean polyhedron is a root of algebraic equation whose coefficients are functions depending of combinatorial type and lengths of polyhedra.

In hyperbolic and spherical spaces the situation is much more complicated. Gauss, who is one of creators of hyperbolic geometry, use the word *"die Dschungel"* in relation with volume calculation in non-Euclidean geometry. Since Lobachevsky [Lb] and Schläfli [Sch1] the volume formula for biorthogonal tetrahedron (orthoscheme) is known. The volume of the Lambert cube and some other polyhedra were calculated by R. Kellerhals [K], D. A. Derevnin, A. D. Mednykh [DM1],

A. D. Mednykh, J. Parker, A. Yu. Vesnin [MPV] and others. The volume of hyperbolic polyhedra with at least one vertex at infinity was found by E. B. Vinberg [V].

The general formula for volume of tetrahedron remained to be unknown for a long time. A few years ago Y. Choi, H. Kim [ChK], J. Murakami, U. Yano [MY] and A. Ushijima [U] were succeeded in finding of a such formula. D. A. Derevnin, A. D. Mednykh [DM2] suggested an elementary integral formula for the volume of hyperbolic tetrahedron. We note that the volume formula for symmetric tetrahedra whose opposite dihedral angles are mutually equal is rather simple. For the first time this phenomena was discovered by Lobachevsky [Lb] for ideal hyperbolic tetrahedra, which is automatically symmetric. The respective result in quite elegant form was presented by J. Milnor [M1]. For general case of symmetric tetrahedron the volume was given by D. A. Derevnin, A. D. Mednykh and M. G. Pashkevich [DMP].

Surprisedly, but a hundred years ago, in 1906 an essential advance in volume calculation for non-Euclidean tetrahedra was achieved by Italian Duke Gaetano Sforza. It came to light during discussion of the third named author with Jose Maria Montesinos-Amilibia at the conference in El Burgo de Osma (Spain), August 2006. Unfortunately, the outstanding work by Sforza [Sf] has been completely forgotten.

One of the key tools in the calculation of convex polyhedra in constant curvature spaces is the Schläfli differential formula. In the present paper we need the case of dimension three.

Theorem 1 (Schläfli formula)**.** *Let P be a convex polyhedron in the spaces $\mathbb{S}^3$ or $\mathbb{H}^3$. If P is deformed in such a way that its combinatorial structure is preserved, while its dihedral angles vary in a differentiable manner, then the volume $V = V(P)$ also varies in a differential manner and the volume differential is given by*

$$K\, dV = \frac{1}{2} \sum_i \ell_i \, d\alpha_i \,,$$

where K is the curvature of the space, the sum is taken over all the edges of P, ℓ_i denotes the length of the i-th edge and α_i denotes dihedral angle along it.

In classical paper by Schläfli [Sch2] this formula was proved for the case of spherical n-simplex. For hyperbolic case it was obtained by H. Kneser [Kn]. For more details see also [V] and [M2].

2. Volume of the Spherical Octahedron with *mmm* Symmetry

Consider a spherical octahedron $\mathcal{O}$ with mmm symmetry, that is symmetric with respect of reflections in three mutually orthogonal planes passing through its edges. We notice that in this case polyhedron has eight congruent faces. Denote the lengths of edges by a, b, c, dihedral angles by A, B, C and the face angles by α, β, γ as shown on Fig. 1. In this notation face angle α is the opposite to the side with length a and the dihedral angle A can be found between two faces meeting in a side with length a.

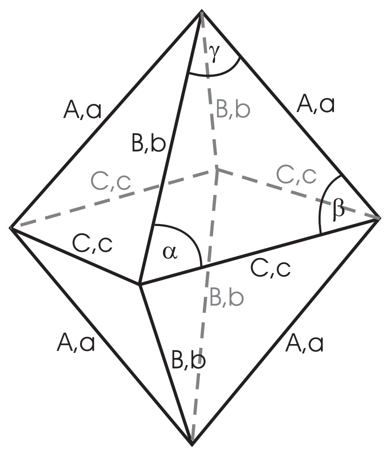

Figure 1. Octahedron $\mathcal{O} = \mathcal{O}(a, b, c, A, B, C)$ with mmm symmetry.

