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The Chern–Simons Invariants of Cone-Manifolds with the Whitehead Link Singular Set

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Abstract—In the present article, we obtain some explicit integral formulas for the generalized Chern–Simons function $I\big(W(\alpha,\beta)\big)$ for Whitehead link cone-manifolds in the hyperbolic and spherical cases. We also give the Chern–Simons invariant for the Whitehead link orbifolds. We find a formula for the Chern–Simons invariant of n-fold coverings of the three-sphere branched over the Whitehead link.

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Key words: *Chern–Simons invariant, generalized Chern–Simons function, complex length, cone-manifold, orbifold, singular set, Whitehead link.*

1. INTRODUCTION

The hyperbolic volume, the Chern–Simons invariant, and the complex lengths of singular geodesics are important characteristics of geometric structure of a cone-manifold. According to the Kojima–Mostow rigidity theorem [9, 12], the above invariants are topological invariants as well.

The Chern–Simons invariants for coverings branched over some two-bridge knots were calculated by H. M. Hilden, M. T. Lozano, and J. M. Montesinos-Amilibia in [5–7]. We consider a two-parameter family of cone-manifolds with a three-sphere as an underlying space and the cone singularity along the Whitehead link. The aim of the present article is to find the Chern–Simons invariants of these cone-manifolds.

Denote by $W(\alpha,\beta)$ a 3-manifold with a three-sphere as an underlying space and the cone singularity along the Whitehead link, where α and β are cone angles along the corresponding components of the singular set. To find the Chern–Simons invariant of a geometric[1)] 3-manifold $W(\alpha,\beta)$, we correspond a real number $I\big(W(\alpha,\beta)\big)$ to each such manifold. We will give a definition of the number $I\big(W(\alpha,\beta)\big)$ in Section 4. This number depends on certain conditions; and, therefore, it is not an invariant of a cone-manifold. The Chern–Simons invariant can be defined as the residue of $I\big(W(\alpha,\ \beta)\big)$ modulo a certain number if the given cone-manifold is an orbifold. Then, for a 2-parameter family of cone-manifolds $W(\alpha,\beta)$, we have a function of two variables $I\big(W(\alpha,\beta)\big) = I(\alpha,\beta)$ which is called the *generalized Chern–Simons function*. This function has the following important property: It satisfies an analog of the classical differential Schläfli formula that enables us to find this function.

The article consists of 5 sections. The second section contains preliminaries and necessary definitions. In Section 3, we represent some results by A. D. Mednykh [10] about trigonometry of the Whitehead link. Grounding on these results, we obtain imaginary parts of complex lengths of singular geodesics. In Section 4, we define the Chern–Simons form, the generalized Chern–Simons function, and the Chern–Simons invariant for the Whitehead link orbifolds. In Section 5, using an analog of the classical differential Schläfli formula, we obtain some explicit integral formulas for the generalized Chern–Simons function $I\big(W(\alpha,\ \beta)\big)$ in the hyperbolic and spherical cases. These formulas enable us to find the Chern–Simons invariant of the Whitehead link orbifolds $(\mathbb{S}^3, W, n, m)$. In addition, in Section 5, we find the Chern–Simons invariants of the n-fold coverings $M_n(W)$ of the three-sphere branched over the Whitehead link and give some results of computations of these invariants.

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1)From now on, by "geometric" we mean a structure that admits a metric of constant (positive, negative, or zero) curvature.

2. PRELIMINARIES

In this article, we consider the Whitehead link cone-manifolds $C = W(\alpha, \beta)$ (see Fig.).

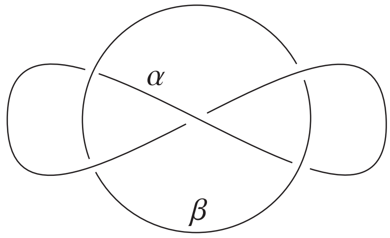

Fig. The Whitehead link $W(\alpha,\, \beta)$

A cone-manifold C is defined by its underlying space $\mathbb{S}^3$, a singular set Σ consisting of two components Σ_α and Σ_β, and cone angles α and β. In the case when these angles have the form $2\pi/n$, $n \in \mathbb{N}$, we deal with an *orbifold*.

An orbifold has the same local structure as a given manifold, and it is a natural object in studying the discrete groups. Thus, the orbifolds, as a particular case of cone-manifolds, are of special interest, and the group theory technique is applicable to them.

