# Volumes for twist link cone-manifolds

D. Derevnin A. Mednykh M. Mulazzani

May 4, 2007

**Abstract**

It was shown by A. Reid that there are exactly four links whose trace field is a quadratic extension of the field of rational numbers. They are: the figure eight knot $4_1$, the Whitehead link $5_1^2$, the link $6_2^2$ and the link $6_3^2$. All of them are two-bridge links with slopes 5/2, 8/3, 10/3 and 12/5 respectively. Recently, the explicit volume formulae for hyperbolic cone–manifolds, whose underlying space is the 3-sphere and the singular set is the knot $4_1$ and the links $5_2^1$ and $6_2^2$, have been obtained by the second named author and his collaborators. In this paper we find explicitly the hyperbolic volume for cone-manifolds with the link $6_3^2$ as singular set. Trigonometrical identities (Tangent, Sine and Cosine Rules) between complex lengths of singular components and cone angles are obtained for an infinite family of two-bridge links containing $5_2^1$ and $6_3^2$.
*Mathematics Subject Classification 2000:* Primary 57M50; Secondary 57M25, 57M27.
*Keywords:* Hyperbolic orbifold, hyperbolic cone-manifold, complex length, Tangent Rule, Sine Rule, Cosine Rule, hyperbolic volume.

## 1 Introduction

Starting from Alexander's works, polynomial invariants became a very convenient instrument for knot investigation. Several kinds of such polynomials were discovered in the last twenty years. Among these, there are Jones-, Kaufmann-, HOMFLY-, A–polynomials and others ([Kauf], [CCGLS], [HLM2]). This relates the knot theory with algebra and algebraic geometry. Algebraic technique is used to find the most important geometrical characteristics of knots such as volume, length of shortest geodesics and others.

In particular, it was shown by A. Reid that there are exactly four links whose trace field is a quadratic extension of the field of rational numbers. They are: the figure eight knot $4_1$, the Whitehead link $5_1^2$, the link $6_2^2$ and the link $6_3^2$. All of them are two-bridge links with slopes 5/2, 8/3, 10/3 and 12/5 respectively.

The explicit volume formulae for hyperbolic cone–manifolds, whose underlying space is the 3-sphere and the singular set is the knot $4_1$ and the links $5_2^1$ and $6_2^2$, have been obtained in [MR2], [MV2] and [Me].

The aim of our paper is to find explicitly the hyperbolic volume for cone-manifolds with the link $6_3^2$ as singular set. In order to do this, we will introduce a family of hyperbolic cone–manifolds $T_p(\alpha, \beta)$, with the two-bridge links $T_p$, with slope $(4p + 4/2p + 1)$ as singular set, and $\alpha, \beta$ as cone angles.

Trigonometrical identities (Tangent, Sine and Cosine Rules) between complex lengths of singular components and cone angles for $T_p(\alpha, \beta)$ are obtained. Then the Schläfli formula applies to find explicit hyperbolic volumes for cone-manifolds $T_2(\alpha, \beta)$.

In the present paper links and knots are considered as singular subsets of the three-sphere endowed by Riemannian metric of negative constant curvature.

# 2 Trigonometrical identities for knots and links

## 2.1 Cone–manifolds, complex distances and lengths

We start with the definition of cone–manifold modelled in hyperbolic, spherical or Euclidian structure.

**Definition 2.1.1**. A 3–dimensional *hyperbolic cone–manifold* is a Riemannian 3–dimensional manifold of constant negative sectional curvature with cone-type singularity along simple closed geodesics. To each component of singular set we associate a real number $n \geq 1$ such that the cone-angle around the component is $\alpha = 2\pi/n$. The concept of the hyperbolic cone-manifold generalizes the hyperbolic manifold which appears in the partial case when all cone-angles are $2\pi$. The hyperbolic cone-manifold is also a generalization of the hyperbolic 3–orbifold which arises when all associated numbers $n$ are integers. Euclidean and spherical cone–manifolds are defined similarly.

In the present paper hyperbolic, spherical or Euclidean cone-manifolds $C$ are considered whose underlying space is the three-dimensional sphere and the singular set $\Sigma = \Sigma^1 \cup \Sigma^2 \cup \ldots \cup \Sigma^k$ is a link consisting of components $\Sigma^j = \Sigma_{\alpha_j}$, $j = 1, 2, \ldots, k$ with cone-angles $\alpha_1, \ldots, \alpha_k$ respectively.

Recall a few well-known facts from the hyperbolic geometry.

Let $\mathbb{H}^3 = \{(z, \xi) \in \mathbb{C} \times \mathbb{R} : \xi > 0\}$ be the upper half model of the 3 -dimensional hyperbolic space endowed by the Riemannian metric

$$ds^2 = \frac{dzd\overline{z} + d\xi^2}{\xi^2}.$$

We identify the group of orientation preserving isometries of $\mathbb{H}^3$ with the group $PSL(2, \mathbb{C})$ consisting of linear fractional transformations

$$A : z \in \mathbb{C} \to \frac{az + b}{cz + d}.$$

By the canonical procedure the linear transformation $A$ can be uniquely extended to the isometry of $\mathbb{H}^3$. We prefer to deal with the matrix $\widetilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ rather than the element $A \in PSL(2, \mathbb{C})$. The matrix $\widetilde{A}$ is uniquely determined by the element $A$ up to a sign. If there will be no confusions we shall use the same letter $A$ for both $A$ and $\widetilde{A}$.

