THE CHERN–SIMONS INVARIANTS OF HYPERBOLIC MANIFOLDS VIA COVERING SPACES

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Introduction

One important invariant of a closed Riemannian 3-manifold is the Chern–Simons invariant [**1**]. The concept was generalized to hyperbolic 3-manifolds with cusps in [**11**], and to geometric (spherical, euclidean or hyperbolic) 3-orbifolds, as particular cases of geometric cone-manifolds, in [**7**]. In this paper, we study the behaviour of this generalized invariant under change of orientation, and we give a method to compute it for hyperbolic 3-manifolds using virtually regular coverings [**10**]. We confine ourselves to virtually regular coverings because a covering of a geometric orbifold is a geometric manifold if and only if the covering is a virtually regular covering of the underlying space of the orbifold, branched over the singular locus. Therefore our work is the most general for the applications in mind; namely, computing volumes and Chern–Simons invariants of hyperbolic manifolds, using the computations for cone-manifolds for which a convenient Schläfli formula holds (see [**7**]). Among other results, we prove that every hyperbolic manifold obtained as a virtually regular covering of a figure-eight knot hyperbolic orbifold has rational Chern–Simons invariant. We give explicit examples with computations of volumes and Chern–Simons invariants for some hyperbolic 3-manifolds, to show the efficiency of our method. We also give examples of different hyperbolic manifolds with the same volume, whose Chern–Simons invariants (mod $\frac{1}{2}$) differ by a rational number, as well as pairs of different hyperbolic manifolds with the same volume and the same Chern–Simons invariant (mod $\frac{1}{2}$). (Examples of this type were also obtained in [**12**] and [**9**], but using mutation and surgery techniques, respectively, instead of coverings as we do here.)

1. *Chern–Simons number of an oriented cone-manifold*

Let $\overrightarrow{M}$ be a closed oriented geometric cone-manifold of dimension three [**6**, **7**]. We recall here the definition of this concept. There exists a subset $\Lambda \subset \overrightarrow{M}$, called the *singular set*, which is a union of curves (*singular geodesics*), such that $\overrightarrow{M} \setminus \Lambda$ is a geometric 3-manifold modelled on some geometric space X. In this paper, X is a simply connected constant curvature space. Points off the singular set have neighbourhoods homeomorphic to neighbourhoods in the model X. Points on the

Received 20 October 1997; revised 28 May 1998.

1991 *Mathematics Subject Classification* 57M50, 51M10, 51M25.

This research was supported by grant PB95-0413 for the second and third authors.

singular set have neighbourhoods homeomorphic to neighbourhoods constructed as follows. Take an angle α wedge in the model X. (A wedge is the intersection or union of two half spaces that intersect; the angle α is the dihedral angle, where $0 < \alpha < 2\pi$.) Then identify the two boundaries of the wedge, using the natural rotation by α, to form a topological space W_α. Points on Λ have neighbourhoods homeomorphic to neighbourhoods in this topological space. The homeomorphism carries the singular set to the axis of rotation in the topological space. Transition functions are isometries. The following lemma generalizes [**2**, Proposition 3.1.1].

LEMMA 1.1. *Let $\overrightarrow{M}$ be an oriented* 3*-manifold without boundary. Assume that $\overrightarrow{M}$ is X-geometric, $X = S^3, E^3, H^3$. Let* Iso^+X *be the Lie group of orientation-preserving isometries of X. Let $H : \pi_1(\overrightarrow{M}, o) \to \mathrm{Iso}^+X$ be the holonomy of the X-geometry. Then H can be lifted to the universal covering of* Iso^+X.

Proof. Choose an orthonormal parallelization $\mathscr{P}$ of the Riemannian manifold M inducing the given orientation in $\overrightarrow{M}$. Take a base point o in M and a base point o' in X. Let r be an orthonormal reference at o'. The developing map $D : \widetilde{M} \to X$ is a local isometry from the universal covering $\widetilde{M}$, which sends paths starting in o to paths starting in o'. Choose r so that the reference of $\mathscr{P}$ at o is sent to r. In this way, to each element $\gamma : [0,1] \to M$, $\gamma \in \pi_1(\overrightarrow{M}, o)$, is associated a path γ' of orthonormal references in X starting at r, that is, a path in Iso^+X starting at the identity, that is, an element of the universal covering Lie group G of Iso^+X. The map $\gamma \to \gamma'$ is the required lifting of the holonomy $\gamma \to \gamma'(1)$. In fact, the map is a homomorphism, because $(\gamma * \mu)' = \gamma' * (\gamma'(1)(\mu'))$ for every $\gamma, \mu \in \pi_1(\overrightarrow{M}, o)$, and the composition of two elements γ', μ' of G is *defined* by $\gamma' * (\gamma'(1)(\mu'))$, where $(\gamma'(1)(\mu'))$ is the image of the path of references μ' under the isometry $\gamma'(1) \in \mathrm{Iso}^+X$.

We shall use the following notation. For each geodesic Σ_α, the subscript α denotes the value of the angle around Σ_α, which we normalize to be non-negative. A geodesic Σ_α is called *regular* if $\alpha = 2\pi$, or *singular* otherwise. For each geodesic Σ_α which bounds an oriented surface $\overrightarrow{S}$ in M, such that $\overrightarrow{S} \cap (\Lambda \cup \Sigma_\alpha) = \Sigma_\alpha$, the *jump*, $\beta(\Sigma)$, and the *twist*, $\mathrm{tw}(\Sigma)$, are defined as follows (compare [**7**]).

Choose an orientation on the curve Σ_α, and denote it by $\overrightarrow{\Sigma_\alpha}$. Consider an oriented meridian disc $\overrightarrow{D}$ of the neighbourhood $U = \{p \in \overrightarrow{M}^3;\ d(p, \overrightarrow{\Sigma_\alpha}) \leqslant \varepsilon\}$ of $\overrightarrow{\Sigma_\alpha}$. (The orientation of $\overrightarrow{D}$ followed by the orientation of $\overrightarrow{\Sigma_\alpha}$ coincides with the orientation of $\overrightarrow{M}$.) Let $\overrightarrow{m} = \partial \overrightarrow{D}$. Call $\overrightarrow{l_c}$ the canonical longitude of $\overrightarrow{\Sigma_\alpha}$, that is, $l_c = \partial U \cap S$, and the curves $\overrightarrow{l_c}$, $\overrightarrow{\Sigma_\alpha}$ are parallel. Let $o = \overrightarrow{m} \cap \overrightarrow{l_c}$.

Next, we distinguish three cases, according to whether the curvature of the Riemannian metric on M is -1, 0 or $+1$.