Further, the cone-manifold C determines a nonsingular but noncomplete manifold $N = C - \Sigma$. Denote by Φ the fundamental group of N. Recall some standard notations: the symbols $\mathbb{H}^3$, $\mathbb{E}^3$, or $\mathbb{S}^3$ denote the three-dimensional hyperbolic, Euclidean, or spherical spaces, $\mathrm{SL}(2, \mathbb{C})$ is the group of complex matrices of the second order with unit determinant, $\mathrm{PSL}(2, \mathbb{C})$ is the corresponding projective group, $\mathrm{SO}(3)$ is the three-dimensional rotation group, and $\mathrm{SO}(4)$ is the four-dimensional rotation group. There are three possible cases with respect to geometry in which a given manifold N is realized.

Case 1. A hyperbolic geometric structure on N defines a holonomic homomorphism

$$\widehat{h} : \Phi \longrightarrow \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}(2, \mathbb{C})$$

up to conjugation in $\mathrm{PSL}(2, \mathbb{C})$.

It is known [2] that the monodromy homomorphism $\widehat{h}$ can be lifted to $\mathrm{SL}(2, \mathbb{C})$ if all cone angles are less than π. Denote by $h : \Phi \longrightarrow \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{SL}(2, \mathbb{C})$ such a homomorphism.

Case 2. A Euclidean geometric structure on N defines a homomorphism

$$h : \Phi \longrightarrow \mathrm{Isom}^+(\mathbb{E}^3) = \mathrm{SO}(3) \cdot \mathbb{R}^3.$$

Case 3. A spherical geometric structure on N defines a homomorphism

$$h : \Phi \longrightarrow \mathrm{Isom}^+(\mathbb{S}^3) = \mathrm{SO}(4).$$

Choose some orientation on the link $\Sigma = \Sigma_\alpha \bigcup \Sigma_\beta$ and fix the pairs $\{m_j,\, l_j\}$, $j = \alpha, \beta$, of longitudes and parallels for each singular component. Then the matrices $M_j = h(m_j)$ and $L_j = h(l_j)$ satisfy the following defining relations:

$$M_j L_j = L_j M_j, \ \text{ where } j = \alpha, \beta.$$

In all three cases (of hyperbolic, Euclidean, and spherical structures) $h(l_j)$ is a skew motion with a displacement δ_j and an angle of rotation φ_j. Then a *jump* of the component Σ_j is a point in $\mathbb{R}/4\pi\mathbb{Z}$ which is defined by an angle φ_j. Denote by $\overline{\varphi}_j$ an equivalence class of φ_j modulo 4π. We call the quantity $\mathrm{tw}\big(\Sigma_j\big) = \varphi_j\, \alpha/2\pi$ the *twist* of the component Σ_j.

See [4] for more details.

Definition 1. We say that the cone-manifold C is obtained by the *orbifold Dehn surgery* with the cone angle $\alpha = 2\pi/m$ on the component Σ_α if $\mathrm{tr}(M_\alpha) = 2\,\cos(\alpha/2)$.

Definition 2. We say that the cone-manifold C is obtained by the *spontaneous Dehn surgery* with the cone angle $\alpha = 2\pi/m$ on the component Σ_α if $\mathrm{tr}(L_\alpha) = 2\,\cos(\alpha/2)$.

Definition 3. A *complex length* of the singular component Σ_j of the cone-manifold C is the complex number $\gamma_\alpha = \delta_\alpha + i\varphi_\alpha$.

From the above definitions it follows immediately that

$$2\cosh\gamma_j = \mathrm{tr}(L_j^2), \quad j = \alpha, \beta, \text{ in the case of the orbifold surgery;}$$
$$2\cosh\gamma_j = \mathrm{tr}(M_j^2), \ j = \alpha, \beta, \text{ in the case of the spontaneous surgery}$$

(see [3, p. 46]).

We note that the meridian-longitude pair $\{m_j, l_j\}$ of the oriented link is uniquely determined up to a common conjugating element of the group Φ. Hence, the complex length $\gamma_j = l_j + i\,\varphi_j$ is uniquely determined up to a sign and $(\mathrm{mod}\ 2\pi i)$. This means that the complex length γ_j satisfies the conditions $\delta_j \geq 0$ and $-2\pi < \varphi_j \leq 2\pi$ which we will hold in what follows.