Let $C$ be a hyperbolic cone–manifold with the singular set $\Sigma$. Then $C$ defines a nonsingular but incomplete hyperbolic manifold $N = C - \Sigma$. Denote by $\Phi$ the fundamental group of the manifold $N$.

The hyperbolic structure of $N$ defines, up to conjugation in $PSL(2, \mathbb{C})$, a holonomy homomorphism

$$\hat{h} : \Phi \to PSL(2, \mathbb{C}).$$

It is shown in [Zhou] that the holonomy homomorphism of a compact orientable cone-orbifold can be lifted to $SL(2, \mathbb{C})$. Denote by $h : \Phi \to SL(2, \mathbb{C})$ this lifting homomorphism. Chose an orientation on the link $\Sigma = \Sigma^1 \cup \Sigma^2 \cup \ldots \cup \Sigma^k$ and fix a meridian-longitude pair $\{m_j, l_j\}$ for each component $\Sigma_j = \Sigma_{\alpha_j}$. Then the matrices $M_j = h(m_j)$ and $L_j = h(l_j)$ satisfy the following properties:

$$\mathrm{tr}\,(M_j) = 2\cos(\alpha_j/2), \quad M_jL_j = L_jM_j,\ j = 1, 2, \ldots, k.$$

Recall some definitions and results from the book [Fench]. A matrix $l \in SL(2, \mathbf{C})$ satisfying $tr(l) = 0$ will be called a (normalized) *line matrix*. We have from definition $l^2 = -I$, where $I$ is the identity matrix. Hence any line matrix determines a half-turn about a line in $\mathbf{H}^3$, and this line determines matrix up to sign. According to [Fench, p.63] there exists a natural one-to-one correspondence between line matrices and oriented lines in $\mathbf{H}^3$. Hereby, if a line matrix $l$ determines an oriented line $[e, e']$ with and points $e$ and $e'$, then the line matrix $(-l)$ determines the line $[e', e]$. Moreover, if matrix $f \in SL(2, \mathbf{C})$ is considered as a motion of $\mathbf{H}^3$, then the matrix $lfl^{-1}$ determines the line $[f(e), f(e')]$.

Let $A$ and $B$ be oriented lines determined by line matrices $a$ and $b$. A complex number $\mu$ is called a *complex distance* from $A$ to $B$ if it real part $\Re\,\mu$ is the distance from $A$ to $B$, and it imaginary part $\Im\,\mu$ is the angle from $L$ to $M$ in the obvious sense. We get [Fench, p.68]

$$\cosh \mu = -\frac{1}{2}\, tr(ab). \tag{2.1}$$

From now on, all lines in this paper will be supposed to be oriented.

Now we consider a family of links which are generalization of the Whitehead link. The link $T_p$, $p \geq 0$, is the two-component link depicted in Figure ??, where $p$ is the number of half twist of one component. For this reason we will call them *twist links*. It is easy to see that $T_0$ is the torus link of type $(2, 4)$ and $T_1$ is the Whitehead link. All twists links are two-bridge links, in particular $T_p$ is the two-bridge link of type $(4p + 4, 2p + 1)$, for al $p$. They are all hyperbolic, except for $T_0$.

Denote by $T_p(\alpha, \beta)$ the cone-manifold whose underlying space is the 3-sphere and whose singular set consists of two components of the twist link $T_p$ with cone angles $\alpha = 2\pi/m$ and $\beta = 2\pi/n$ (see Fig.1). It follows from Thurston theorem that $T_p(\alpha, \beta)$ admits a hyperbolic structure for all sufficiently small $\alpha$ and $\beta$.

By the Kojima rigidity theorem [Kj] the hyperbolic structure is unique, up to isometry, if $0 \leq \alpha, \beta \leq \pi$.

In our paper we deal only with such range of angles.

Let us investigate the hyperbolic structure of the cone-manifold $T_p(\alpha, \beta)$ The singular set $\Sigma = \Sigma^1 \cup \Sigma^2$ of $T_p(\alpha, \beta)$ consists of two components $\Sigma^1 = \Sigma^1(\alpha)$ and $\Sigma^2 = \Sigma^2(\beta)$ with cone-angles $\alpha$ and $\beta$ respectively. $T_p(\alpha, \beta)$ defines a nonsingular but incomplete hyperbolic manifold $N = T_p(\alpha, \beta) - \Sigma$.

The fundamental group of the manifold $N$ has the following presentation

$$\Phi_p = \langle s, t \mid \; sl_p = l_p s \rangle,$$

where $s$ and $t$ are meridians of the components $\Sigma^1$ and $\Sigma^2$ respectively, and $l_p$ is a longitude of $\Sigma^1$. The expression of $l_p$ in term of $s$ and $t$ is

$$l_p = s^2[s^{-1}, t]^{p-2}[s^{-1}, t^{-1}]^{p-2}, \quad \text{if } \; p \; \text{ is even,}$$

$$l_p = s[t, s]^{p-3} tst[s^{-1}, t^{-1}]^{p-3}, \quad \text{if } \; p \; \text{ is odd.}$$

The hyperbolic structure of $N$ defines, up to conjugation in $PSL(2, \mathbf{C})$, a holonomy homomorphism

$$\hat{h} = \hat{h}_{\alpha,\beta} : \Phi_p \to PSL(2, \mathbf{C}).$$

The images $\hat{h}(s)$ and $\hat{h}(t)$ of $s$ and $t$ are rotations in $\mathbf{H}^3$ of angles $\alpha$ and $\beta$ respectively.