For the Euclidean case the following result is given in [GMS].

Theorem 2 (Galiulin, Mikhalev, Sabitov, 2004)**.** *Let V be the volume of an Euclidean octahedron $\mathcal{O}(a, b, c, A, B, C)$ with mmm symmetry. Then V is a positive root of equation*

$$9V^2 = 2(a^2 + b^2 - c^2)(a^2 + c^2 - b^2)(b^2 + c^2 - a^2)\,.$$

To find the volume of such an octahedron in spherical space we obtain the following trigonometrical identity at first.

Theorem 3 (The Sine-Tangent Rule)**.** *Let $\mathcal{O}(a,b,c,A,B,C)$ be a spherical octahedron with mmm symmetry, then the following trigonometric rule holds*

$$\frac{\sin A}{\tan a} = \frac{\sin B}{\tan b} = \frac{\sin C}{\tan c} = T = 2\,\frac{K}{\mathcal{C}}\,,$$

where K and $\mathcal{C}$ are positive numbers defined by

$$K^2 = (z-xy)(x-yz)(y-xz), \quad \mathcal{C} = 2xyz - x^2 - y^2 - z^2 + 1$$

and $x = \cos a$, $y = \cos b$ and $z = \cos c$.

Proof. Consider an intersection $\mathcal{O} = \mathcal{O}(a,b,c,A,B,C)$ with a sufficiently small sphere centered on a vertex of $\mathcal{O}$ (see Fig. 2(i)). Without loss of generality, we assume that the intersection is a spherical quadrilateral with interior angles B, C, B and C. Since the polyhedron under consideration admits the mmm symmetry the respective quadrilateral is a spherical romb with side α (Fig. 2(ii)).

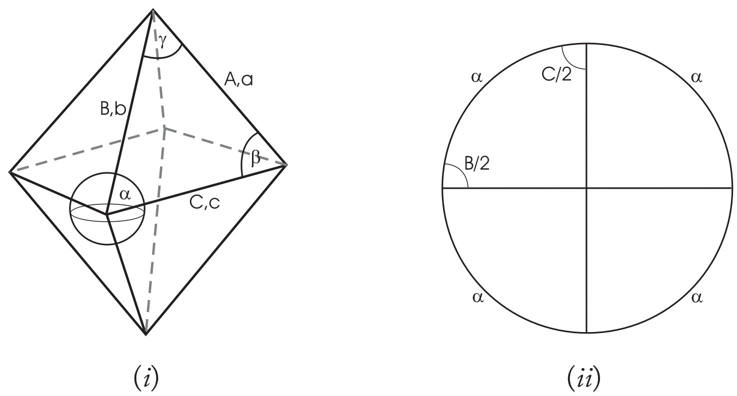

FIGURE 2. Octahedron $\mathcal{O} = \mathcal{O}(a,b,c,A,B,C)$ intersected with a horosphere, and the resulting spherical quadrilateral.

Given the symmetry hypothesis, the quadrilateral can be divided in four right spherical triangles with angles $B/2$ and $C/2$ and hypotenuse with length α. Applying Pythagoras Theorem for spherical right triangles, the identity

$$\cos\alpha = \cot\frac{B}{2}\cot\frac{C}{2}$$

takes place. Similar relations hold for the angles β and γ.

