The Seidel problem on the volume of hyperbolic tetrahedra

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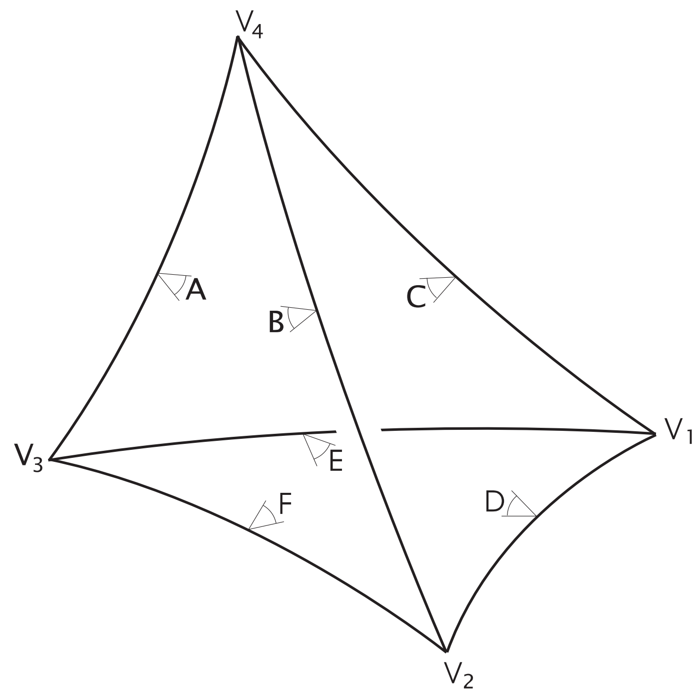
J.J. Seidel, *On the volume of a hyperbolic simplex* — [J] Stud. Sci. Math. Hung., **21** (1986), 243–249.

Conjecture 1. (Original Seidel conjecture) *Let* $\triangle$ *be an ideal hyperbolic tetrahedron. Then the following statements are true:*

(i) *the volume* $\mathrm{Vol}(\triangle)$ *can be expressed as a function of the determinant and the permanent of its Gram matrix* G*;*

(ii) *the volume* $\mathrm{Vol}(\triangle)$ *is increasing in the absolute value of the determinant* $|\det G|$ *for any fixed value of the permanent* $\operatorname{per} G$*;*

(iii) *the volume* $\mathrm{Vol}(\triangle)$ *is decreasing in the permanent* $\operatorname{per} G$ *for any fixed value of the determinant* $\det G$*.*



Tetrahedron $T(A, B, C, D, E, F)$

Definition 1. *The Gram matrix of a tetrahedron* $T(A, B, C, D, E, F)$ is defined as

$$G = \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}.$$

Definition 2. *The permanent of a matrix* $M = \langle m_{ij} \rangle_{i,j=1,\dots,n}$ is defined in the following way:

Take an arbitrary $j \in \{1, \dots, n\}$. Then

$$\begin{cases} \operatorname{per}\{m_{ij}\} = m_{ij}, \\ \operatorname{per} M = \sum_{i=1}^{n} m_{ij} \operatorname{per} M_{ij}, \end{cases}$$

where M_{ij} is the matrix obtained from M by omitting the i-th row and the j-th column.

Definition 3. *An ideal hyperbolic tetrahedron* is a hyperbolic tetrahedron with all vertices at infinity.

Advances in volume calculation for Euclidean polyhedra:

Theorem 1. (Tartaglia, 1494) *Let T be a Euclidean tetrahedron with edge lengths d_{ij}, $1 \leqslant i < j \leqslant 4$. Then $V = \mathrm{Vol}(T)$ is given by*

$$288\,V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix}.$$

Theorem 2. (I.Kh. Sabitov, 1997) *Let P be a Euclidean polyhedron with simplicial faces and edge lengths d_{ij}. Then $V = \mathrm{Vol}(P)$ is a root of an even degree algebraic equation whose coefficients are polynomials with integer coefficients in d_{ij}^2 that depends on combinatorial type of P only.*