It is shown in [Zhou] that the holonomy homomorphism can be lifted to $SL(2, \mathbf{C})$ if all cone-angles are at most $\pi$. Denote by $h = h_{\alpha,\beta} : \Phi_p \to SL(2, \mathbf{C})$ this lifting homomorphism and set $\Gamma_{\alpha,\beta} = h_{\alpha,\beta}(\Phi_p)$. The group $\Gamma_{\alpha,\beta}$ is generated by the two matrices $S = h_{\alpha,\beta}(s)$ and $T = h_{\alpha,\beta}(t)$ with the following properties:

$$tr(S) = 2\cos\frac{\alpha}{2}, \quad tr(T) = 2\cos\frac{\beta}{2},$$

$$SL = LS, \quad L = h_{\alpha,\beta}(l_p). \tag{2.2}$$

Recall that a subgroup $G$ of $SL(2, \mathbf{C})$ is called *elementary* if it has a finite orbit in $\mathbf{H}^3 \cup \widehat{\mathbf{C}}$. We remark that the group (2.2) is non-elementary, is not conjugated to a subgroup of $SL(2, \mathbf{R})$ and is not necessary discrete [CCGLS].

Introduce the following definitions:

**Definition 2.1**. Let $M$ be an isometry of $\mathbf{H}^3$ different from identity. It has two fixed points $u$ and $v$ in $\widehat{\mathbf{C}}$ which may coincide. Consider the oriented axis $[u, v]$. A complex number $\delta(M)$ is call to be the *displacement* of $M$ if its real part is the distance which the axis is translated and its imaginary part is the angle through which the half–planes bounded by the axis are rotated.

The isometry $M$ without an orientation of its axis determines $\delta(M)$ only up to sign. By [Fench, p.46] for the isometry is given by matrix $M \in SL(2, \mathbf{C})$ we have

$$2\cosh\delta(M) = \mathrm{tr}(\mathrm{M}^2) = \mathrm{tr}^2(\mathrm{M}) - 2. \tag{2.3}$$

We remark that if $\delta(M) \neq 0$ then $M$ has two different fixed points, so it admits an axis determined by these points. The line matrix $l(M)$ of this axis is defined by

$$l(M) = \frac{M - M^{-1}}{2i \sinh \frac{\delta(M)}{2}} \tag{2.4}$$

(see [Fench]).

**Definition 2.2**. A *complex length* $\gamma_j$ of the singular component $\Sigma^j$ of the cone-manifold $C$ is defined as displacement of the isometry $L_j$ of $\mathbb{H}^3$ , where $L_j = h(l_j)$ is represented by the longitude $l_j$ of $\Sigma^j$.

Immediately from the definition we get [Fench, p.46]

$$2 \cosh \gamma_j = \mathrm{tr}\ (L_j^2) \tag{1.1.1}$$

We note [BZie, p.38] that the meridian-longitude pair $\{m_j,\, l_j\}$ of the oriented link is uniquely determined up to a common conjugating element of the group $\Phi$. Hence, the complex length $\gamma_j = l_j + i\, \varphi_j$ is uniquely determined up to a sign and (mod $2\pi i$) by the above definition.

We need two conventions to choose correctly real and imaginary parts of $\gamma_j$. The first convention is the following. Since $\Sigma^j$ does not shrink to a point, $l_j \neq 0$. Hence, we choose $\gamma_j$ in such a way that $l_j = \Re\, \gamma_j > 0$. The second convention is concerned with the imaginary part $\varphi_j = \Im\, \gamma_j$. We want to choose $\varphi_j$ such that the following identity holds

$$\cosh \frac{\gamma_j}{2} = -\frac{1}{2}\, \mathrm{tr}\, (L_j) \tag{1.1.2}$$

By virtue of identity $\mathrm{tr}\, (L_j)^2 - 2 = \mathrm{tr}\, (L_j^2)$ equality (1.1.1) is a consequence of (1.1.2). The converse, in general, is true only up to a sign. Under the second convention (1.1.1) and (1.1.2) are equivalent. The two above conventions lead to convenient analytic formulas for calculation of $\gamma_j$ and $l_j$. More precisely, there are simple relations between these numbers and eigenvalues of matrix $L_j$. Recall that $\det L_j = 1$. Since matrix $L_j$ is loxodromic it has two eigenvalues $f_j$ and $1/f_j$. We choose $f_j$ so that $|f_j| > 1$. The case $|f_j| = 1$ is impossible because in this case the matrix $L_j$ is elliptic and $l_j = 0$. Hence

$$f_j = -e^{\frac{\gamma_j}{2}},\ \ |f_j| = e^{\frac{l_j}{2}}. \tag{1.1.3}$$

## 2.2 Complex distance equation for two-bridge links

Recall that the fundamental group of a link $K$ is generated by two meridians if and only if $K$ is a two-bridge link [BZ]. The two-bridge link is hyperbolic if and only if its slope is different of $p/1$ and $p/(p-1)$.