3. TRIGONOMETRIC IDENTITIES AND THEIR COROLLARIES

Theorem 1 (the sine rule [10]). *Let $\gamma_\alpha = \delta_\alpha + i\varphi_\alpha$ (resp. γ_β) be a complex length of the singular geodesic of a hyperbolic cone-manifold $W(\alpha,\beta)$ with a cone angle α (resp. β). Then*

$$\frac{\sin(\varphi_\alpha/2)}{\sinh(\delta_\alpha/2)} = \frac{\sin(\varphi_\beta/2)}{\sinh(\delta_\beta/2)}.$$

Moreover, as was shown in [10, p. 300], the following relations hold:

$$iB\coth(\gamma_\alpha/4) = iA\coth(\gamma_\beta/4) = u, \tag{3.1}$$

where $A = \cot(\alpha/2)$, $B = \cot(\beta/2)$, and u, $\mathrm{Im}(u) > 0$, is a root of the cubic equation

$$u^3 - ABu^2 + \frac{1}{2}(A^2B^2 + A^2 + B^2 - 1)u + AB = 0. \tag{3.2}$$

Similar results take place in the spherical case as well.

The relations (3.1) and (3.2) gives us a practical way to calculate the real part δ_α (resp. δ_β) and the imaginary part φ_α (resp. φ_β) of the complex lengths γ_α (resp. γ_β) of singular geodesics. Indeed, for a suitable choice of analytic branches, from (3.1) it follows that

$$\varphi_\alpha = \frac{\gamma_\alpha - \overline{\gamma}_\alpha}{2i} = 2\arctan(\overline{u}/B) + 2\arctan(u/B) = 2\arctan(A/z) + 2\arctan(A/\overline{z}),$$

where $z = A\,B/\bar{u}$, $\mathrm{Im}(z) > 0$, satisfies the equation

$$z^3 + \frac{1}{2}(A^2B^2 + A^2 + B^2 - 1)z^2 - A^2B^2z + A^2B^2 = 0. \tag{3.3}$$

Thus, we have proven the following:

Proposition 1. *Let $W(\alpha,\beta)$ be a hyperbolic Whitehead link cone-manifold. Denote by φ_α and φ_β the imaginary parts of complex lengths of singular geodesics of $W(\alpha,\beta)$ with cone angles α and β respectively. Then*

$$\varphi_\alpha = 2\arctan(A/z) + 2\arctan(A/\overline{z}),$$
$$\varphi_\beta = 2\arctan(B/z) + 2\arctan(B/\overline{z}),$$

where z, $\mathrm{Im}(z) > 0$, is a root of the equation (3.3), $A = \cot(\alpha/2)$, and $B = \cot(\beta/2)$.

Similar reasoning can be carried out for the spherical cone-manifold $W(\alpha,\,\beta)$ too. In the spherical case (see [7]), all roots of the cubic equation (3.3) are real, and the imaginary parts of complex lengths of singular geodesics are given by the formulas

$$\varphi_\alpha = 2\arctan(A/\zeta_1) + 2\arctan(A/\zeta_2), \tag{3.4}$$
$$\varphi_\beta = 2\arctan(B/\zeta_1) + 2\arctan(B/\zeta_2), \tag{3.5}$$

where ζ_1 and ζ_2, $0 \leq \zeta_1 < \zeta_2$, are nonnegative roots of (3.3).

4. THE GENERALIZED CHERN–SIMONS FUNCTION

Define a number $I(\vec{C},\vec{\Sigma})$ (see [6]) associated with an oriented cone-manifold $(\vec{C},\vec{\Sigma})$. From now on, the sign "arrow" indicates that the orientation is specified. We will omit this sign in those cases when the orientation is immaterial.

Let $(\vec{C},\vec{\Sigma})$ be an oriented cone-manifold with a singular set $\Sigma = \Sigma_{\alpha_1} \cup \cdots \cup \Sigma_{\alpha_k}$. Choose $\vec{m}_1, \ldots, \vec{m}_k$ as meridians of the singular components whose orientation agrees with the orientation of $\vec{C}$. From [11, Theorem 4.3] it follows that there exists a frame field $F(\vec{C} - \Sigma - \cup_j m_j)$ with special singularities at $\Sigma \cup (\cup_j m_j)$ (see [6]). In what follows, we also call the map

$$s : \vec{C} - \Sigma - \cup_j m_j \to F\big(\vec{C} - \Sigma - \cup_j m_j\big),$$

with singularities at $\Sigma \cup (\cup_j m_j)$, a frame field (see [6]). Here, the minus stands for the set-theoretic difference.