Advances in volume calculation for Euclidean polyhedra:

1494 — Tartaglia

1997 — I.Kh. Sabitov

Advances in volume calculation for non-Euclidean polyhedra:

1836 — N.I. Lobachevsky

1858 — L. Schläfli

1935 — H.S.M. Coxeter

1989–2004 — R. Kellerhals, D.A. Derevnin, A.D. Mednykh, J. Parker, A.Yu. Vesnin, G. Martin, E.B. Vinberg

Advances in volume calculation for non-Euclidean tetrahedra:

A general algorithm for obtaining a general volume formula was outlined by Wu-Yi Hsiang (1988). Recently, a tetrahedron volume formula was obtained by Yu. Cho and H. Kim (1999) and also by J. Murakami and M. Yano (2001). An easy proof for this formula which also covers the case of the volume of a truncated tetrahedron was given by A. Ushijima (2002).

All these results deal with analytical formulas which express the volume in terms of 16 dilogarithms, i.e. Lobachevsky functions, of the dihedral angles with an additional parameter which is a root of a complicated quadratic equation with complex coefficients.

Advances in volume calculation for non-Euclidean tetrahedra:

A geometric interpretation of the Murakami-Yano formula was pointed out by Leibon (2002) from the point of view of Regge symmetry. An excellent exposition of these ideas and a complete geometrical proof of the the Murakami-Yano formula can be found in Yana Mohanty's Ph.D. thesis (2003).

It is worth mentioning that the ideas of Regge symmetry and scissors congruence were partially used by Cho and Kim who obtained the very first general formula for the volume of a hyperbolic tetrahedron.

Finally, an explicit integral formula for the volume of a hyperbolic tetrahedron was obtained by Derevnin and Mednykh.

Advances in volume calculation for non-Euclidean tetrahedra:

Theorem 3. (D.A. Derevnin, A.D. Mednykh, 2005) *Let* T *be a hyperbolic tetrahedron. Put* V_i *for the sum of the dihedral angles at the edges incident to the vertex* v_i*. Set* H_1, H_2 *and* H_3 *to be the sums of the dihedral angles along the three Hamiltonian cycles of* T *and* $H_4 = 0$*. Then*

$$\mathrm{Vol}(T) = -\frac{1}{4}\int_{z_1}^{z_2} \log \prod_{i=1}^{4} \frac{\cos\frac{V_i+z}{2}}{\sin\frac{H_i+z}{2}}\, dz,$$

where z_1 *and* z_2 *are roots of the integrand so that* $0 < z_2 - z_1 < \pi$.

The formula involves some parameters depending only on the dihedral angles and is helpful in evaluating the volume numerically.

Advances in volume calculation for non-Euclidean tetrahedra:

The following Murakami-Yano result can be obtained as an easy consequence of the formula above.

Theorem 4. (J. Murakami, M. Yano, 2001) *Let* T *be a hyperbolic tetrahedron. Then*

$$\mathrm{Vol}(T) = \frac{1}{2}(U(T,z_1) - U(T,z_2)),$$

where

$$U(T,z) = \sum_{j=1}^{4}\left(\Lambda\left(\frac{V_i + z}{2}\right) - \Lambda\left(\frac{\pi + H_i + z}{2}\right)\right),$$

$\Lambda(x) = -\int\limits_0^x \log|2\sin t|\,dt$ *is the Lobachevsky function and* V_i, H_i, z_1, z_2 *are as in Theorem 3.*

Advances in volume calculation for non-Euclidean tetrahedra:

Theorem 5. (J. Milnor, 1982) *Let* $T(A,B,C)$ *be an ideal hyperbolic tetrahedron (which is automatically symmetric). Then*

$$\mathrm{Vol}\, T(A,B,C) = \Lambda(A) + \Lambda(B) + \Lambda(C),$$

where $\Lambda(A) = -\int_0^x \log|2\sin t|dt$ *is the Lobachevsky function.*

Theorem 6. (D.A. Derevnin, A.D. Mednykh, M.G. Pashkevich, 2004) *Let* $T(A,B,C)$ *be a symmetric hyperbolic tetrahedron. Then*

$$\mathrm{Vol}\, T(A,B,C) =$$

$$\frac{1}{2}\int_\theta^{\frac{\pi}{2}} \frac{\sin^{-1}(\cos A\cos t)+\sin^{-1}(\cos B\cos t)+\sin^{-1}(\cos C\cos t)-\sin^{-1}(\cos t)}{\sin 2t}dt,$$ *where*

$\theta \in (0,\pi/2)$ *satisfies* $\tan\theta = \frac{1-a^2-b^2-c^2-2abc}{\sqrt{(1-a+b+c)(1+a-b+c)(1+a+b-c)(-1+a+b+c)}}$

with $a = \cos A, b = \cos B$ *and* $c = \cos C$.

Advances in volume calculation for non-Euclidean tetrahedra:

1906 — Gaetano Sforza

1988 — Wu–Yi Hsiang

1999 — Yu. Cho, H. Kim

2001 — J. Murakami, M. Yano

2002 — A. Ushijima

2002 — G. Leibon

2003 — Y. Mohanty

2005 — D. A. Derevnin, A. D. Mednykh

Advances in volume calculation for non-Euclidean tetrahedra:

Theorem 7. (Gaetano Sforza, 1906) *Let T be a hyperbolic tetrahedron with Gram matrix G. Then*

$$\mathrm{Vol}(T) = -\frac{1}{4}\int_{A_0}^{A} \log \frac{c_{34} + \sqrt{-\det G}\sin A}{c_{34} - \sqrt{-\det G}\sin A}\, dA,$$

where A_0 is a root of the equation $\det G = 0$ in the variable A and c_{34} is the corresponding minor of G.

Igor Rivin and Feng Luo asked the following question:

Stronger Seidel problem: *Is it true that the volume of a tetrahedron (hyperbolic or spherical) can be expressed as a function of the determinant of its Gram matrix?*

Spherical case:

Theorem 8. (A., A.D. Mednykh, 2006) *There exists a 2-parameter family of spherical tetrahedra whose Gram matrix has constant determinant and which have varying volume.*

In other words, the volume of a spherical tetrahedron cannot be expressed as a function of the determinant of its Gram matrix.

Hyperbolic case:

Theorem 9. (A., A.D. Mednykh, 2006) *There exists a 2-parameter family of hyperbolic tetrahedra whose Gram matrix has constant determinant and which have varying volume.*

In other words, the volume of a hyperbolic tetrahedron cannot be expressed as a function of the determinant of its Gram matrix.

Conjecture 1. (Original Seidel conjecture) *Let* $\triangle$ *be an ideal hyperbolic tetrahedron. Then*

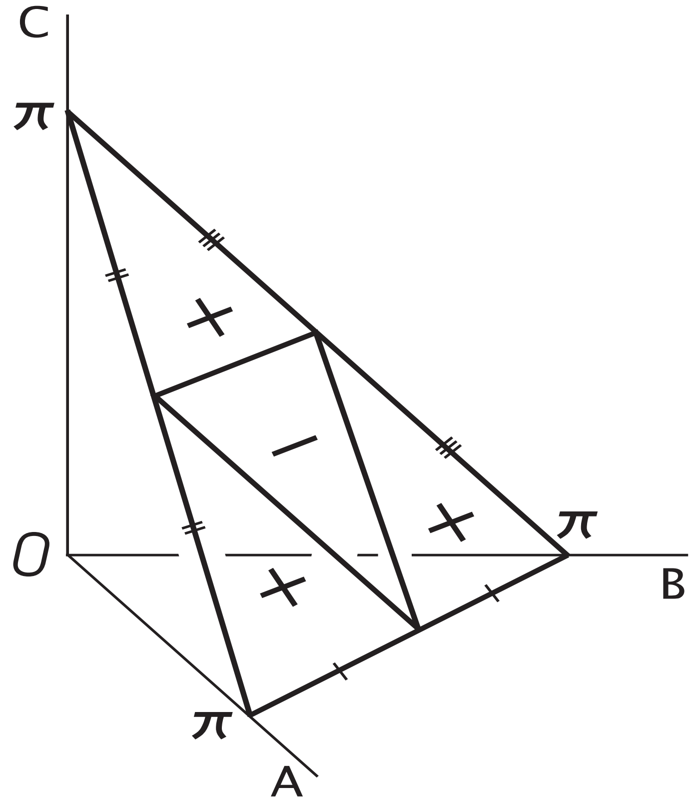
(i) *the volume* $\mathrm{Vol}(\triangle)$ *can be expressed as a function of the determinant and the permanent of its Gram matrix* G*;*