**Proposition 1** *Let $\Phi = \langle s,t \rangle$ be the fundamental group a hyperbolic two-bridge link $K$ generated by two meridians. Denote by $\Gamma_{\alpha,\beta} = h_{\alpha,\beta}(\Phi)$ the image of $\Phi$ under holonomy homomorphism of a hyperbolic cone manifold $K(\alpha,\beta)$. Then, up to conjugation in $SL(2,\mathbb{C})$, the generators $S = h_{\alpha,\beta}(s)$ and $T = h_{\alpha,\beta}(t)$ of the group $\Gamma_{\alpha,\beta}$ can be chosen in such a way that*

$$S = \begin{pmatrix} \cos\frac{\alpha}{2} & i\,e^{\frac{\rho}{2}}\sin\frac{\alpha}{2} \\ i\,e^{-\frac{\rho}{2}}\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}, \quad T = \begin{pmatrix} \cos\frac{\beta}{2} & i\,e^{-\frac{\rho}{2}}\sin\frac{\beta}{2} \\ i\,e^{\frac{\rho}{2}}\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}, \qquad (2.12)$$

*where $\rho$ is a complex distance between axis of $S$ and $T$.*

**Proof.** After a suitable conjugation in the group $SL(2,\mathbb{C})$, one can assume that the axes of elliptic elements $S$ and $T$ are lines with endpoints $\{\pm e^{\frac{\rho}{2}}\}$ and $\{\pm e^{-\frac{\rho}{2}}\}$ respectively. Since $\mathrm{tr}\,(S) = 2\,\cos\frac{\alpha}{2}$ and $\mathrm{tr}\,(T) = 2\,\cos\frac{\beta}{2}$, the matrices $S$ and $T$ are given by the formulas (2.5) . Check that $\rho$ coincides with the complex distance $\rho(S,T)$ between oriented axes $[-e^{\frac{\rho}{2}}, e^{\frac{\rho}{2}}]$ and $[-e^{-\frac{\rho}{2}}, e^{-\frac{\rho}{2}}]$ of $S$ and $T$. The line matrices $l(S)$ and $l(T)$, corresponding to these axes can be obtained from (2.4).

Since $\delta(S) = i\,\alpha$ and $\delta(T) = i\,\beta$ we have $l(S) = \begin{pmatrix} 0 & -ie^{\frac{\rho}{2}} \\ -ie^{-\frac{\rho}{2}} & 0 \end{pmatrix}$ and $l(T) = \begin{pmatrix} 0 & -ie^{-\frac{\rho}{2}} \\ -ie^{\frac{\rho}{2}} & 0 \end{pmatrix}$, respectively. By [Fench, p.68] we get $\cosh\rho(S,T) = -\frac{1}{2}\,tr(l(S)l(T)) = \cosh\rho$. ■

The following two propositions can be obtained by direct calculation from the above statement.

**Proposition 2** *Let $\Phi_2 = \langle s,t : sl = ls,\, l = stst^{-1}s^{-1}tsts^{-1}t^{-1}st \rangle$ be the fundamental group of the two-bridge link $T_2$ with the slope $12/5$ and $\Gamma_{\alpha,\beta} = h_{\alpha,\beta}(\Phi)$ is the image of $\Phi_2$ under holonomy homomorphism of a hyperbolic cone manifold $T_2(\alpha,\beta)$ generated by $S = h_{\alpha,\beta}(s)$ and $T = h_{\alpha,\beta}(t)$. Denote by $\rho = \rho(S,T)$ the complex distance between axes of $S$ and $T$. Then $u = \cosh\rho$ is a non-real root of the complex distance equation*

$$4u^3 - 4ABu^2 + (3A^2B^2 + 3A^2 + 3B^2 - 1)u - AB(A^2B^2 + A^2 + B^2 - 3) = 0. \quad (2.13)$$

*where* $A = \cot\frac{\alpha}{2}$ *and* $B = \cot\frac{\beta}{2}$.

**Proof.** Denote by $L = STST^{-1}S^{-1}TSTS^{-1}T^{-1}ST$ the image of the longitude $l$ under holonomy homomorphism $h = h_{\alpha,\beta} : \Phi_2 \to SL(2, \mathbf{C})$. Then we have $SL = LS$.

Let $a$ be a line matrix corresponding to common normal to axes of $S$ and $T$. If matrices $S$ and $T$ are represented in the form (2.12) then one can take $a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. It is not difficult to verify that $aSa^{-1} = S^{-1}$ and $aTa^{-1} = T^{-1}$.

To complete the proof, we need the following lemma, which gives simple criteria for matrices $S$ and $L$ to be permutable.

**Lemma 3** *The following conditions are equivalent: (i)* $SL = LS$; *(ii)* $aLa^{-1} = L^{-1}$; *(iii)* $\mathrm{tr}\,(aL) = 0$.

**Proof.** In the first we show that (i) and (ii) are equivalent. Indeed, since $L = STS^{-1}T^{-1}ST^{-1}S^{-1}T$ we get $aLa^{-1} = S^{-1}T^{-1}STS^{-1}TST^{-1} = S^{-1}L^{-1}S$. Hence (ii) holds if and only if $S$ and $L^{-1}$ are permutable. The last property is equivalent to (i). Because of $a^2 = -I$ the condition (ii) can be rewritten in the form $aLaL = -I$; that is equivalent to (iii). ∎

By this lemma and direct calculations we have

$$\mathrm{tr}\,(aL) = \frac{??\sinh\rho}{??(1+A^2)^2(1+B^2)^2}\cdot$$

$$\cdot(4u^3 - 4ABu^2 + (3A^2B^2 + 3A^2 + 3B^2 - 1)u - AB(A^2B^2 + A^2 + B^2 - 3)) = 0,$$

where $u = \cosh\rho$.