Rewriting the previous relation, it is immediate to obtain

$$\cot^2\frac{A}{2} = \frac{\cos\beta\cos\gamma}{\cos\alpha}\,,\ \cot^2\frac{B}{2} = \frac{\cos\alpha\cos\gamma}{\cos\beta}\,,\ \cot^2\frac{C}{2} = \frac{\cos\alpha\cos\beta}{\cos\gamma}\,.$$

The previous identities relate the dihedral angles and the face angles of $\mathcal{O}$. Similarly, it is possible to find relations between lengths and angles. Applying the first cosine rule to a face, the identities

$$\begin{aligned}
\cos a &= \cos b \cos c + \sin b \sin c \cos \alpha \,, \\
\cos b &= \cos a \cos c + \sin a \sin c \cos \beta \,, \\
\cos c &= \cos a \cos b + \sin a \sin b \cos \gamma
\end{aligned}$$

are valid. Rewriting the previous relation, the equivalent identities are obtained

$$\begin{aligned}
\cos \alpha &= \frac{\cos a - \cos b \cos c}{\sin b \sin c} \,, \\
\cos \beta &= \frac{\cos b - \cos a \cos c}{\sin a \sin c} \,, \\
\cos \gamma &= \frac{\cos c - \cos a \cos b}{\sin a \sin b} \,.
\end{aligned}$$

Defining new variables

$$\begin{array}{lll}
x = \cos a \,, & y = \cos b \,, & z = \cos c \,, \\
X = \cos A \,, & Y = \cos B \,, & Z = \cos C
\end{array}$$

it is immediate to observe that

$$\begin{aligned}
\cot^2 \frac{A}{2} &= \frac{(y - xz)(z - xy)}{(x - yz)(1 - x^2)} \,, \\
\cot^2 \frac{B}{2} &= \frac{(x - yz)(z - xy)}{(y - xz)(1 - y^2)} \,, \\
\cot^2 \frac{C}{2} &= \frac{(x - yz)(y - xz)}{(z - xy)(1 - z^2)} \,.
\end{aligned}$$

And, from the previous relations, the following equations are valid

$$\begin{aligned}
\sin A &= 2 \, \frac{(1 - x^2)^{1/2}}{x} \, \frac{K}{\mathcal{C}} \,, \\
\sin B &= 2 \, \frac{(1 - y^2)^{1/2}}{y} \, \frac{K}{\mathcal{C}} \,, \\
\sin C &= 2 \, \frac{(1 - z^2)^{1/2}}{z} \, \frac{K}{\mathcal{C}} \,,
\end{aligned}$$

where

$$K^2 = (z - xy)(x - yz)(y - xz) \quad \textit{and} \quad \mathcal{C} = 2xyz - x^2 - y^2 - z^2 + 1.$$

□

We note that in the Sine-Tangent Rule the parameter T is given in terms of edges of octahedron $\mathcal{O}$. To find the volume of $\mathcal{O}$ we have to express T in terms of dihedral angles.

Remark 1. *The value T in the Sine-Tangent Rule satisfies the following equation*

$$T^2 + \frac{(1+X)(1+Y)(1+Z)}{1+X+Y+Z} = 0\,,$$

where $X = \cos A, Y = \cos B$ and $Z = \cos C$.

Proof. To achieve this, we apply the second cosine rule to a face of $\mathcal{O}$, obtaining

$$\cos a = \frac{\cos\beta\cos\gamma + \cos\alpha}{\sin\beta\sin\gamma}\,.$$

By rather simple trigonometric identities, we obtain

$$\cot^2 a = \frac{(\cos\alpha + \cos\beta\cos\gamma)^2}{1 - (2\cos\alpha\cos\beta\cos\gamma + \cos^2\alpha + \cos^2\beta + \cos^2\gamma)}\,.$$

Using the relations between face angles and dihedral angles, we rewrite the latter in terms of X, Y and Z as

$$\cot^2 a = \frac{(1+Y)(1+Z)}{(X-1)(1+X+Y+Z)}\,.$$

Then the statement follows from identity $T^2 = \cot^2 a \sin^2 A\,.$ $\square$

It follows from Theorem 3 that a *mmm* symmetric spherical octahedron is completely determined by its dihedral angles, hence $\mathcal{O} = \mathcal{O}(A, B, C)$.

Before proving the volume formula, the following technical lemma is needed.