Case 1. $k = -1$. Let $H : \pi_1(M \setminus (\Lambda \cup \Sigma_\alpha), o) \to \mathrm{PSL}(2, \mathbb{C})$ be the holonomy of the hyperbolic manifold $M \setminus (\Lambda \cup \Sigma_\alpha)$. Then H can be lifted to a map h into $\mathrm{SL}(2, \mathbb{C})$ (Lemma 1.1). If $\pi_1(M \setminus (\Lambda \cup \Sigma_\alpha), o)$ is presented by $|a_1, \ldots, a_n; r_1, \ldots, r_s|$, then the map h assigns matrices to the generators $h(a_i) = A_i$, $i = 1, \ldots, n$, such that the relations hold. The element of the group $\pi_1(M \setminus (\Lambda \cup \Sigma_\alpha), o)$ represented by the loop $\overrightarrow{l_c}$ is a word in the alphabet $\{a_1^{\pm 1}, \ldots, a_n^{\pm 1}\}$. Since $\overrightarrow{l_c}$ is nullhomologous in $M \setminus (\Lambda \cup \Sigma_\alpha)$, it is a product of commutators of $\pi_1(M \setminus (\Lambda \cup \Sigma_\alpha), o)$. Therefore the number of appearances of the symbol a_i plus the number of appearances of the symbol a_i^{-1} in the word for $\overrightarrow{l_c}$ is even. Therefore, because any other lifting of H differs from h by changing signs in the matrices A_i, one sees that the image of $\overrightarrow{l_c}$ under any of these liftings is the same. So, up to conjugation in $\mathrm{SL}(2,\mathbb{C})$,

$$h(\overrightarrow{m}) = \pm \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}, \quad h(\overrightarrow{l_c}) = \begin{bmatrix} e^{v/2} & 0 \\ 0 & e^{-v/2} \end{bmatrix}.$$

(The conjugation puts the matrix $h(\overrightarrow{l_c})$ in diagonal form, with $\delta > 0$ in the complex number $v = \delta + i\beta$.) The number $\delta > 0$ is the *length* of $\overrightarrow{\Sigma_\alpha}$, and β, $-2\pi \leqslant \beta < 2\pi$, is the *angle of the lifted holonomy* of $\overrightarrow{\Sigma_\alpha}$.

Case 2. $k = 0$. The holonomy $H : \pi_1(M \setminus \Lambda, o) \to \mathrm{Iso}^+(E^3)$ can be lifted to a map h into $\mathbb{R}^3 \ltimes \mathrm{SU}(2)$ and, as in Case 1, the images of $\overrightarrow{l_c}$ under any of these liftings coincide. Up to conjugation, we have

$$h(\overrightarrow{m}) = \left(\overrightarrow{0}, \pm \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}\right), \quad h(\overrightarrow{l_c}) = \left(\delta \overrightarrow{k}, \begin{bmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{bmatrix}\right),$$

where $\delta > 0$ is the *length* of $\overrightarrow{\Sigma_\alpha}$, and β, $-2\pi \leqslant \beta < 2\pi$, is the *angle of the lifted holonomy* of $\overrightarrow{\Sigma_\alpha}$. (We are using quaternion notation $\overrightarrow{0}, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ for vectors in $\mathbb{R}^3$.)

Case 3. $k = 1$. The holonomy $H : \pi_1(M \setminus \Lambda, o) \to \mathrm{SO}(4)$ can be lifted to a map h into $\mathrm{SU}(2) \times \mathrm{SU}(2)$. As before, we can assume, up to conjugation in $\mathrm{SU}(2) \times \mathrm{SU}(2)$, that

$$h(\overrightarrow{m}) = \left(\pm \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}, \pm \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}\right), \quad h(\overrightarrow{l_c}) = \left(\begin{bmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{bmatrix}, \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}\right),$$

where $\gamma > \phi$. In this case, $\delta = \gamma - \phi$ is the *length* of $\overrightarrow{\Sigma_\alpha}$, and $\beta = \gamma + \phi$, $-2\pi \leqslant \beta < 2\pi$, is the *angle of the lifted holonomy* of $\overrightarrow{\Sigma_\alpha}$.

For any $k \neq 0$ we can normalize by multiplying the metric by a constant such that the new cone-manifold, $\overrightarrow{M_n}$, belongs to Cases 1 or 3. This process, of course, does not change angles.

LEMMA 1.2. *The angle of the lifted holonomy of* $\overrightarrow{\Sigma_\alpha}$ *does not depend on the orientation of the geodesic* Σ_α.

Proof. Let $\overrightarrow{\Sigma_\alpha}$ be an oriented nullhomologous knot in an oriented 3-manifold $\overrightarrow{M}$. Let $\overleftarrow{\Sigma_\alpha}$ denote the same knot with the opposite orientation. Observe that the oriented meridian of $\overleftarrow{\Sigma_\alpha}$ is $\overleftarrow{m}$, that is, the meridian of $\overrightarrow{\Sigma_\alpha}$ with the opposite orientation. We give the complete proof of this lemma for the hyperbolic case; the other two cases are analogous. We have

$$h(\overleftarrow{m}) = (h(\overrightarrow{m}))^{-1} = \pm \begin{bmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{bmatrix}, \quad h(\overleftarrow{l_c}) = (h(\overrightarrow{l_c}))^{-1} = \begin{bmatrix} e^{-v/2} & 0 \\ 0 & e^{v/2} \end{bmatrix}.$$

Then, to normalize α, δ and β, we conjugate by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{SL}(2,\mathbb{C})$. We obtain

$$h(\overleftarrow{m}) = \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \pm \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix},$$

$$h(\overleftarrow{l_c}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{-v/2} & 0 \\ 0 & e^{v/2} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} e^{v/2} & 0 \\ 0 & e^{-v/2} \end{bmatrix}.$$

DEFINITION 1.3. Let Σ_α be a geodesic (regular or singular) in an oriented geometric cone-manifold $\overrightarrow{M}^3$, of constant curvature k, where Σ_α is nullhomologous in $\overrightarrow{M} \setminus (\Lambda \setminus \Sigma_\alpha)$. The *jump* of Σ_α is the equivalence class $\overline{\beta}$ in $\mathbb{R}/4\pi\mathbb{Z}$ represented by the angle β, $-2\pi \leqslant \beta < 2\pi$, of the lifted holonomy of $\overrightarrow{\Sigma_\alpha}$ in the normalized cone-manifold $\overrightarrow{M_n}$, of constant curvature $1, 0, -1$. The *twist* of Σ_α, $\mathrm{tw}(\Sigma_\alpha)$, is the real number $\beta\alpha/2\pi$, and $\overline{\mathrm{tw}}(\Sigma_\alpha) = \overline{\beta}\alpha/2\pi$ is an equivalence class in $\mathbb{R}/2\alpha\mathbb{Z}$.

REMARKS. (1) The invariant $\mathrm{tw}(\Sigma_\alpha)$ should be understood as a kind of 'torsion' of the geodesic. When the geodesic is singular, it contains a correction factor because of the angle α. It is equivalent to $\mathrm{torsion}(\Sigma_{2\pi})$ (mod 2π) if the geodesic is not singular (see [**16**, Definition 1.2]). This is because the torsion of a regular geodesic computed in a suitable frame field is β.

(2) If Σ_α is any geodesic (not necessarily nullhomologous) in a geometric cone-manifold, then it is always possible to define an invariant $j(\Sigma_\alpha)$ as the equivalence class (mod α) of the angle β of the lifted holonomy of any longitude of the knot Σ_α. This is because if two longitudes differ by, say, k meridians, then the angles of the lifted holonomy differ by $k\alpha$. In particular, if $\alpha = 2\pi$, then this invariant is also equivalent to the ordinary torsion of Σ_α (mod 2π). However, we shall not work with this invariant in the present article.