Let Q be the Chern–Simons form defined on the positively-oriented orthonormal frame bundle $F(\vec{C},\vec{\Sigma})$; and

$$Q = \frac{1}{4\pi^2}(\theta_{12} \wedge \theta_{13} \wedge \theta_{23} + \theta_{12} \wedge \Omega_{12} + \theta_{13} \wedge \Omega_{13} + \theta_{23} \wedge \Omega_{23}),$$

where (θ_{ij}) is the connection 1-form and (Ω_{ij}) is the curvature 2-form of the Riemannian connection on the 3-manifold $\vec{C} - \Sigma$.

Proposition 2 [6]. *Let Q be the Chern–Simons form defined on the positively-oriented orthonormal frame bundle $F(\vec{C},\vec{\Sigma})$. Then the number*

$$\frac{1}{2} \int\limits_{s(\vec{C}-\Sigma-\cup m_j)} Q \pmod 1$$

is an invariant of $(\vec{C},\vec{\Sigma})$.

Let $s' = (f_1, f_2, f_3)$ be an orthonormal system on a subset of $\vec{C} - \Sigma$ containing meridians which possesses the following property: At each point $y \in m_j$, the tangent vector $f_1(y)$ is the tangent vector to m_j, and $f_2(y)$ is tangent to the meridian disc that is bounded by m_j. Then $\tau(m_j, s') = -\int_{s'(m_j)} \theta_{23}$. In fact, $\tau(m_j, s') \equiv \tau(m_j) \pmod{2\pi}$, where $\tau(m_j)$ is the torsion of the curve m_j.

Definition 4. Suppose that

$$I(\vec{C},\vec{\Sigma}) = \frac{1}{2} \int\limits_{s(\vec{C}-\Sigma-\cup m_j)} Q - \frac{1}{4\pi}\tau(m, s') - \frac{1}{4\pi}\mathrm{tw}(\Sigma),$$

$$I_1(\vec{C},\vec{\Sigma}) \equiv I(\vec{C},\vec{\Sigma}) \pmod 1,$$

$$I_{\alpha/2\pi}(\vec{C},\vec{\Sigma}) \equiv I(\vec{C},\vec{\Sigma}) \pmod{\alpha/2\pi}.$$

Definition 5. The number $I(\vec{C},\vec{\Sigma})$, expressed as a function of cone angles corresponding to the components of a singular set Σ of a cone-manifold C, is the *generalized Chern–Simons function*.

Proposition 3. *The number $I_{1/\mathrm{l.c.m.}\{n,m\}}(\vec{C},\vec{\Sigma}_{2\pi/n} \cup \vec{\Sigma}_{2\pi/m})$ is an invariant of the orbifold $(\vec{C}, \vec{\Sigma}_{2\pi/n} \cup \vec{\Sigma}_{2\pi/m})$.*

Proof. The real number $I(\vec{C},\vec{\Sigma})$ depends only on the choice of the frame field s provided that the values φ_j are chosen so that $-2\pi \le \varphi_j < 2\pi$. From Proposition 2 it follows that the class $I_1(\vec{C},\vec{\Sigma})$ is independent of the choice of s. The class $I_{\alpha/2\pi}(\vec{C},\vec{\Sigma})$ depends on the choice of s, but this class does not depend on a representative of the equivalence class $\overline{\varphi}_j$. In the case of $\alpha_j = 2\pi/n_j$, the cone-manifold is an orbifold.

In order to avoid conglomeration of indices, consider the case of $\Sigma = \Sigma_\alpha \cup \Sigma_\beta$ (the singular set is a two-component link with cone angles α and β). This particular case can be easily extended to the case of a k-component singular set. Moreover, in this article, we study the Whitehead link case in detail. Let $\alpha = 2\pi/n$ and $\beta = 2\pi/m$; i.e., $\alpha/2\pi = 1/n$ and $\beta/2\pi = 1/m$. Then the number $I_1\big(\vec{C}, \vec{\Sigma}_\alpha \cup \vec{\Sigma}_\beta\big)$ is independent of the choice of s. The number $I_{1/n}\big(\vec{C}, \vec{\Sigma}_\alpha \cup \vec{\Sigma}_\beta\big)$ is independent of a representative in the equivalence class $\overline{\varphi}_\alpha$. The number $I_{1/m}\big(\vec{C}, \vec{\Sigma}_\alpha \cup \vec{\Sigma}_\beta\big)$ is independent of a representative in the equivalence class $\overline{\varphi}_\beta$. □

Following [6], denote by $(\mathbb{S}^3, W, n, m)$ the Whitehead link orbifold with the cone angles $2\pi/n$ and $2\pi/m$, where $n, m \in \mathbb{N}$.