Lemma 1. *If*

$$\tan^2\theta + \frac{(1+X)(1+Y)(1+Z)}{1+X+Y+Z} = 0$$

then

$$\operatorname{arth}(X\cos\theta) + \operatorname{arth}(Y\cos\theta) + \operatorname{arth}(Z\cos\theta) + \operatorname{arth}(\cos\theta) = 0\,.$$

Proof. Since

$$\frac{1}{\cos^2\theta} = 1 - \frac{(1+X)(1+Y)(1+Z)}{1+X+Y+Z} = -\frac{XY + XZ + YZ + XYZ}{1+X+Y+Z}$$

we have

$$X + Y + (XZ + YZ)\cos^2\theta + Z + 1 + (XY + XYZ)\cos^2\theta = 0.$$

Equivalently,
$$\frac{X+Y}{1+XY\cos^2\theta}=-\frac{Z+1}{1+Z\cos^2\theta}\,.$$
Multiplying the latter equation by $\cos\theta$ we obtain
$$\mathrm{th}(\mathrm{arth}(\mathrm{X}\cos\theta)+\mathrm{arth}(\mathrm{Y}\cos\theta))=-\mathrm{th}(\mathrm{arth}(\mathrm{Z}\cos\theta)+\mathrm{arth}(\cos\theta))\,.$$
Since both arguments of hyperbolic tangent are real the statement of Lemma follows. ☐

Now we are able to obtain the following theorem.

Theorem 4 (Volume of the *mmm* Octahedron)**.** *Let $\mathcal{O}=\mathcal{O}(A,B,C)$ be a spherical octahedron with mmm symmetry. Then the volume $V=V(\mathcal{O})$ is given by*
$$2\int_{\frac{\pi}{2}}^{\theta}\Big(\mathrm{arth}(X\cos\tau)+\mathrm{arth}(Y\cos\tau)+\mathrm{arth}(Z\cos\tau)+\mathrm{arth}(\cos\tau)\Big)\frac{d\tau}{\cos\tau}\,,$$
where $X=\cos A$, $Y=\cos B$, $Z=\cos C$ and $0\le\theta\le\pi/2$ is a root of equation
$$\tan^2\theta+\frac{(1+X)(1+Y)(1+Z)}{1+X+Y+Z}=0.$$
Moreover, θ is determined by the Sine-Tangent Rule
$$\frac{\sin A}{\tan a}=\frac{\sin B}{\tan b}=\frac{\sin C}{\tan c}=\tan\theta\,.$$

Proof. To obtain this result, it is necessary to verify the Schläfli formula, that is
$$dV=2(a\;dA+b\;dB+c\;dC)\,,$$
or equivalently, to check that
$$\frac{\partial V}{\partial A}=2a\,,\quad\frac{\partial V}{\partial B}=2b\,,\quad\frac{\partial V}{\partial C}=2c\,.$$
We notice that the volume V is the unique solution of the above system of differential equations satisfying the condition $V\to 0$ as $a=b=c\to 0$.

Denote the integrand
$$\frac{\mathrm{arth}(\mathrm{X}\cos\tau)+\mathrm{arth}(\mathrm{Y}\cos\tau)+\mathrm{arth}(\mathrm{Z}\cos\tau)+\mathrm{arth}(\cos\tau)}{\cos\tau}$$
by $F(X,Y,Z,\tau)$. Then by the Leibniz rule we get
$$\frac{\partial V}{\partial A}=2F(X,Y,Z,\theta)\frac{\partial\theta}{\partial A}+2\int_{\pi/2}^{\theta}\frac{\partial F}{\partial A}\,d\tau\,.$$

By Lemma 1 we $F(X,Y,Z,\theta)=0$. Hence by Sine-Tangent Rule (Theorem 3) we obtain

$$\frac{\partial V}{\partial A}=\int_{\pi/2}^{\theta}\frac{2\,\sin A}{\cos^2\tau\cos^2 A-1}\,d\tau=2\arctan\frac{\sin A}{\tan\theta}=2a\,.$$

The equations $\dfrac{\partial V}{\partial B}=2b$ and $\dfrac{\partial V}{\partial C}=2c$ can be derived in a similar way.