DEFINITION 1.4. Let $(\overrightarrow{M}, \Sigma_\alpha)$ be a cone-manifold, where the singular set Σ_α is a nullhomologous knot in M. Choose an orientation in Σ_α, and denote the resulting oriented curve by $\overrightarrow{\Sigma_\alpha}$. Define the oriented meridian $\overrightarrow{m}$, as before. Consider the number

$$I(\overrightarrow{M}, \overrightarrow{\Sigma_\alpha}, s) = \frac{1}{2}\int_{s(\overrightarrow{M}-\overrightarrow{\Sigma_\alpha}-\overrightarrow{m})} Q - \frac{1}{4\pi}\tau(m, s') - \frac{1}{4\pi}\mathrm{tw}(\Sigma_\alpha),$$

where Q is the Chern–Simons form defined on the positively-oriented orthonormal frame bundle, $F(\overrightarrow{M} \setminus \Sigma_\alpha)$, and $s : \overrightarrow{M} \setminus (\Sigma_\alpha \cup m) \to F(\overrightarrow{M} \setminus (\Sigma_\alpha \cup m))$ is a frame field having special singularities at $\overrightarrow{\Sigma_\alpha} \cup \overrightarrow{m}$ (see [**16**, **11**]), and, finally, s' is a positively-oriented orthonormal frame field defined in a neighbourhood of m such that the first vector of the frame $s'(x)$, $x \in m$, is tangent to m, and the second vector is tangent to the meridian disc of Σ_α bounded by m and points in the direction of Σ_α. The number $\tau(m, s')$ is the torsion of the meridian in the sense of Yoshida [**16**, Definition 1.1]. In fact, $\tau(m, s')$ (mod 2π) is the torsion of m in the Riemannian geometry of the manifold $\overrightarrow{M} \setminus \overrightarrow{\Sigma_\alpha}$ [**16**, Definition 1.2]. Note that $\tau(m, s')$ does not depend on the orientation of m.

This number, $I(\overrightarrow{M}, \overrightarrow{\Sigma_\alpha}, s)$, was defined in [**7**], where it was shown that it generalizes the Chern–Simons invariant of Riemannian 3-manifolds. Using a subscript, I_λ denotes the equivalence class of I (mod λ). We showed in [**7**] that the equivalence class $I_1(\overrightarrow{M}, \overrightarrow{\Sigma_\alpha}, s) :\equiv I(\overrightarrow{M}, \overrightarrow{\Sigma_\alpha}, s)$ (mod 1) does not depend on the section s, and therefore can be denoted by $I_1(\overrightarrow{M}, \overrightarrow{\Sigma_\alpha})$. Specifically, in [**7**] we defined the Chern–Simons invariant of the orbifold $(\overrightarrow{M}, \overrightarrow{\Sigma}, n) := (\overrightarrow{M}, \overrightarrow{\Sigma}_{2\pi/n})$ as

$$\mathrm{CS}(\overrightarrow{M}, \overrightarrow{\Sigma}, n) :\equiv I_{1/n}(\overrightarrow{M}, \overrightarrow{\Sigma}_{2\pi/n}) \equiv I_1(\overrightarrow{M}, \overrightarrow{\Sigma}_{2\pi/n}) \equiv I(\overrightarrow{M}, \overrightarrow{\Sigma}_{2\pi/n}, s) \pmod{1/n}.$$

Suppose that we choose for Σ_α the other possible orientation, $\overleftarrow{\Sigma_\alpha}$. Then the frame field s can be modified in a neighbourhood of the link in such a way that the new frame field $\bar{s}$ is a frame field having special singularities at $\overleftarrow{\Sigma_\alpha} \cup \overleftarrow{m}$ and such that

$$\int_{s(\overrightarrow{M}-\overrightarrow{\Sigma_\alpha}-\overrightarrow{m})} Q = \int_{\bar{s}(\overrightarrow{M}-\overleftarrow{\Sigma_\alpha}-\overleftarrow{m})} Q;$$

see, for instance, [**12**, p. 114], where $\bar{s}$ is obtained by rotating s (near the singular locus) in the plane perpendicular to the e_2-vector. Then

$$I(\overrightarrow{M}, \overrightarrow{\Sigma_\alpha}, s) = I(\overrightarrow{M}, \overleftarrow{\Sigma_\alpha}, \bar{s}).$$

PROPOSITION 1.5. $I_1(\overrightarrow{M}, \Sigma_\alpha) :\equiv I(\overrightarrow{M}, \overrightarrow{\Sigma_\alpha})$ (mod 1) *is independent of the orientation given to the singular curve* Σ_α. *In particular, the Chern–Simons invariant of the orbifold* $(\overrightarrow{M}, \Sigma_{2\pi/n})$, *denoted now by* $\mathrm{CS}(\overrightarrow{M}, \Sigma, n)$, *is independent of the orientation of* Σ_α.

Proof. By the above observations and Lemma 1.2, it follows that $I_1(\overrightarrow{M}, \overrightarrow{\Sigma_\alpha})$ (mod 1) is well defined regardless of the orientation of Σ_α.

LEMMA 1.6. *Let* $(\overrightarrow{M}, \Sigma_\alpha)$ *be an oriented geometric cone-manifold, where the singular set* Σ_α *is a nullhomologous knot in* M. *Denote by* $(\overleftarrow{M}, \Sigma_\alpha)$ *the same cone-manifold with the opposite orientation. Then*

$$I(\overrightarrow{M}, \Sigma_\alpha, s) = -I(\overleftarrow{M}, \Sigma_\alpha, \hat{s}),$$

where $\hat{s}(x)$ *is the frame obtained by changing the sign of the first vector in the frame* $s(x)$.

Proof. Consider the diffeomorphism

$$\phi : F(\overrightarrow{M} - \Sigma_\alpha) \to F(\overleftarrow{M} - \Sigma_\alpha),$$
$$(e_1(x), e_2(x), e_3(x)) \mapsto (-e_1(x), e_2(x), e_3(x)).$$

If we denote by $\hat{Q}$ the Chern–Simons form in $F(\overleftarrow{M} - \Sigma_\alpha)$, then it is easy to check that $\delta\phi(\hat{Q}) = Q$, where $\delta\phi$ denotes the map induced by ϕ on forms. Observe that $\hat{s} = \phi \circ s$. Then the result is a consequence of the following equalities:

$$\int_{s(\overrightarrow{M}-\Sigma_\alpha-m)} Q = \int_{(\overrightarrow{M}-\Sigma_\alpha-m)} \delta s(Q) = \int_{(\overrightarrow{M}-\Sigma_\alpha-m)} \delta s(\delta\phi(\hat{Q}))$$
$$= \int_{(\overrightarrow{M}-\Sigma_\alpha-m)} \delta(\phi \circ s)(\hat{Q}) = -\int_{(\overleftarrow{M}-\Sigma_\alpha-m)} \delta\hat{s}(\hat{Q}) = -\int_{\hat{s}(\overleftarrow{M}-\Sigma_\alpha-m)} \hat{Q},$$

$$\tau(m, s') = -\tau(m, \hat{s}'),$$
$$\mathrm{tw}(\Sigma_\alpha \subset \overrightarrow{M}) = -\mathrm{tw}(\Sigma_\alpha \subset \overleftarrow{M}).$$

COROLLARY 1.7. *Let* $(\overrightarrow{M}, \Sigma_\alpha)$ *be an oriented geometric cone-manifold, where the singular set* Σ_α *is a nullhomologous knot in* M. *Denote by* $(\overleftarrow{M}, \Sigma_\alpha)$ *the same cone-manifold with the opposite orientation. Then*

$$I_1(\overrightarrow{M}, \Sigma_\alpha) \equiv -I_1(\overleftarrow{M}, \Sigma_\alpha) \text{ (mod 1)},$$
$$\mathrm{CS}(\overrightarrow{M}, \Sigma, n) \equiv -\mathrm{CS}(\overleftarrow{M}, \Sigma, n) \text{ (mod } 1/n).$$

COROLLARY 1.8. *Let* $(\overrightarrow{M}, \Sigma_\alpha)$ *be an oriented geometric cone-manifold, where the singular set* Σ_α *is a nullhomologous knot in* M. *If there exists an orientation-reversing isometry, then* $I_1(\overrightarrow{M}, \Sigma_\alpha) \equiv 0 \pmod{1/2}$.