Now we have to show that $u$ is a non-real root of (2.13). Since $\Gamma_{\alpha,\beta}$ is the holonomy group of a hyperbolic cone–manifold, it is non-elementary and is not conjugated to a subgroup of $SL(2, \mathbf{R})$ [CCGLS].

If $\sinh\rho = 0$ then the axes $S$ and $T$ coincide, and the group $\Gamma_{\alpha,\beta}$ is elementary. Consequently, $u = \cosh\rho$ is a root of equation (2.13).

Suppose that $u = \cosh\rho$ is a real root. Let

$$R(a, b, c, d) = \frac{(c-a)(d-b)}{(c-b)(d-a)}$$

be the cross ratio of four points. Then $R(-e^{\frac{\rho}{2}}, e^{\frac{\rho}{2}}, -e^{-\frac{\rho}{2}}, e^{-\frac{\rho}{2}}) = \frac{\cosh\rho - 1}{\cosh\rho + 1} \in \mathbb{R}\cup\{\infty\}$. We have that the axes $[-e^{\frac{\rho}{2}}, e^{\frac{\rho}{2}}]$ and $[-e^{-\frac{\rho}{2}}, e^{-\frac{\rho}{2}}]$ of $S$ and $T$ lie in a common plane. If the axes intersect then the group $\Gamma_{\alpha,\beta} = \langle S, T\rangle$ has a fixed point and is elementary. If they do not intersect, $\Gamma_{\alpha,\beta}$ is conjugated to a subgroup of $SL(2,\mathbb{R})$.

Therefore, we have shown that $u$ is a non-real root of (2.13) and the proof of Proposition 2 is completed. ∎

The next proposition can be proved by similar arguments.

**Proposition 4** *Let $\Phi_3 = \langle s, t : sl = ls,\ l = stst^{-1}s^{-1}tst^{-1}s^{-1}t^{-1}sts^{-1}t^{-1}st\rangle$ be the fundamental group of the two-bridge link $T_3$ with the slope $16/7$ and $\Gamma_{\alpha,\beta} = h_{\alpha,\beta}(\Phi)$ is the image of $\Phi_2$ under holonomy homomorphism of a hyperbolic cone manifold $T_3(\alpha,\beta)$ generated by $S = h_{\alpha,\beta}(s)$ and $T = h_{\alpha,\beta}(t)$. Denote by $\rho = \rho(S,T)$ the complex distance between axes of $S$ and $T$. Then $u = \cosh\rho$ is a non-real root of the complex distance equation*

$$0 = 8u^5 - 8ABu^4 + 8(A^2B^2 + A^2 + B^2 - 1)u^3 - 4AB(A^2B^2 + A^2 + B^2 - 3)u^2 +$$

$$(A^4B^4 + 2A^4B^2 + 2A^2B^4 - 4A^2B^2 + A^4 + B^4 - 6A^2 - 6B^2 + 1)u + 4AB(A^2B^2 + A^2 + B^2 - 1),$$

*where $A = \cot\frac{\alpha}{2}$ and $B = \cot\frac{\beta}{2}$.*

## 2.3 Tangent, Sine and Cosine rules

If we set $z = \mathrm{tr}\,(S^{-1}T)$, then from presentation in Proposition 1 we have

$$z = 2(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - u\sin\frac{\alpha}{2}\sin\frac{\beta}{2})$$

where $u = \cosh\rho$.

In several papers ([CCGLS], citeHLM2 and others), devoted to $PSL(2,\mathbf{C})$ representation of two-generators groups, the parameter $z$ is considered as a main one.

In general, the algebraic equation for $u$ (as well as for $z$) is very complicated, even for twist links. In spite of this since $u = \cosh(\rho)$ has a very clear geometric sense, we are able to produce some general result for twist links without calculation of $u$.

We start with a preliminary result.

**Proposition 5** *Let $T_p(\alpha,\beta)$ be a hyperbolic twist link cone–manifold. Denote by $S = h_{\alpha,\beta}(s)$ and $T = h_{\alpha,\beta}(t)$ the images of the generators of the group $\Phi_p = \langle s, t \mid sl_p = l_ps\rangle$ under the holonomy homomorphism*

*$h_{\alpha,\beta} : \Phi_p \to SL(2,\mathbf{C})$. Set $u = \cosh\rho$, where $\rho$ is the complex distance between axes $S$ and $T$, chosen thus that $\Im u > 0$. Moreover, denote by $\gamma_\alpha$ and $\gamma_\beta$ the complex lengths of the singular components of $T_p(\alpha,\beta)$ with cone-angles $\alpha$ and $\beta$ respectively. Then*

$$u = i\,\cot\frac{\alpha}{2}\coth\frac{\gamma_\beta}{4} = i\,\cot\frac{\beta}{2}\coth\frac{\gamma_\alpha}{4}. \tag{2.29}$$

**Proof.** To prove the statement we need to calculate the complex distance between axes of elliptic elements $S$ and $T$ in two ways. Denote by $L = L_S$ and $V = L_T$ the longitudes of cone-manifold of $T_p(\alpha,\beta)$ represented by singular components with cone angles $\alpha$ and $\beta$ respectively.