Definition 6. The Chern–Simons invariant (CS-invariant in a shorter form) for the orbifold $(\mathbb{S}^3, W, n, m)$ is given by the formula

$$\mathrm{CS}(\mathbb{S}^3, W, n, m) = I\left(W\left(\frac{2\pi}{n}, \frac{2\pi}{m}\right)\right) \quad \left(\mathrm{mod}\ \frac{1}{\mathrm{l.c.m.}\{n, m\}}\right).$$

Observe that the above definition is correct according to Proposition 3.

Remark 1 [6]. Suppose that $\alpha = \beta = 2\pi$. Then the cone-manifold $\big(\vec{M}, \Sigma_{2\pi}\big)$ is a geometric manifold $\vec{M}$ (the singular set degenerates in this case). The classes I_1 and $I_{\alpha/2\pi}$ are equal, since $\alpha/2\pi = 1$. Hence,

$$I_1\big(\vec{M}, \Sigma\big) \equiv \mathrm{CS}\big(\vec{M}\big) \pmod 1.$$

Denote by “tilde” the n-fold cyclic covering of the cone-manifold $(\vec{C}, \vec{\Sigma}_\alpha)$ branched over $\vec{\Sigma}_\alpha$. We have

$$\Big(\vec{\tilde{C}}, \vec{\tilde{\Sigma}}_{n\alpha}\Big) \to (\vec{C}, \vec{\Sigma}_\alpha).$$

Remark 2 [6]. Suppose that $\alpha = \beta$. Then

$$\begin{aligned} I\Big(\vec{\tilde{C}}, \vec{\tilde{\Sigma}}_{n\alpha}\Big) &= nI\big(\vec{C}, \vec{\Sigma}_\alpha\big), \\ I_1\Big(\vec{\tilde{C}}, \vec{\tilde{\Sigma}}_{n\alpha}\Big) &\equiv nI_1\big(\vec{C}, \vec{\Sigma}_\alpha\big) \qquad \pmod 1, \\ I_{n\alpha/2\pi}\Big(\vec{\tilde{C}}, \vec{\tilde{\Sigma}}_{n\alpha}\Big) &\equiv nI_{\alpha/2\pi}\big(\vec{C}, \vec{\Sigma}_\alpha\big) \quad \pmod{n\alpha/2\pi}. \end{aligned}$$

5. THE SCHLÄFLY FORMULA AND THE CHERN–SIMONS INVARIANT

In this section, we will obtain explicit formulas for the Chern–Simons invariant for certain cone-manifolds in the hyperbolic and spherical geometries. As in the case of volumes, the starting point for the CS-invariant calculation is the Schläfli formula (for more details see, for example, [8] or [6]).

Theorem 2 (the Schläfly formula for torsion)**.** *Suppose that C_t is a smooth one-parameter family of cone-manifold structures (of curvature K) on a 3-manifold with a singular set Σ. Then the derivative of the generalized Chern–Simons function I for C_t satisfies the relation*

$$K\, dI(C_t) = \frac{1}{4\pi^2} \sum_i \varphi_{\theta_i}\, d\theta_i,$$

where the summation is taken over all components of the singular set Σ with imaginary parts of the complex length φ_{θ_i} and the cone angles θ_i.

Theorem 3. *Let $W(\alpha, \beta)$ be a hyperbolic cone-manifold. Then the generalized Chern–Simons function is given by the formula*

$$I\big(W(\alpha, \beta)\big) = \int_{\zeta_1}^{-1} F(\zeta, A, B)\, d\zeta + \int_{\zeta_2}^{-1} F(\zeta, A, B)\, d\zeta - \left(\frac{\pi - \alpha}{2\pi}\right)^2 - \left(\frac{\pi - \beta}{2\pi}\right)^2 + C,$$

where

$$F(\zeta, A, B) = \frac{1}{2\pi^2(\zeta^2-1)} \log\left[\frac{2(\zeta^2+A^2)(\zeta^2+B^2)}{(1+A^2)(1+B^2)(\zeta^2-\zeta^3)}\right],$$