COROLLARY 1.9. *Let* K *be a hyperbolic and amphicheiral knot in* S^3. *Then* $I_1(\overrightarrow{S}^3, K_\alpha) \equiv 0 \pmod{1/2}$ *for any angle* $0 \leqslant \alpha \leqslant \pi$.

2. *Chern–Simons invariant and virtually regular coverings over orbifolds*

The concept of virtually regular covering was defined in [**10**], where some of its properties were explored.

DEFINITION 2.1 ([**10**]). A branched covering $p : N \to M$ is virtually regular if there exists an unbranched covering $u : P \to N$ such that $p \circ u : P \to M$ is regular. It is surprising that this is equivalent to the property that the branching index is constant along the fibre over each point of the branching set. Notice that if one can prove that every simply connected 3-manifold is a virtually regular covering of S^3, then the Poincaré Conjecture follows by Thurston's geometrization results for orbifolds; indeed, a simply connected virtually regular covering is regular.

We are interested here in virtually regular coverings over an oriented geometric 3-orbifold $(\overrightarrow{M}, \Sigma, n)$, $p : N \to M$, such that the branching set is the singular set $\Sigma_{2\pi/n}$, and the branching indices are all equal to n. This guarantees that N is a geometric manifold. Indeed, the geometric structure on the base $(\overrightarrow{M}, \Sigma, n)$ lifts to a geometric structure on the cover N, where the only possible singularity is the link $p^{-1}(\Sigma_{2\pi/n})$. But because the branching indices are all n, the angle around each component of $p^{-1}(\Sigma_{2\pi/n})$ is $n \cdot 2\pi/n = 2\pi$, that is, N is a geometric 3-manifold. Denote by $\overrightarrow{N}$ the manifold N with the orientation induced by the orientation of $\overrightarrow{M}$.

THEOREM 2.2. *Let* $(\overrightarrow{M}, \Sigma, n)$ *be an oriented geometric* 3*-orbifold, where the singular set* $\Sigma_{2\pi/n}$ *is a nullhomologous knot in* M. *Let* $p : N \to M$ *be an* h*-fold virtually regular covering, branched over* $\Sigma_{2\pi/n}$ *with branching index* n. *Then the volume and the Chern–Simons invariant of the geometric manifold* $\overrightarrow{N}$ *are related to those of the orbifold* $(\overrightarrow{M}, \Sigma, n)$ *by the formulas*

$$\mathrm{Vol}(\overrightarrow{N}) = h\,\mathrm{Vol}(\overrightarrow{M}, \Sigma, n),$$
$$\mathrm{CS}(\overrightarrow{N}) \equiv h\,\mathrm{CS}(\overrightarrow{M}, \Sigma, n) - k/2n \pmod{1/2},$$
$$\mathrm{CS}(\overrightarrow{N}) \equiv h\,\mathrm{CS}(\overrightarrow{M}, \Sigma, n) \pmod{1/2n},$$

where $k \pmod{n}$ *is the intersection number of the union of any set of canonically oriented longitudes, for* $p^{-1}(\overrightarrow{\Sigma}_{2\pi/n})$, *with* $p^{-1}(\overrightarrow{l_c})$.

Proof. The fundamental group of N is isomorphic to a discrete group Γ_N of isometries of S^3, E^3 or H^3, according to whether the geometry is spherical, euclidean or hyperbolic. This group Γ_N is a subgroup of index h of Γ_M, the group of the orbifold $(\overrightarrow{M}, \Sigma, n)$. The volume of N is the volume of a fundamental domain for the action of the group Γ_N, which is h times the volume of the fundamental domain for the group Γ_M. This proves that

$$\mathrm{Vol}(\overrightarrow{N}) = h\,\mathrm{Vol}(\overrightarrow{M}, \Sigma, n).$$

The Chern–Simons invariant of the orbifold $(\overrightarrow{M}, \Sigma, n)$ is, by definition,

$$\mathrm{CS}(\overrightarrow{M}, \Sigma, n) \equiv I(\overrightarrow{M}, \Sigma_{2\pi/n}, s) \pmod{1/n},$$

where s is a frame field having special singularities at the link $\overrightarrow{\Sigma}_{2\pi/n} \cup \overrightarrow{m}$. This frame field s lifts to a frame field $\tilde{s}$ in $\overrightarrow{N}$, having special singularities at the link $p^{-1}(\overrightarrow{\Sigma}_{2\pi/n}) \cup p^{-1}(\overrightarrow{m})$. Then

$$\int_{\tilde{s}(\overrightarrow{N} - p^{-1}(\overrightarrow{\Sigma}_{2\pi/n}) - p^{-1}(\overrightarrow{m}))} Q = h \int_{s(\overrightarrow{M} - \Sigma_{2\pi/n} - m)} Q.$$

The link $p^{-1}(\overrightarrow{m})$ has $l = h/n$ components, $\tilde{m}_1, \ldots, \tilde{m}_l$, such that the restriction of p to a neighbourhood of each one of them is an n-cyclic unbranched covering onto a neighbourhood of m. This implies that

$$\tau(\tilde{m}_i) \equiv n\tau(m, s') \pmod{2\pi}.$$

The link $p^{-1}(\overrightarrow{\Sigma}_{2\pi/n})$ has, say, d components, $\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_d$. The restriction of p to a suitable neighbourhood N_i of $\tilde{\Sigma}_i$ is a $q_i n$ covering onto the neighbourhood $(p(N_i))$ of $\Sigma_{2\pi/n}$, branched over $\Sigma_{2\pi/n}$, where $\sum_{i=1}^{d} q_i = h/n$. To obtain the jump of $\tilde{\Sigma}_i$, consider the unbranched covering between tori $p_{|\partial(N_i)} : \partial(N_i) \to \partial(p(N_i))$. The pair $(\overrightarrow{m}, \overrightarrow{l_c})$ is a homology basis on $\partial(p(N_i))$. Let $\overrightarrow{\tilde{m}_i}$ be a component of $p^{-1}(\overrightarrow{m})$ which is a meridian on $\partial(N_i)$. Let $\overrightarrow{L_i}$ be an oriented simple closed curve in $\partial(N_i)$ such that $(\overrightarrow{\tilde{m}}_i, \overrightarrow{L_i})$ is a homology basis on $\partial(N_i)$, and $p(\overrightarrow{\tilde{m}_i}) = n\overrightarrow{m}$ and $p(\overrightarrow{L_i}) = q_i \overrightarrow{l_c} + k_i \overrightarrow{m}$. Notice that

$$k_i = p(\overrightarrow{L_i}) \cdot \overrightarrow{l_c} = \overrightarrow{L_i} \cdot (\text{component of } p^{-1}(\overrightarrow{l_c}) \text{ in } \partial(N_i)).$$