We note that in the case $a=b=c$ by Theorem 3 we have

$$T=\tan\theta=2\,\frac{\sqrt{(x-x^2)^3}}{2x^3-3x^2+1}\,,\quad \text{where } \ x=\cos a\,.$$

Hence, $\tan\theta\to+\infty$ and $\theta\to\pi/2$ as $x\to 1-0$. Then the condition $V\to 0$ as $a=b=c\to 0$ follows from the convergence of integral

$$V=\int_{\pi/2}^{\theta}F(A,B,C,\tau)\,d\tau\,.$$

□

3. Hexahedron Dual to the Octahedron with *mmm* Symmetry

Let P be a spherical polyhedron, and let P^* be its dual. If an edge of length a has associated a dihedral angle A, then the corresponding edge of P^* has length $a^*=\pi-A$ and dihedral angle $A^*=\pi-a$. The previous arguments allow to obtain the following analogues of Theorems 3 and 4 for hexahedron $\mathcal{H}(a,b,c,A,B,C)$ with *mmm* symmetry dual to octahedron $\mathcal{O}(a^*,b^*,c^*,A^*,B^*,C^*)$.

Theorem 5 (The Sine-Tangent Rule)**.** *Let $\mathcal{H}(a,b,c,A,B,C)$ be a spherical hexahedron with mmm symmetry, then the following trigonometric rule holds*

$$\frac{\sin a}{\tan A}=\frac{\sin b}{\tan B}=\frac{\sin c}{\tan C}=2\,\frac{K}{\mathcal{C}}\,,$$

where $X=\cos A$, $Y=\cos B$, $Z=\cos C$, K and $\mathcal{C}$ are defined by $K^2=-(X+YZ)(Y+XZ)(Z+XY)$ and $\mathcal{C}=2XYZ+X^2+Y^2+Z^2-1$.

Theorem 6 (Volume of Hexahedron)**.** *Let $\mathcal{H}=\mathcal{H}(A,B,C)$ be a spherical hexahedron with mmm symmetry. Then the volume $V=V(\mathcal{H})$ is given by*

$$2\mathrm{Re}\int_{\theta}^{\frac{\pi}{2}}\left(\mathrm{arch}\frac{X}{\cos\tau}+\mathrm{arch}\frac{Y}{\cos\tau}+\mathrm{arch}\frac{Z}{\cos\tau}+\mathrm{arch}\frac{1}{\cos\tau}\right)\frac{d\tau}{\sin\tau}\,,$$

where $X = \cos A$, $Y = \cos B$, $Z = \cos C$ *and* $0 \le \theta \le \pi/2$ *is a root of equation*

$$\tan^2\theta + \frac{(2XYZ + X^2 + Y^2 + Z^2 - 1)^2}{4(X+YZ)(Y+XZ)(Z+XY)} = 0\,.$$

4. Volume of the Spherical Octahedron with $2|m$ Symmetry

We consider octahedron $\mathcal{O} = \mathcal{O}(a,b,c,d,A,B,C,D)$ with lengths of edges a,b,c,d and corresponding dihedral angles A,B,C,D which has $2|m$ symmetry (Fig. 3).

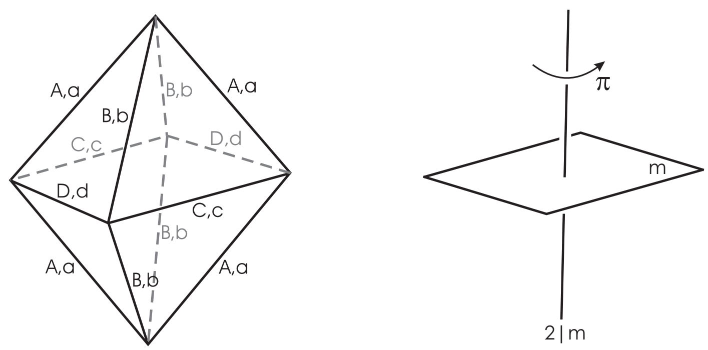

Figure 3. Octahedron $\mathcal{O} = \mathcal{O}(a,b,c,d,A,B,C,D)$ with $2|m$ symmetry.