First of all we fix an orientation on the axes $S$ and $T$ by the following line matrices

$$l(S) = \frac{S - S^{-1}}{2\,i\,\sinh\frac{i\,\alpha}{2}}, \qquad l(T) = \frac{T - T^{-1}}{2\,i\,\sinh\frac{i\,\beta}{2}}.$$

Then the complex distance $\rho(S,T)$ between oriented axes of $S$ and $T$ is defined by (2.1):

$$\cosh\rho(S,T) = -\frac{1}{2}\mathrm{tr}\,(l(S)l(T)) = \cosh\rho(T,S).$$

Using (2.4) we define the line matrices for $L_S$ and $L_T$ as

$$l(L_S) = \frac{L_S - L_S^{-1}}{2i\sinh\frac{\gamma_\alpha}{2}}, \qquad l(L_T) = \frac{L_T - L_T^{-1}}{2i\sinh\frac{\gamma_\beta}{2}}.$$

To continue the proof, we need two lemmas:

**Lemma 6** *For every $S, T$ we have $l(S) = -l(L_S)$ and $l(T) = -l(L_T)$*

**Proof.** Up to conjugation in $SL(2,\mathbf{C})$, we can assume that $S$ is given by

$$S = \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{pmatrix}.$$

Then, since $L_S$ is a loxodromic element, with displacement $\gamma_\alpha$, permutable with $S$, we have

$$L_S = \begin{pmatrix} \pm e^{\frac{\gamma_\alpha}{2}} & 0 \\ 0 & \pm e^{-\frac{\gamma_\alpha}{2}} \end{pmatrix}$$

By convention (see formula (2.6)) we have

$$\mathrm{tr}\,(L_S) = -2\cosh\frac{\gamma_\alpha}{2}.$$

Hence

$$L_S = \begin{pmatrix} -e^{\frac{\gamma_\alpha}{2}} & 0 \\ 0 & -e^{-\frac{\gamma_\alpha}{2}} \end{pmatrix}$$

and we obtain

$$l(L_S) = \frac{L_S - L_S^{-1}}{2i\sinh\frac{\gamma_\alpha}{2}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and

$$l(S) = \frac{S - S^{-1}}{2\,i\sinh\frac{i\,\alpha}{2}} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

■

**Lemma 7** *For every* $S, T$ *we have* $\mathrm{tr}\,(S) = \mathrm{tr}\,(S^{-1}L_T)$ *and* $\mathrm{tr}\,(T) = \mathrm{tr}\,(T^{-1}L_S)$.

**Proof.** For the proof of this lemma we need only two facts — the existence of automorphism $\theta : S \to S$, $T \to T^{-1}L_S$, $L_S \to L_S$, induced by order two rotation of $T_p(\alpha,\beta)$ along the longitude $L_S$ and Kojima's Rigidity theorem for cone-manifolds. Since $T_p(\alpha,\beta)$ is hyperbolic cone-manifold with cone-angles at most $\pi$, by [Kj] there exists an element $D \in SL(2,\mathbf{C})$ such that

$$DSD^{-1} = S, \quad DTD^{-1} = T^{-1}L_S, \quad DL_SD^{-1} = L_S.$$

Hence $T$ and $T^{-1}L_S$ are conjugate and we have the second statement of lemma. The first can be obtained in a similar way. ■

To complete the proof of Proposition 5, we note that $\mathrm{tr}\,(XY) = \mathrm{tr}\,(X)\mathrm{tr}\,(Y) - \mathrm{tr}\,(X^{-1}Y)$, $\mathrm{tr}\,(X^{-1}) = \mathrm{tr}\,(X)$ and $\mathrm{tr}\,(XY) = \mathrm{tr}\,(X^{-1}Y^{-1})$ holds for all $X, Y \in SL(2,\mathbf{C})$. By Lemma 6, Lemma 7 and formulae $\mathrm{tr}\,(S) = 2\cos\frac{\alpha}{2}$, $\mathrm{tr}\,(L_S) = -2\cosh\frac{\gamma_\alpha}{2}$, we have

$$\cosh\rho(S,T) = -\frac{1}{2}\mathrm{tr}\,(l(S)l(T)) = \frac{1}{2}\mathrm{tr}\,(l(S)l(L_T)) =$$

$$= \frac{1}{2}\mathrm{tr}\left(\frac{S - S^{-1}}{2\sin\frac{\alpha}{2}}\,\frac{L_T - L_T^{-1}}{2i\sinh\frac{\gamma_\beta}{2}}\right) = \frac{\mathrm{tr}\,(SL_T - S^{-1}L_T - SL_T^{-1} + S^{-1}L_T^{-1})}{8i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}} =$$

$$= \frac{2(\mathrm{tr}\,(SL_T) - \mathrm{tr}\,(S^{-1}L_T))}{8i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}} = \frac{\mathrm{tr}\,(S)\mathrm{tr}\,(L_T) - 2\mathrm{tr}\,(S^{-1}L_T)}{4i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}} =$$

$$= \frac{\mathrm{tr}\,(S)\mathrm{tr}\,(L_T) - 2\mathrm{tr}\,(S)}{4i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}} = \frac{\mathrm{tr}\,(S)(2 - \mathrm{tr}\,(L_T))}{-4i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}}$$

$$= \frac{2\cos\frac{\alpha}{2}(2 + 2\cosh\frac{\gamma_\beta}{2})}{-4i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}} = i\cot\frac{\alpha}{2}\coth\frac{\gamma_\beta}{4}.$$

■

As immediate consequence of previous proposition, we have the following results.