(Any other possible $\overrightarrow{L_i}'$ differs from $\overrightarrow{L_i}$ by a multiple of the meridian $\overrightarrow{\tilde{m}_i}$.) Thus the number k_i is well defined (mod n). Therefore (recall that $\overrightarrow{\Sigma_i}$ is not singular)

$$\tau(\overrightarrow{\Sigma_i}) \equiv \mathrm{tw}(\overrightarrow{\Sigma_i}) = q_i n\,\mathrm{tw}(\overrightarrow{l_c}) + k_i\, 2\pi/n \pmod{2\pi}.$$

The Torsion Formula for the Chern–Simons invariant [**11**] states that

$$\mathrm{CS}(\overrightarrow{N}) \equiv \frac{1}{2}\int_{\tilde{s}(\overrightarrow{N} - p^{-1}(\Sigma_{2\pi/n} - m))} Q - \frac{1}{4\pi}\left(\sum_{i=1}^{l} \tau(\tilde{m}_i)\right) - \frac{1}{4\pi}\left(\sum_{i=1}^{d} \tau(\overrightarrow{\Sigma_i})\right) \pmod{\tfrac{1}{2}}.$$

Then

$$\begin{aligned}
\mathrm{CS}(\overrightarrow{N}) &\equiv \frac{1}{2}h\int_{s(\overrightarrow{M} - \Sigma_{2\pi/n} - m)} Q - \frac{1}{4\pi}\left(\sum_{i=1}^{l} n\,\tau(m, s')\right) - \frac{1}{4\pi}\left(\sum_{i=1}^{d}\left(q_i n\,\mathrm{tw}(\overrightarrow{l_c}) + k_i\frac{2\pi}{n}\right)\right) \\
&\equiv \frac{1}{2}h\int_{s(\overrightarrow{M} - \Sigma_{2\pi/n} - m)} Q - \frac{1}{4\pi}\big(ln\,\tau(m, s')\big) - \frac{1}{4\pi}\left(\sum_{i=1}^{d}\big(q_i n\,\mathrm{tw}(\overrightarrow{l_c})\big)\right) - \left(\sum_{i=1}^{d} k_i\frac{1}{2n}\right) \\
&\equiv h\left(\frac{1}{2}\int_{s(\overrightarrow{M} - \Sigma_{2\pi/n} - m)} Q - \frac{1}{4\pi}\tau(m, s') - \frac{1}{4\pi}\mathrm{tw}(\Sigma_{2\pi/n})\right) - \left(\sum_{i=1}^{d} k_i\frac{1}{2n}\right) \\
&\equiv h\big(\mathrm{CS}(\overrightarrow{M}, \Sigma, n)\big) - \left(\sum_{i=1}^{d} k_i\frac{1}{2n}\right) \pmod{\tfrac{1}{2}}.
\end{aligned}$$

Therefore

$$\mathrm{CS}(\overrightarrow{N}) \equiv h\big(\mathrm{CS}(\overrightarrow{M}, \Sigma, n)\big) - \frac{k}{2n} \ (\mathrm{mod}\ \tfrac{1}{2}),$$

where $k = \sum_{i=1}^{d} k_i = (\bigcup \overrightarrow{L_i}) \cdot p^{-1}(\overrightarrow{l_c})$. Thus

$$\mathrm{CS}(\overrightarrow{N}) \equiv h\big(\mathrm{CS}(\overrightarrow{M}, \Sigma, n)\big) \ (\mathrm{mod}\ 1/2n).$$

Notice that the above theorem gives an easy method to obtain the Chern–Simons invariant (mod $\frac{1}{2}$) of geometric manifolds obtained by virtually regular coverings of orbifolds, if the monodromy of the covering is given. We shall use this method in some examples explained later.

Corollary 2.3. *Let K be a hyperbolic and amphicheiral knot in S^3. Then any hyperbolic manifold M obtained as a virtually regular covering of the orbifold (S^3, K, n) has rational Chern–Simons invariant. In fact,*

$$\mathrm{CS}(\overrightarrow{M}) \equiv 0 \ (\mathrm{mod}\ 1/2n).$$

Proof. The result follows from Corollary 1.9 and Theorem 2.2.

This is true, in particular, for the figure-eight knot (rational knot 5/3), which is a universal knot [**4**]. Recall also that the hyperbolic orbifold $(S^3, 5/3, 12)$ is a universal orbifold [**5**].

3. *Some examples*

Virtually regular coverings of geometric orbifolds are a source of examples of the following two kinds:

(1) different geometric manifolds with the same volume and Chern–Simons invariant (mod $\frac{1}{2}$);

(2) geometric manifolds with the same volume whose Chern–Simons invariants differ by a rational number.

In this section we give various examples as a sample of the method of construction. Notice that by computing the Chern–Simons invariant in this way, we have a very easy method for distinguishing certain hyperbolic manifolds with the same volume.

Example 1. *Some virtually regular coverings of the figure-eight knot orbifolds.* Consider the figure-eight knot, which is the 2-bridge knot 5/3 (4_1 in [**13**]). The group $G(5/3) = \pi_1(S^3 - (5/3))$ has the presentation $|a, b : aba^{-1}b^{-1}a = ba^{-1}b^{-1}ab|$, where the generators a, b are represented by meridians.

The reflection through O (see Fig. 1) defines an orientation-reversing involution u of S^3 which fixes the knot 5/3 as a set. This exhibits the strong amphicheirality of 5/3. The involution u defines the automorphism $u_\sharp$ on $G(5/3)$ given by

$$u_\sharp(a) = a, \quad u_\sharp(b) = c = b^{-1}ab.$$

The π-turn around the axis A (see Fig. 1) defines an orientation-preserving involution v of S^3 which fixes the knot as a set but reverses its orientation. This implies the strong invertibility of the knot 5/3, as it does for any 2-bridge knot. The induced automorphism $v_\sharp$ on $G(5/3)$ is given by

$$v_\sharp(a) = a^{-1}, \quad v_\sharp(b) = b^{-1}.$$

Both automorphisms, $u_\sharp$ and $v_\sharp$, will be taken into account in studying all the virtually regular coverings of a given type.