(ii) *the volume* $\mathrm{Vol}(\triangle)$ *is increasing in absolute value of the determinant* $|\det G|$ *for any fixed value of the permanent* $\mathrm{per}\, G$*;*

(iii) *the volume* $\mathrm{Vol}(\triangle)$ *is decreasing in the permanent* $\mathrm{per}\, G$ *for any fixed value of the determinant* $\det G$*.*

(iii^*) *if the sum of two non-opposite dihedral angles exceeds* $\pi/2$*, then the volume* $\mathrm{Vol}(\triangle)$ *is decreasing in the permanent* $\mathrm{per}\, G$ *for any fixed value of the determinant* $\det G$*, otherwise the volume is increasing.*

Theorem 10. (A., A.D. Mednykh, 2006) *Statements* $(i), (ii)$ *and* (iii^*) *are true.*



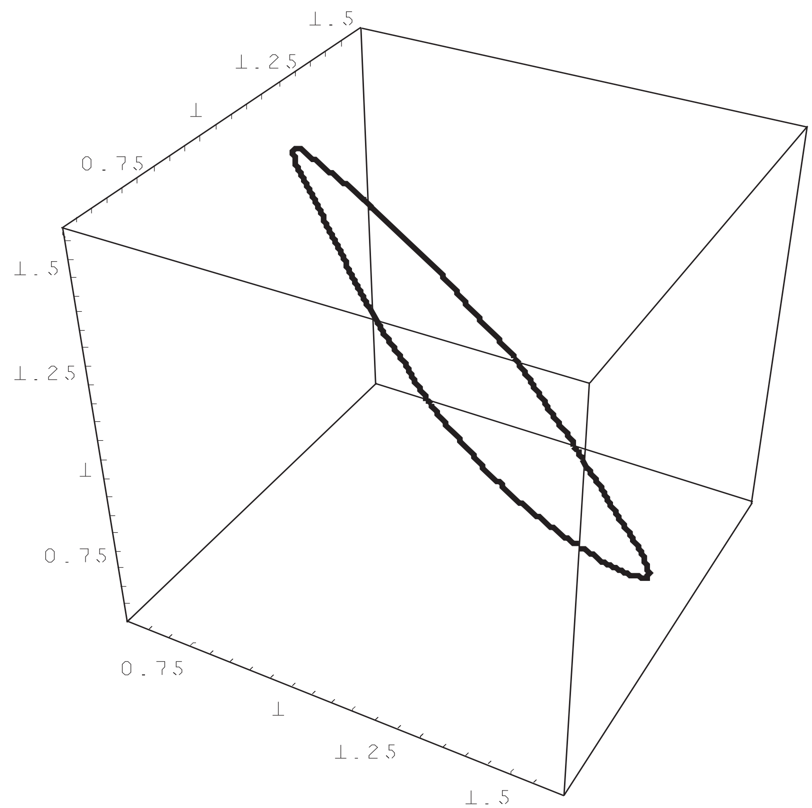
Constant-sign areas of the function $\dfrac{\partial V}{\partial\, \mathrm{per}\, G}$

Question: *Is it possible to express the volume of a symmetric (spherical or hyperbolic) tetrahedron as a function of the determinant and the permanent of its Gram matrix?*

The following statement gives an answer to this question:

Theorem 11. (A., A.D. Mednykh, 2006) *There exists a family of symmetric spherical tetrahedra whose Gram matrix has constant determinant and constant permanent and which have varying volume.*

In other words, the volume of a symmetric spherical tetrahedron cannot be expressed as a function of the determinant and the permanent of its Gram matrix.