$A = \cot(\alpha/2),\ B = \cot(\beta/2),\ C = 11/24,\ \zeta_1 = z,\ \zeta_2 = \overline{z},\ \mathrm{Im}(z) > 0,$ *and z is a root of the cubic equation*

$$z^3 + \frac{1}{2}(A^2B^2 + A^2 + B^2 - 1)z^2 - A^2B^2z + A^2B^2 = 0.$$

Proof. According to the Schläfli formula for torsion, we have

$$\frac{\partial I}{\partial \alpha} = -\frac{\varphi_\alpha}{4\pi^2}, \quad \frac{\partial I}{\partial \beta} = -\frac{\varphi_\beta}{4\pi^2}, \tag{5.6}$$

where φ_α and φ_β are the imaginary parts of complex lengths of singular geodesics with the corresponding cone angles α and β.

Put

$$\widetilde{I} = \int_{\zeta_1}^{-1} F(\zeta, A, B)d\zeta + \int_{\zeta_2}^{-1} F(\zeta, A, B)d\zeta - \left(\frac{\pi-\alpha}{2\pi}\right)^2 - \left(\frac{\pi-\beta}{2\pi}\right)^2 + C_1,$$

where C_1 is some integration constant, and show that $\widetilde{I}$ satisfies the condition (5.6). Then $\widetilde{I} = I$, and the theorem is proven.

By the Newton–Leibniz formula, we obtain

$$\begin{aligned}\frac{\partial \widetilde{I}}{\partial \alpha} &= -F(\zeta_1, A, B)\frac{\partial \zeta_1}{\partial \alpha} + \int_{\zeta_1}^{-1} \frac{\partial F(\zeta, A, B)}{\partial A}\frac{\partial A}{\partial \alpha}\, d\zeta - F(\zeta_2, A, B)\frac{\partial \zeta_2}{\partial \alpha} \\ &\quad + \int_{\zeta_2}^{-1} \frac{\partial F(\zeta, A, B)}{\partial A}\frac{\partial A}{\partial \alpha}\, d\zeta + \frac{\pi-\alpha}{2\pi^2}.\end{aligned} \tag{5.7}$$

Observe that $F(\zeta_1, A, B) = F(\zeta_2, A, B) = 0$ if ζ_1, ζ_2, A, and B are the same as in the statement of the theorem. Moreover, since $\alpha = 2\,\mathrm{arccot}\,A$, we infer

$$\frac{\partial A}{\partial \alpha} = -\frac{1+A^2}{2},$$
$$\frac{\partial F(\zeta, A, B)}{\partial A}\frac{\partial A}{\partial \alpha} = \frac{A}{2\pi^2(\zeta^2+A^2)}.$$

Hence, by Proposition 1, from the equation (5.7) it follows that

$$\begin{aligned}\frac{\partial \widetilde{I}}{\partial \alpha} &= \frac{1}{2\pi^2}\int_{\zeta_1}^{-1} \frac{A\,d\zeta}{\zeta^2+A^2} + \frac{1}{2\pi^2}\int_{\zeta_2}^{-1} \frac{A\,d\zeta}{\zeta^2+A^2} + \frac{\pi-\alpha}{2\pi^2} \\ &= -\frac{1}{2\pi^2}\left[\arctan(A/\zeta_1) + \arctan(A/\zeta_2) + 2\arctan A + \alpha - \pi\right] \\ &= -\frac{1}{2\pi^2}\left[\arctan(A/z) + \arctan(A/\overline{z})\right] \\ &= -\frac{\varphi_\alpha}{4\pi^2},\end{aligned}$$

since $2\arctan A = 2\arctan\big(\cot(\alpha/2)\big) = 2\arctan\big(\tan(\pi/2 - \alpha/2)\big) = \pi - \alpha$.

The equation $\dfrac{\partial I}{\partial \beta} = -\dfrac{\varphi_\beta}{4\pi^2}$ can be obtained by analogy.

In [1], the constant $C = 11/24$ was found as a constituent part of the generalized Chern–Simons function independent of the variables α and β. Letting the constant C_1 equal to this value, we obtain the claim of the theorem. □

The analogous theorem is valid in the spherical case as well.