**Theorem 8** *(The Tangent Rule) Suppose that cone–manifold $T_p(\alpha,\beta)$ is hyperbolic. Denote by $\gamma_\alpha$ and $\gamma_\beta$ complex lengths of the singular geodesics of $W(\alpha,\beta)$ with cone angles $\alpha$ and $\beta$ respectively. Then*

$$\frac{\tanh\frac{\gamma_\alpha}{4}}{\tanh\frac{\gamma_\beta}{4}} = \frac{\tan\frac{\alpha}{2}}{\tan\frac{\beta}{2}}.$$

The following two theorems are consequences of the Tangent Rule.

**Theorem 9** *(The Sine Rule) Let $\gamma_\alpha = l_\alpha + i\,\varphi_\alpha$ (resp.$\gamma_\beta$) be a complex length of the singular geodesic of a hyperbolic cone-manifold $T_p(\alpha,\beta)$ with cone angle $\alpha$ (resp. $\beta$). Then*

$$\frac{\sin\frac{\varphi_\alpha}{2}}{\sinh\frac{l_\alpha}{2}} = \frac{\sin\frac{\varphi_\beta}{2}}{\sinh\frac{l_\beta}{2}}.$$

**Proof.** By the Tangent Rule we have

$$\frac{\tanh\frac{\gamma_\alpha}{4}}{A} = \frac{\tanh\frac{\gamma_\beta}{4}}{B},$$

where $A = \tan\frac{\alpha}{2}$ and $B = \tan\frac{\beta}{2}$ are real numbers. Hence

$$\frac{\Re(\tanh\frac{\gamma_\alpha}{4})}{A} = \frac{\Re(\tanh\frac{\gamma_\beta}{4})}{B},$$

and

$$\frac{\Im(\tanh\frac{\gamma_\alpha}{4})}{A} = \frac{\Im(\tanh\frac{\gamma_\beta}{4})}{B}.$$

By dividing one equation by the other we obtain

$$\frac{\Re(\tanh\frac{\gamma_\alpha}{4})}{\Im(\tanh\frac{\gamma_\alpha}{4})}=\frac{\Re(\tanh\frac{\gamma_\beta}{4})}{\Im(\tanh\frac{\gamma_\beta}{4})}.$$

By direct calculations we have

$$\Re(\tanh\frac{\gamma_\alpha}{4})=\frac{1}{2}(\tanh\frac{\gamma_\alpha}{4}+\tanh\frac{\bar{\gamma}_\alpha}{4})=\frac{\sinh\frac{l_\alpha}{2}}{\cosh\frac{l_\alpha}{2}-\cos\frac{\varphi_\alpha}{2}}$$

and

$$\Im(\tanh\frac{\gamma_\alpha}{4})=\frac{1}{2}(\tanh\frac{\gamma_\alpha}{4}-\tanh\frac{\bar{\gamma}_\alpha}{4})=\frac{\sin\frac{\varphi_\alpha}{2}}{\cosh\frac{l_\alpha}{2}-\cos\frac{\varphi_\alpha}{2}}.$$

Since $l_\alpha>0$, we have $\cosh\frac{l_\alpha}{2}>1$. Therefore $\cosh\frac{l_\alpha}{2}-\cos\frac{\varphi_\alpha}{2}>0$ and the result follows. ∎

**Theorem 10** *(The Cosine Rule) Let $\gamma_\alpha=l_\alpha+i\,\varphi_\alpha$ (resp.$\gamma_\beta$) be a complex length of the singular geodesic of a hyperbolic cone-manifold $T_p(\alpha,\beta)$ with cone angle $\alpha$ (resp. $\beta$). Then*

$$\frac{\cos\frac{\varphi_\alpha}{2}\cosh\frac{l_\beta}{2}-\cos\frac{\varphi_\beta}{2}\cosh\frac{l_\alpha}{2}}{\cosh\frac{l_\alpha}{2}\cosh\frac{l_\beta}{2}-\cos\frac{\varphi_\alpha}{2}\cos\frac{\varphi_\beta}{2}}=\frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}.$$

**Proof.** By the Tangent Rule

$$\frac{\tanh\frac{\gamma_\alpha}{4}\tanh\frac{\bar{\gamma}_\alpha}{4}}{A^2}=\frac{\tanh\frac{\gamma_\beta}{4}\tanh\frac{\bar{\gamma}_\beta}{4}}{B^2},$$

where $A=\tan\frac{\alpha}{2}$ and $B=\tan\frac{\beta}{2}$. Hence

$$\frac{1+\cos\alpha}{1-\cos\alpha}\,\frac{\cosh\frac{l_\alpha}{2}-\cos\frac{\varphi_\alpha}{2}}{\cosh\frac{l_\alpha}{2}+\cos\frac{\varphi_\alpha}{2}}=\frac{1+\cos\beta}{1-\cos\beta}\,\frac{\cosh\frac{l_\beta}{2}-\cos\frac{\varphi_\beta}{2}}{\cosh\frac{l_\beta}{2}+\cos\frac{\varphi_\beta}{2}}.$$