For the Euclidean case the following result is given in [GMS]

Theorem 7 (Galiulin, Mikhalev, Sabitov, 2004)**.** *Let V be the volume of an Euclidean octahedron $\mathcal{O}(a,b,c,d,A,B,C,D)$ with $2|m$ symmetry. Then V is a positive root of equation*

$$9V^2 = (2a^2 + 2b^2 - c^2 - d^2)(a^2 - b^2 + cd)(b^2 - a^2 + cd)\,.$$

To find the volume of such an octahedron in spherical space, it is necessary to obtain a trigonometrical identity relating lengthes and angles. The following theorem holds.

Theorem 8 (The Sine-Tangent Rule)**.** *Let $\mathcal{O}(a,b,c,d,A,B,C,D)$ be a spherical octahedron with $2|m$ symmetry. Then the following trigonometric rule holds*

$$\frac{\sin A}{\tan a} = \frac{\sin B}{\tan b} = \frac{\sin\dfrac{C+D}{2}}{\tan\dfrac{c+d}{2}} = \frac{\sin\dfrac{C-D}{2}}{\tan\dfrac{c-d}{2}} = \tan\theta\,,$$

where $0 \le \theta \le \pi/2$ *is a number defined by*

$$\cos^2\theta + \frac{X+Y+Z+W}{XYZ+XYW+XZW+YZW} = 0,$$

$X = \cos A,\ Y = \cos B,\ Z = \cos\dfrac{C+D}{2}$ *and* $W = \cos\dfrac{C-D}{2}$.

Proof. This relations are obtained by straightforward calculations similar to those in the proof of Theorem 3. □

Thus, what we have realized is the important fact that a $2|m$ symmetric spherical octahedron is completely determined by its dihedral angles, hence $\mathcal{O} = \mathcal{O}(A, B, C, D)$.

The following technical lemma (similar to Lemma 1) takes place.

Lemma 2. *If*

$$\cos^2\theta + \frac{X+Y+Z+W}{XYZ+XYW+XZW+YZW} = 0,$$

then

$$\operatorname{arth}(X\cos\theta) + \operatorname{arth}(Y\cos\theta) + \operatorname{arth}(Z\cos\theta) + \operatorname{arth}(W\cos\theta) = 0\ .$$

Proof. By the data we have

$$X+Y+(XZW+YZW)\cos^2\theta+Z+W+(XYZ+XYW)\cos^2\theta = 0\ .$$

Equivalently,

$$\frac{X+Y}{1+XY\cos^2\theta} = -\frac{Z+W}{1+ZW\cos^2\theta}\ .$$

Multiplying the latter equation by $\cos\theta$ we obtain

$$\operatorname{th}(\operatorname{arth}(\mathrm{X}\cos\theta)+\operatorname{arth}(\mathrm{Y}\cos\theta)) = -\operatorname{th}(\operatorname{arth}(\mathrm{Z}\cos\theta)+\operatorname{arth}(\mathrm{W}\cos\theta))\ .$$

Since both arguments of hyperbolic tangent are real the statement of Lemma follows. □

With this considerations, we are able to prove the following

Theorem 9 (Volume of the $2|m$ Octahedron)**.** *Let* $\mathcal{O} = \mathcal{O}(A, B, C, D)$ *be a spherical octahedron with* $2|m$ *symmetry. Then the volume* $V = V(\mathcal{O})$ *is given by*

$$2\int_{\frac{\pi}{2}}^{\theta} \Big(\operatorname{arth}(X\cos\tau)+\operatorname{arth}(Y\cos\tau)+\operatorname{arth}(Z\cos\tau)+\operatorname{arth}(W\cos\tau)\Big)\frac{d\tau}{\cos\tau}\ ,$$

where $X = \cos A,\ Y = \cos B,\ Z = \cos\dfrac{C+D}{2},\ W = \cos\dfrac{C-D}{2}$ *and* $0 \le \theta \le \dfrac{\pi}{2}$ *is a root of equation*

$$\cos^2\theta + \frac{X+Y+Z+W}{XYZ+XYW+XZW+YZW} = 0.$$

Moreover, θ is determined by the Sine-Tangent Rule

$$\frac{\sin A}{\tan a} = \frac{\sin B}{\tan b} = \frac{\sin\dfrac{C+D}{2}}{\tan\dfrac{c+d}{2}} = \frac{\sin\dfrac{C-D}{2}}{\tan\dfrac{c-d}{2}} = \tan\theta\ .$$