Theorem 4. *Let $W(\alpha,\beta)$ be a spherical Whitehead link cone-manifold. Then the generalized Chern–Simons function is given by the formula*

$$I\big(W(\alpha,\beta)\big) = \int_{\zeta_1}^{-1} F(\zeta,A,B)\,d\zeta + \int_{\zeta_2}^{-1} F(\zeta,A,B)\,d\zeta - \left(\frac{\pi-\alpha}{2\pi}\right)^2 - \left(\frac{\pi-\beta}{2\pi}\right)^2 + C,$$

where

$$F(\zeta,A,B) = \frac{1}{2\pi^2(\zeta^2-1)}\,\log\left[\frac{2(\zeta^2+A^2)(\zeta^2+B^2)}{(1+A^2)(1+B^2)(\zeta^2-\zeta^3)}\right],$$

$A=\cot(\alpha/2)$, $B=\cot(\beta/2)$, $C=11/24$, $\zeta_1=z_1$, $\zeta_2=z_2$, $0\le z_1<z_2$, *and z_1 and z_2 are the real roots of the cubic equation*

$$z^3+\frac{1}{2}(A^2B^2+A^2+B^2-1)z^2 - A^2B^2z + A^2B^2 = 0.$$

Proof. According to the Schläfli formula for torsion, we have

$$\frac{\partial I}{\partial\alpha} = -\frac{\varphi_\alpha}{4\pi^2},\quad \frac{\partial I}{\partial\beta} = -\frac{\varphi_\beta}{4\pi^2}, \tag{5.8}$$

where φ_α and φ_β are the imaginary parts of complex lengths of singular geodesics with the corresponding cone angles α and β.

Put

$$\widetilde{I} = \int_{\zeta_1}^{-1} F(\zeta,A,B)\,d\zeta + \int_{\zeta_2}^{-1} F(\zeta,A,B)\,d\zeta - \left(\frac{\pi-\alpha}{2\pi}\right)^2 - \left(\frac{\pi-\beta}{2\pi}\right)^2 + C_2,$$

where C_2 is an integration constant, and show that $\widetilde{I}$ satisfies the condition (5.8). Then $\widetilde{I}=I$, and the theorem is proven.

By the Newton–Leibniz formula, we obtain

$$\begin{aligned}\frac{\partial\widetilde{I}}{\partial\alpha} &= -F(\zeta_1,A,B)\frac{\partial\zeta_1}{\partial\alpha} + \int_{\zeta_1}^{-1}\frac{\partial F(\zeta,A,B)}{\partial A}\,\frac{\partial A}{\partial\alpha}\,d\zeta \\ &\quad - F(\zeta_2,A,B)\frac{\partial\zeta_2}{\partial\alpha} + \int_{\zeta_2}^{-1}\frac{\partial F(\zeta,A,B)}{\partial A}\,\frac{\partial A}{\partial\alpha}\,d\zeta + \frac{\pi-\alpha}{2\pi^2}.\end{aligned} \tag{5.9}$$

Note that $F(\zeta_1,A,B)=F(\zeta_2,A,B)=0$ if ζ_1, ζ_2, A, and B are the same as in the statement of the theorem. Moreover, since $\alpha = 2\,\mathrm{arccot}\,A$, we obtain

$$\frac{\partial A}{\partial\alpha} = -\frac{1+A^2}{2},$$

$$\frac{\partial F(\zeta,A,B)}{\partial A}\,\frac{\partial A}{\partial\alpha} = \frac{A}{2\pi^2(\zeta^2+A^2)}.$$

Hence, (3.4), (3.5), and (5.9) yield

$$\begin{aligned}\frac{\partial\widetilde{I}}{\partial\alpha} &= \frac{1}{2\pi^2}\int_{\zeta_1}^{-1}\frac{A\,d\zeta}{\zeta^2+A^2} + \frac{1}{2\pi^2}\int_{\zeta_2}^{-1}\frac{A\,d\zeta}{\zeta^2+A^2} + \frac{\pi-\alpha}{2\pi^2} \\ &= -\frac{1}{2\pi^2}\big[\arctan(A/\zeta_1) + \arctan(A/\zeta_2) + 2\,\arctan A + \alpha - \pi\big] \\ &= -\frac{1}{2\pi^2}\big[\arctan(A/z_1) + \arctan(A/z_2)\big] \\ &= -\frac{\varphi_\alpha}{4\pi^2}.\end{aligned}$$

The equation $\dfrac{\partial I}{\partial\beta} = -\dfrac{\varphi_\beta}{4\pi^2}$ can be derived similarly.