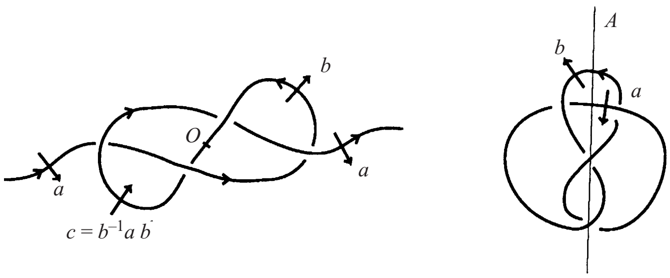

FIG. 1. Two projections of the figure-eight knot

(1) *Virtually regular coverings of* 6 *sheets.* The representations $\omega_i : G(5/3) \to S_6$, $i = 1, 2, 3, 4$, where S_6 is the permutation group of 6 elements, given by

$$\begin{aligned} \omega_1(a) &= (1\,2\,3\,4\,5\,6), & \omega_1(b) &= (1\,2\,3\,4\,5\,6), \\ \omega_2(a) &= (1\,2\,3\,4\,5\,6), & \omega_2(b) &= (1\,2\,6\,4\,5\,3), \\ \omega_3(a) &= (1\,2\,3\,4\,5\,6), & \omega_3(b) &= (1\,2\,4\,3\,6\,5), \\ \omega_4(a) &= (1\,2\,3\,4\,5\,6), & \omega_4(b) &= (1\,3\,6\,4\,2\,5), \end{aligned}$$

are monodromies of virtually regular coverings of 6 sheets and branching index 6 over the knot 5/3. In fact, any other monodromy of a virtually regular covering of 6 sheets and branching index 6, over the knot 5/3, is conjugate to one of them. (A virtually regular covering with $h = n$ is called a locally cyclic covering. This kind of branched covering was first considered by Kneser, and quoted by Seifert and Threlfall as a footnote [**15**].) Denote the virtually regular covering with monodromy ω_i by $p_i : M_i \to S^3$, $i = 1, 2, 3, 4$. Observe that

$$\begin{aligned} \omega_1 \circ u_\sharp &\cong \omega_1, & \omega_1 \circ v_\sharp &\cong \omega_1, \\ \omega_2 \circ u_\sharp &\cong \omega_2, & \omega_2 \circ v_\sharp &\cong \omega_2, \\ \omega_4 \circ u_\sharp &\cong \omega_3, & \omega_3 \circ v_\sharp &\cong \omega_3. \end{aligned}$$

Therefore u lifts to an isometry between $\overrightarrow{M_3}$ and $\overrightarrow{M_4}$, thus we have to consider as different examples only $\overrightarrow{M_1}$, $\overrightarrow{M_2}$ and $\overrightarrow{M_3}$. We see also that u lifts to an orientation-reversing homeomorphism of $\overrightarrow{M_1}$, and to an orientation-reversing homeomorphism of $\overrightarrow{M_2}$. Thus $\mathrm{CS}(\overrightarrow{M_1}) \equiv \mathrm{CS}(\overleftarrow{M_1}) \equiv -\mathrm{CS}(\overrightarrow{M_1}) \equiv 0 \pmod{\frac{1}{2}}$, and analogously $\mathrm{CS}(\overrightarrow{M_2}) \equiv 0 \pmod{\frac{1}{2}}$. This shows that $\overrightarrow{M_1}$ and $\overrightarrow{M_2}$ have the same Chern–Simons invariant $\mathrm{CS}(\overrightarrow{M_1}) \equiv \mathrm{CS}(\overrightarrow{M_2}) \equiv 0 \pmod{\frac{1}{2}}$. In what follows, we shall say that a manifold is *amphicheiral* if it admits an orientation-reversing homeomorphism. Thus $\overrightarrow{M_1}$ and $\overrightarrow{M_2}$ are amphicheiral. The volume of the three hyperbolic manifolds $\overrightarrow{M_i}$, $i = 1, 2, 3$, is the same:

$$\mathrm{Vol}(\overrightarrow{M_1}) = \mathrm{Vol}(\overrightarrow{M_2}) = \mathrm{Vol}(\overrightarrow{M_3}) = 6\,\mathrm{Vol}(\overrightarrow{S}^3, 5/3, 6) = 7.327725\ldots.$$

(The volumes of the hyperbolic orbifolds $\mathrm{Vol}(\overrightarrow{S}^3, 5/3, n)$ were obtained in [**6**].) Therefore $\overrightarrow{M_1}$ and $\overrightarrow{M_2}$ have the same volume and the same Chern–Simons invariant. However, these two manifolds are not homeomorphic. In fact, their first homology groups are different (see, for instance, [**3**]):

$$H_1(M_1) = C_{40} \oplus C_8 \quad \text{and} \quad H_1(M_2) = C_8 \oplus C_8,$$

where C_l is the cyclic group of l elements.

Let us compute the Chern–Simons invariant of $\overrightarrow{M_3}$ using Theorem 2.2. The canonical longitude l_c of the knot is a representative of the element $[l_c] = ba^{-1}b^{-1}aab^{-1}a^{-1}b$ in the group $G(5/3)$. In this case (as in all locally cyclic coverings), the preimage of the knot has only one component. To compute the Chern–Simons invariant of the hyperbolic manifold M_3, we have to consider the covering between tori $p_{3|\hat{T}} : \hat{T} \to T$, where T is the boundary of a tubular neighbourhood of the knot 5/3, and $\hat{T} = p_3^{-1}(T)$. The monodromy of this covering is given by

$$\omega_3(a) = (1\,2\,3\,4\,5\,6), \quad \omega_3([l_c]) = (1\,3\,5)(2\,4\,6).$$

Figure 2 shows the torus $\hat{T}$, and the homology basis $(\overrightarrow{\tilde{m}}, \overrightarrow{L})$. Vertical lines of $\hat{T}$ in Fig. 2 project onto the canonical longitude l_c, and horizontal lines project on the meridian m.

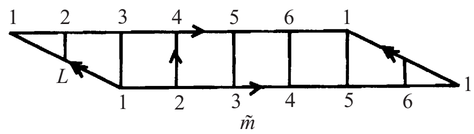

FIG. 2. The torus $\hat{T}$

The value of k for the covering p_3 ($p_3(\overrightarrow{L}) = \overrightarrow{l_c} + k\overrightarrow{m}$) can be obtained geometrically, directly from Fig. 1. An alternative algebraic method is the following. Observe that $\omega_3([l_c])$ and $\omega_3(a)$ always generate an abelian group. (This is because a and $[l_c]$ commute in $G(K)$ for any knot K.) In fact, for any locally cyclic covering of h sheets, $\omega([l_c])$ and $\omega(a)$ generate the cyclic group of h elements. Therefore $\omega([l_c])$ is a power of $\omega(a)$. The number n minus this power (mod n) is k: $\omega_3([l_c]) = (\omega_3(a))^{n-k}$. Then $k = 4$. Therefore

$$\mathrm{CS}(\overrightarrow{M_3}) \equiv 6\,\mathrm{CS}(\overrightarrow{S}^3, 5/3, 6) - \tfrac{2}{6} \equiv \tfrac{1}{6} \pmod{\tfrac{1}{2}}.$$

Therefore the Chern–Simons invariant distinguishes $\overrightarrow{M_3}$ from the manifolds with equal volume, $\overrightarrow{M_1}$ and $\overrightarrow{M_2}$. We can also deduce from this invariant that the manifold $\overrightarrow{M_3}$ is not amphicheiral. This is because $\mathrm{CS}(\overleftarrow{M_3}) \equiv -\mathrm{CS}(\overrightarrow{M_3}) \equiv \frac{1}{3} \pmod{\frac{1}{2}}$, and $\frac{1}{6} \not\equiv \frac{1}{3} \pmod{\frac{1}{2}}$, so $\mathrm{CS}(\overleftarrow{M_3}) \not\equiv \mathrm{CS}(\overrightarrow{M_3})$. This is not the case in the next example.