Proof. The Schläfli formula, restricted to our case, ensures that

$$\begin{aligned} dV &= 2\arctan\left(\frac{\sin A}{\tan\theta}\right) dA + 2\arctan\left(\frac{\sin B}{\tan\theta}\right) dB \\ &+\ 2\arctan\left(\frac{\sin C}{\tan\theta}\right) dC + 2\arctan\left(\frac{\sin D}{\tan\theta}\right) dD\ . \end{aligned}$$

We note that volume V is the unique solution of the above differential equation satisfying the condition $V \to 0$ as $a = b = c = d \to 0$.

As in Theorem 4, denote the integrand

$$\frac{\mathrm{arth}(\mathrm{X}\cos\tau) + \mathrm{arth}(\mathrm{Y}\cos\tau) + \mathrm{arth}(\mathrm{Z}\cos\tau) + \mathrm{arth}(\mathrm{W}\cos\tau)}{\cos\tau}$$

by $F(A,B,C,D,\tau)$. Then

$$V(\theta) = 2\int_{\pi/2}^{\theta} F(A,B,C,D,\tau)\, d\tau\ .$$

By the Leibniz rule we get

$$\frac{\partial V(\theta)}{\partial A} = 2\, F(A,B,C,D,\theta)\frac{\partial\theta}{\partial A} + 2\int_{\pi/2}^{\theta} \frac{\partial F(A,B,C,D,\tau)}{\partial A}\, d\tau\ ,$$

but from Lemma 2 we have that $F(A,B,C,D,\theta) = 0$, thus

$$\begin{aligned} \frac{\partial V(\theta)}{\partial A} &= 2\int_{\pi/2}^{\theta} \frac{\partial F(A,B,C,D,\tau)}{\partial A}\, d\tau \\ &= 2\int_{\pi/2}^{\theta} \frac{-\sin A}{1-\cos^2 A\,\cos^2\tau}\, d\tau = 2\arctan\left(\frac{\sin A}{\tan\theta}\right) = 2a\ . \end{aligned}$$

A similar argument follows for the dihedral angle B, hence $\dfrac{\partial V}{\partial B} = 2b$. For the dihedral angle C (and similarly for D), it is clear that

$$\frac{\partial V(\theta)}{\partial C} = 2F(A,B,C,D,\theta)\frac{\partial\theta}{\partial C} + 2\int_{\pi/2}^{\theta} \frac{\partial F(A,B,C,D,\tau)}{\partial C}\, d\tau\ ,$$

but from Lemma 2 we have that $F(A,B,C,D,\theta)=0$, thus

$$\frac{\partial V(\theta)}{\partial C} = 2\int_{\pi/2}^{\theta} \frac{\partial F(A,B,C,D,\tau)}{\partial C}\, d\tau$$

$$= 2\int_{\pi/2}^{\theta} \left(\frac{-\sin\left(\dfrac{C+D}{2}\right)}{1-\cos^2\left(\dfrac{C+D}{2}\right)\cos^2\tau} + \frac{-\sin\left(\dfrac{C-D}{2}\right)}{1-\cos^2\left(\dfrac{C-D}{2}\right)\cos^2\tau} \right) d\tau$$

$$= 2\arctan\frac{\sin\left(\dfrac{C+D}{2}\right)}{\tan\theta} + 2\arctan\frac{\sin\left(\dfrac{C-D}{2}\right)}{\tan\theta}$$

$$= 2\left(\frac{c+d}{2}\right) + 2\left(\frac{c-d}{2}\right) = 2c\,.$$

As in the proof of Theorem 4, we have $V \to 0$ as $a=b=c=d\to 0$. $\square$

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