In [1], the constant $C = 11/24$ was found as a constituent part of the generalized Chern–Simons function independent of the variables α and β. Letting the constant C_2 equal to this value, we obtain the claim of the theorem. □

According to Definition 6, Theorems 3 and 4 allow us to find the CS-invariant of the orbifolds $(\mathbb{S}^3, W, n, m)$ in the hyperbolic and spherical cases:

$$\mathrm{CS}(\mathbb{S}^3, W, n, m) = I\left(W\left(\frac{2\pi}{n}, \frac{2\pi}{m}\right)\right) \quad \left(\mathrm{mod}\ \frac{1}{\mathrm{l.c.m.}\{n, m\}}\right). \tag{5.10}$$

The following theorem gives a useful application of the CS-invariant of the orbifolds $(\mathbb{S}^3, W, n, m)$ to finding the classical CS-invariant.

Theorem 5. *Let $M_n(W)$ be the n-fold cyclic covering of the three-sphere branched over the Whitehead link. Then the* CS-*invariant of the manifold $M_n(W)$ can be obtained by the formula*

$$\mathrm{CS}\big(M_n(W)\big) = n\,\mathrm{CS}(\mathbb{S}^3, W, n, n) \quad (\mathrm{mod}\ 1).$$

The *proof* of the last theorem is immediate from Remarks 1 and 2. □

In the table below, we present the results of computations of the CS-invariant for some n-fold cyclic coverings of the three-sphere branched over the Whitehead link. The data of this table were computed on using the program package Mathematica 5.0 (Wolfram Research).

$\mathrm{CS}\big(M_2(W)\big)$	$0,375000\ldots$
$\mathrm{CS}\big(M_4(W)\big)$	$0,618178\ldots$
$\mathrm{CS}\big(M_6(W)\big)$	$0,320379\ldots$
$\mathrm{CS}\big(M_{10}(W)\big)$	$0,539523\ldots$
$\mathrm{CS}\big(M_{12}(W)\big)$	$0,093624\ldots$

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REFERENCES

1. Coulson D., Goodman O. A., Hodgson C. D., and Neumann W. D., “Computing arithmetic invariants of 3-manifolds,” Experiment. Math. **9** (1), 127–152 (2000).
2. Culler M., “Lifting representations to covering groups,” Adv. Math. **59** (1), 64–79 (1986).
3. Fenchel W., *Elementary Geometry in Hyperbolic Space*, Vol. 11 (Walter de Gruyter & Co., Berlin, 1989).
4. Hilden H. M., Lozano M. T., and Montesinos-Amilibia J. M., “On a remarkable polyhedron geometrizing the figure eight knot cone manifolds,” J. Math. Sci. Univ. Tokyo **2** (3), 501–561 (1995).
5. Hilden H. M., Lozano M. T., and Montesinos-Amilibia J. M., “On the arithmetic 2-bridge knots and link orbifolds and a new knot invariant,” J. Knot Theory Ramifications **4** (1), 81–114 (1995).
6. Hilden H. M., Lozano M. T., and Montesinos-Amilibia J. M., “On volumes and Chern–Simons invariants of geometric 3-manifolds,” J. Math. Sci. Univ. Tokyo **3** (3), 723–744 (1996).
7. Hilden H. M., Lozano M. T., and Montesinos-Amilibia J. M., “Volumes and Chern–Simons invariants of cyclic covering over rational knots” in *Topology and Teichmüller Spaces*. Proc. of the 37th Taniguchi Symposium held in Finland, July 1995 (Singapore: World Scientific, 1996), pp. 31–35.
8. Hodgson C. D., *Schläfli Revisited: Variation of Volume in Constant Curvature Spaces*, Manuscript (University of Melbourne, Melbourne, 1991).
9. Kojima S. “Deformations of hyperbolic 3-cone-manifolds,” J. Differential Geom. **49** (3), 469–516 (1998).

10. Mednykh A. "On the remarkable properties of the hyperbolic Whitehead link cone-manifold" in *Knots in Hellas'98*, Vol. 24 of Proc. of the Int. Conf. on Knot Theory and Its Ramifications (World Scientific, Singapore, 2000), pp. 290–305.
11. Meyerhoff R., "Density of the Chern–Simons invariant for hyperbolic 3-manifolds" in *Low Dimensional Topology and Kleinian Groups*, Vol. 112 of *London Math. Soc. Lecture Note Ser.* (Cambridge Univ. Press, Cambridge, 1986), pp. 217–239.
12. Mostow G. D., "Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms," Inst. Hautes Études Sci. Publ. Math. **34**, 53–104 (1968).