(2) *Virtually regular coverings of* 8 *sheets with branching index* 4. The representations $\omega_5, \omega_6 : G(5/3) \to S_8$, where S_8 is the permutation group of 8 elements, given by

$$\omega_5(a) = (1\,2\,3\,4)(5\,6\,7\,8), \quad \omega_5(b) = (1\,3\,5\,2)(4\,6\,8\,7),$$
$$\omega_6(a) = (1\,2\,3\,4)(5\,6\,7\,8), \quad \omega_6(b) = (1\,2\,5\,7)(3\,8\,4\,6),$$

are the monodromies of virtually regular coverings with $h = 8$ sheets and branching index $n = 4$. Denote them by $p_i : M_i \to S^3$, $i = 5, 6$. Observe that $\omega_5 \circ u_\sharp \cong \omega_6$. Therefore $\overrightarrow{M_5}$ is equal to $\overleftarrow{M_6}$, so we study only $\overrightarrow{M_5}$.

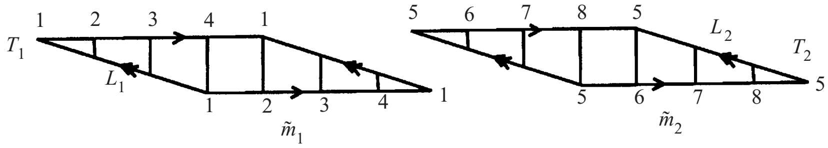

FIG. 3. The tori T_1 and T_2

Because $\omega_5([l_c]) = (1\,4\,3\,2)(5\,8\,7\,6)$, $p_5^{-1}(T)$ consists of two components: denote them by T_1 and T_2. Figure 3 shows these tori, and the homology basis $(\overrightarrow{\tilde{m}_i}, \overrightarrow{L_i})$ in each case. Computing as before, $k_1 = k_2 = 5$. Therefore

$$\mathrm{CS}(\overrightarrow{M_5}) \equiv 8\,\mathrm{CS}(\overrightarrow{S}^3, 5/3, 4) - \left(\tfrac{5}{8} + \tfrac{5}{8}\right) \equiv \tfrac{1}{4} \pmod{\tfrac{1}{2}}.$$

Since $\frac{1}{4} \equiv -\frac{1}{4} \pmod{\frac{1}{2}}$, the invariant CS (mod $\frac{1}{2}$) fails to show whether or not $\overrightarrow{M_5}$ is amphicheiral. In fact, $\overrightarrow{M_5}$ is *not* amphicheiral, and this is shown by the linking form of $\overrightarrow{M_5}$. Its first homology group is $H_1(M_5) \cong C_{24}$ (see [**3**]), and some calculations show that the linking form for $\overrightarrow{M_5}$, $\mathscr{L}_{\overrightarrow{M_5}} : C_{24} \times C_{24} \to \mathbb{Q}/\mathbb{Z}$, is given in any base g of C_{24} by $\mathscr{L}_{\overrightarrow{M_5}}(g, g) = \frac{5}{24}$. Since $\mathscr{L}_{\overleftarrow{M_5}}(g, g) = -\frac{5}{24}$, the linking forms $\mathscr{L}_{\overrightarrow{M_5}}$ and $\mathscr{L}_{\overleftarrow{M_5}}$ are not equivalent. This implies that the manifold $\overrightarrow{M_5}$ is not amphicheiral.

EXAMPLE 2. *Some virtually regular coverings of the* 2*-bridge knot* 7/3 *orbifolds.* Consider the 2-bridge knot 7/3 (5_2 in [**13**]). The group $G(7/3) = \pi_1(S^3 - (7/3))$ has the presentation $|a, b : abab^{-1}a^{-1}ba = bab^{-1}a^{-1}bab|$, where the generators a, b are represented by meridians. Here $[l_c] = bab^{-1}a^{-1}baaba^{-1}b^{-1}aba^{-1}a^{-1}a^{-1}a^{-1}$. The strong invertibility of the knot 7/3 defines an automorphism $v_\sharp$ on $G(7/3)$ given by $v_\sharp(a) = a^{-1}$, $v_\sharp(b) = b^{-1}$.

Virtually regular coverings of 5 *sheets.* The representation $v_1 : G(K_{7/3}) \to S_5$, where S_5 is the permutation group of 5 elements, given by $v_1(a) = v_1(b) = (1\,2\,3\,4\,5)$, is the monodromy of the 5-cyclic covering $p_1 : N_1 \to S^3$ branched over the knot $K_{7/3}$. The monodromy of the canonical longitude is $v_1([l_c]) = (1)(2)(3)(4)(5)$. The representation $v_2 : G(K_{7/3}) \to S_5$ given by $v_2(a) = (1\,2\,3\,4\,5)$, $v_2(b) = (1\,2\,4\,5\,3)$ is the monodromy of a locally cyclic covering $p_2 : N_2 \to S^3$ branched over the same knot $K_{7/3}$. The monodromy of the canonical longitude is $v_2([l_c]) = (1\,3\,5\,2\,4)$. Both coverings are virtually regular coverings of 5 sheets. Computations as before yield $k_1 = 0$ and $k_2 = 5$. Thus

$$\mathrm{Vol}(N_1) = \mathrm{Vol}(N_2) = 5\,\mathrm{Vol}(S^3, K_{7/3}, 5) = 8.6124\ldots,$$

$$\mathrm{CS}(N_1) \equiv 5\,\mathrm{CS}(S^3, K_{7/3}, 5) \equiv 0.0083333\ldots \pmod{\tfrac{1}{2}},$$

$$\mathrm{CS}(N_2) \equiv 5\,\mathrm{CS}(S^3, K_{7/3}, 5) - \tfrac{3}{10} \equiv 0.0083333\ldots + \tfrac{1}{5} \equiv 0.208333\ldots \pmod{\tfrac{1}{2}}.$$

Therefore N_1 and N_2 are topologically different.

Virtually regular coverings of 6 *sheets.* The representations $v_3, v_4 : G(K_{7/3}) \to S_6$ given by

$$v_3(a) = (1\,2\,3\,4\,5\,6), \qquad v_3(b) = (1\,2\,3\,4\,5\,6),$$

$$v_4(a) = (1\,2\,3\,4\,5\,6), \qquad v_4(b) = (1\,2\,4\,3\,6\,5),$$

are the monodromies of locally cyclic coverings of 6 sheets. (The monodromy v_3 corresponds to the 6-cyclic covering.) Denote them by $p_i : N_i \to S^3$, $i = 3, 4$. Because $v_3([l_c]) = (1)(2)(3)(4)(5)(6)$ and $v_4([l_c]) = (1\,3\,5)(2\,4\,6)$, $k_3 = 0$ and $k_4 = 4$. Therefore

$$\mathrm{Vol}(N_3) = \mathrm{Vol}(N_4) = 6\,\mathrm{Vol}(S^3, K_{7/3}, 6) = 12.2552\ldots,$$
$$\mathrm{CS}(N_3) \equiv 6\,\mathrm{CS}(S^3, K_{7/3}, 6) \equiv 0.30277\ldots \pmod{\tfrac{1}{2}},$$
$$\mathrm{CS}(N_4) \equiv 6\,\mathrm{CS}(S^3, K_{7/3}, 6) - \tfrac{4}{12} \equiv 0.30277\ldots + \tfrac{1}{6} \equiv 0.46944\ldots \pmod{\tfrac{1}{2}}.$$

Therefore N_3 and N_4 are topologically different.