The family of tetrahedra satisfying **Theorem 11**

Computational examples:

Dihedral angles			det	per	Vol
$\pi/2$	$\pi/3$	$\pi/4$	$-7/16$	$49/16$	0.222229
1.47063	1.27233	0.569501	$-7/16$	$49/16$	0.322981
1.36944	1.39004	0.526664	$-7/16$	$49/16$	0.35945
1.2661	1.47495	0.574059	$-7/16$	$49/16$	0.319517

Some of the milestones in our proofs:

- the differential Schläfli formula;
- the formula for the edge lengths in terms of dihedral angles;
- existence theorems for hyperbolic and spherical tetrahedra in terms of the signature of the Gram matrix.

Some of the milestones in our proofs:

Theorem 12. (Schläfli formula, 1860) *Let X be a three-dimensional space of constant curvature K. Consider a family of tetrahedra $\triangle \subset X$ which depends on one or more parameters in a differentiable manner. Then the volume differential $dV(\triangle)$ satisfies the condition*

$$2K\,dV(\triangle) = \sum_F \ell_F\,d\alpha_F,$$

where the sum is taken over all edges F of $\triangle$, ℓ_F is the length of the edge F, and α_F is the dihedral angle along F.

This formula was proved in the classical paper of Schläfli, but in the spherical case only. For the hyperbolic case it was proved by H. Kneser in 1936.

Some of the milestones in our proofs:

Theorem 13. (Cosine Rule [Mednykh, Pashkevich, 2005]) *Let* σ^n *be an* $n-$*dimensional hyperbolic simplex with finite vertices* v_i *and edge lengths* l_{ij}, *where* $i, j = 1, 2, ..., n+1$. *Then*

$$\cosh l_{ij} = \frac{c_{ij}}{\sqrt{c_{ii}c_{jj}}},$$

where c_{ij} *are corresponding minors of the Gram matrix of* σ^n.

Some of the milestones in our proofs:

Theorem 14. (F. Luo, 1997) *Given a set of numbers* $0 < A, B, C, D, E, F < \pi$, *the following statements are equivalent:*

(i) *There exists a spherical tetrahedron with dihedral angles* A, B, C, D, E, F.

(ii) *The matrix* G *defined as the Gram matrix of a tetrahedron* $T(A, B, C, D, E, F)$ *satisfies the following conditions:*

(a) $\det G$ *is positive;*

(b) *all principal minors of* G *are positive;*

(c) *the condition* $\frac{\sin \theta_{ij}}{\sin \ell_{ij}} = \frac{\sqrt{c_{kk}\, c_{ll}}}{\sqrt{\det G}}$ *holds for any set of pairwise distinct indices* $i, j, k, l \in \{1, 2, 3, 4\}$, *where* c_{ij} *is the* ij*-th minor of* G.

Some of the milestones in our proofs:

Theorem 15. (A. Ushijima, 2003) *Given a set of numbers* $0 < A, B, C, D, E, F < \pi$, *the following two statements are equivalent:*

(i) *There exists a hyperbolic tetrahedron with dihedral angles* A, B, C, D, E, F.

(ii) *The matrix* G *defined as the Gram matrix of a tetrahedron* $T(A, B, C, D, E, F)$ *satisfies the following conditions:*

- (a) G *has one negative and three positive eigenvalues;*
- (b) *all non-principal minors of* G *are positive.*

Results: we have

- refined and proved the original Seidel conjecture;
- gave a counterexamples to the stronger Seidel problem (put forward by Igor Rivin and Feng Luo);
- considered the possibility of extending the original Seidel conjecture to a more general class of tetrahedra;
- obtained a computational examples confirming the results.

Thank you for your kind attention!