Virtually regular coverings of 7 *sheets.* The representations $v_i : G(K_{7/3}) \to S_7$, $i = 5, 6, 7, 8, 9$, given by

$$\begin{aligned} v_5(a) &= (1\,2\,3\,4\,5\,6\,7), & v_5(b) &= (1\,2\,3\,4\,5\,6\,7),\\ v_6(a) &= (1\,2\,3\,4\,5\,6\,7), & v_6(b) &= (1\,2\,4\,3\,7\,6\,5),\\ v_7(a) &= (1\,2\,3\,4\,5\,6\,7), & v_7(b) &= (1\,2\,5\,4\,7\,6\,3),\\ v_8(a) &= (1\,2\,3\,4\,5\,6\,7), & v_8(b) &= (1\,2\,5\,4\,3\,7\,6),\\ v_9(a) &= (1\,2\,3\,4\,5\,6\,7), & v_9(b) &= (1\,2\,7\,4\,3\,6\,5), \end{aligned}$$

are the monodromies of locally cyclic coverings of 7 sheets. (The monodromy v_5 corresponds to the 7-cyclic covering.) Denote them by $p_i : \overrightarrow{N_i} \to \overrightarrow{S}^3$, $i = 5, 6, 7, 8, 9$. Observe that $v_6 \circ v_\sharp \cong v_8$ and $v_7 \circ v_\sharp \cong v_9$. Therefore $\overrightarrow{N_8} \cong \overrightarrow{N_6}$ and $\overrightarrow{N_9} \cong \overrightarrow{N_7}$. Thus we study only $\overrightarrow{N_5}$, $\overrightarrow{N_6}$ and $\overrightarrow{N_7}$. The images of the canonical longitude and the corresponding values of k_i are

$$\begin{aligned} v_5([l_c]) &= (1)(2)(3)(4)(5)(6)(7), & k_5 &= 0,\\ v_6([l_c]) &= (1\,4\,7\,3\,6\,2\,5), & k_6 &= 4,\\ v_7([l_c]) &= (1)(2)(3)(4)(5)(6)(7), & k_7 &= 0. \end{aligned}$$

Therefore

$$\mathrm{Vol}(N_5) = \mathrm{Vol}(N_6) = \mathrm{Vol}(N_7) = 7\,\mathrm{Vol}(S^3, K_{7/3}, 7) = 15.7081\ldots,$$
$$\mathrm{CS}(\overrightarrow{N_5}) \equiv \mathrm{CS}(\overrightarrow{N_7}) \equiv 7\,\mathrm{CS}(S^3, K_{7/3}, 7) \equiv 0.11441\ldots \pmod{\tfrac{1}{2}},$$
$$\mathrm{CS}(\overrightarrow{N_6}) \equiv 7\,\mathrm{CS}(S^3, K_{7/3}, 7) - \tfrac{4}{14} \equiv 0.11441\ldots + \tfrac{3}{14} \equiv 0.32869\ldots \pmod{\tfrac{1}{2}}.$$

We see that the Chern–Simons invariant distinguishes $\overrightarrow{N_6}$ from the manifolds with the same volume, $\overrightarrow{N_5}$ and $\overrightarrow{N_7}$. These two manifolds can be distinguished by the first homology group:

$$H_1(\overrightarrow{N_5}) = C_{13} \oplus C_{13}, \quad H_1(\overrightarrow{N_7}) = C_4 \oplus \mathbb{Z}.$$

REMARK. It is interesting to compare the homological invariants (homology groups and linking form) with the topological invariants, volume and Chern–Simons invariant. It transpires that volume is relatively weak in distinguishing manifolds that can be cut into hyperbolic polyhedra which are cut-and-paste equivalent. This occurs with branched coverings, with the same numbers of sheets, of the same knot. The Chern–Simons invariant is sensitive to the different ways of pasting together the faces of the defining polyhedron: it measures the total amount of twisting in the pasting process (the proof of Case 2 is in Theorem 3.5 of [**7**]). The symmetry of the knot limits this amount of twisting. Therefore CS is more effective in distinguishing manifolds via coverings when the knot has few symmetries. From the above examples, we can deduce that homological invariants are probably stronger than the volume and the Chern–Simons invariant, when working with branched coverings over the same knot.

We used the program GAP [**14**] to find the above monodromies. Volumes and Chern–Simons invariants of the orbifolds (S^3, K, n) are taken from [**8**].

References

1. S. S. Chern and J. Simons, 'Characteristic forms and geometric invariants', *Ann. of Math.* 99 (1974) 48–69.
2. M. Culler and P. B. Shalen, 'Varieties of group representations and splitting of 3-manifolds', *Ann. of Math.* 117 (1983) 109–146.
3. J. Hempel, 'The lattice of branched covers over the figure-eight knot', *Topology Appl.* 34 (1990) 183–201.
4. H. M. Hilden, M. T. Lozano and J. M. Montesinos, 'On knots that are universal', *Topology* 24 (1985) 499–509.
5. H. M. Hilden, M. T. Lozano and J. M. Montesinos-Amilibia, 'Universal 2-bridge knot and link orbifolds', *J. Knot Theory Ramifications* 2 (1993) 141–148.
6. H. M. Hilden, M. T. Lozano and J. M. Montesinos-Amilibia, 'On a remarkable polyhedron geometrizing the figure-eight knot cone manifolds', *J. Math. Sci. Univ. Tokyo* 2 (1995) 501–561.
7. H. M. Hilden, M. T. Lozano and J. M. Montesinos-Amilibia, 'On volumes and Chern–Simons invariants of geometric 3-manifolds', *J. Math. Sci. Univ. Tokyo* 3 (1996) 723–744.
8. H. M. Hilden, M. T. Lozano and J. M. Montesinos-Amilibia, 'Volumes and Chern–Simons invariants of cyclic coverings over rational knots', *Topology and Teichmüller spaces* (ed. Sadayoshi Kojima *et al*., World Scientific, Singapore, 1996) 31–55.
9. C. D. Hodgson, G. R. Meyerhoff and J. R. Weeks, 'Surgeries on the Whitehead link yield geometrically similar manifolds', *Topology '90* (ed. B. Apanasov, W. D. Neumann, A. W. Reid and L. Siebenmann, de Gruyter, Amsterdam, 1992) 195–206.
10. M. T. Lozano and C. Safont, 'Virtually regular coverings', *Proc. Amer. Math. Soc.* 106 (1989) 207–214.
11. R. Meyerhoff, 'Density of the Chern–Simons invariant for hyperbolic 3-manifolds', *Low-dimensional topology and Kleinian groups*, London Math. Soc. Lecture Note Ser. 112 (ed. D. B. A. Epstein, Cambridge University Press, 1987) 217–240.
12. R. Meyerhoff and D. Ruberman, 'Mutation and the η-invariant', *J. Differential Geom.* 31 (1990) 101–130.
13. D. Rolfsen, *Knots and links* (Publish or Perish, Berkeley, CA, 1976).
14. M. Schönert *et al*., *GAP*, version 3.4, 4th edn (D für Mathematik, RWTH Aachen, 1995).
15. H. Seifert and W. Threlfall, *Lehrbuch der Topologie* (Teubner, Leipzig, 1934).
16. T. Yoshida, 'The η-invariant of hyperbolic 3-manifolds', *Invent. Math.* 8 (1985) 473–514.

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