Set $p=\cos\alpha$, $q=\cos\beta$, $a=\frac{\cos\frac{\varphi_\alpha}{2}}{\cosh\frac{l_\alpha}{2}}$, $b=\frac{\cos\frac{\varphi_\beta}{2}}{\cosh\frac{l_\beta}{2}}$ and rewrite the above equation in the form

$$\frac{1+p}{1-p}\,\frac{1-a}{1+a}=\frac{1+q}{1-q}\,\frac{1-b}{1+b},$$

or, equivalently, as

$$\log \frac{1+p}{1-p} + \log \frac{1-a}{1+a} = \log \frac{1+q}{1-q} + \log \frac{1-b}{1+b}.$$

Since $\operatorname{arctanh} p = \frac{1}{2} \log \frac{1+p}{1-p}$ we have

$$\operatorname{arctanh} p - \operatorname{arctanh} a = \operatorname{arctanh} q - \operatorname{arctanh} b.$$

and

$$\operatorname{arctanh} p - \operatorname{arctanh} q = \operatorname{arctanh} a - \operatorname{arctanh} b.$$

Hence

$$\frac{p-q}{1-pq} = \frac{a-b}{1-ab}$$

and after the putting $a, b, p, q$ in the obtained formula we are done. ■

We remark that, in the case of Whitehead link cone–manifolds, Tangent and Sine rules are obtained in [M].

# 3 Explicit volume calculation for twist link cone–manifolds

## 3.1 The Schläfli formula

In this section we will obtain explicit formulas for volume of some special cone–manifolds in the hyperbolic and spherical geometries. In the case of complete hyperbolic structure on the simplest knot and link complements such formulas in terms of Lobachevsky function are well-known and widely represented in [T]. In general situation, a hyperbolic cone–manifold can be obtained by completion of non-complete hyperbolic structure on a suitable knot or link complement. If the cone–manifold is compact explicit formulas are know just in a few cases [Hds] , [HLM3], [MV2], []. In all these cases the starting point for the volume calculation is the Schläfli formula (see, for example [Hds] )

**Theorem 11** *(The Schläfli volume formula) Suppose that $C_t$ is a smooth 1–parameter family of (curvature $K$) cone–manifold structures on a $n$-manifold,* *with singular locus $\Sigma$ of a fixed topological type. Then the derivative of volume of $C_t$ satisfies*

$$(n-1)KdV(C_t) = \sum_{\sigma} V_{n-2}(\sigma) d\theta(\sigma)$$

*where the sum is over all components $\sigma$ of the singular locus $\Sigma$, and $\theta(\sigma)$ is the cone angle along $\sigma$.*

In the present paper we will deal mostly with three-dimensional cone–manifold structures of negative constant curvature $K = -1$, or positive constant curvature $K = 1$. The Schläfli formula in this case reduces to

$$KdV = \frac{1}{2} \sum_{i} l_{\theta_i} d\theta_i, \qquad (2.1.1)$$

where the sum is taken over all components of the singular set $\Sigma$ with lengths $l_{\theta_i}$ and cone angles $\theta_i$.

Our aim is to obtain the volume formulas for twist link hyperbolic cone–manifolds $T_2(\alpha,\beta)$. We note that formula for $T_1(\alpha,\beta)$ were obtained earlier in [Me1] and [MV2].

Thus, we can prove the following:

**Proposition 12** *Suppose that the cone–manifold $T_2(\alpha,\beta)$ is hyperbolic. Let $l_\alpha$ and $l_\beta$ be the real lengths of the singular geodesics of $T_2(\alpha,\beta)$, with cone angles $\alpha$ and $\beta$ respectively. Then*

$$l_\alpha = 2i \text{ arctan} \frac{A}{\zeta} - 2i \text{ arctan} \frac{A}{\overline{\zeta}}, \qquad (12.1)$$

$$l_\beta = 2i \text{ arctan} \frac{B}{\zeta} - 2i \text{ arctan} \frac{B}{\overline{\zeta}}, \qquad (12.2)$$

*where $\zeta$, $\Im(\zeta) > 0$ is a root of the equation*

$$4(z^2 + A^2)(z^2 + B^2) = (1 + A^2)(1 + B^2)(z - z^2)^2, \qquad (12.3)$$

*with $A = \cot\frac{\alpha}{2}$ and $B = \cot\frac{\beta}{2}$.*

**Proof.** By Proposition 5 we have

$$i\, B \coth \frac{\gamma_\alpha}{4} = i\, A \coth \frac{\gamma_\beta}{4} = u, \qquad (1.2.1),$$

where $u = \cosh\rho$, and $\rho$ is a complex distance between axes $S$ and $T$ chosen thus that $\Im u > 0$, $A = \cot\frac{\alpha}{2}$, $B = \cot\frac{\beta}{2}$. By Proposition 1 $u$ is a root of the cubic equation

$$4u^3 - 4ABu^2 + (3A^2B^2 + 3A^2 + 3B^2 - 1)u - AB(A^2B^2 + A^2 + B^2 - 3) = 0.$$

From (1.2.1), for a suitable choice of analytical branches

$$l_\alpha = \frac{\gamma_\alpha}{2} + \frac{\overline{\gamma}_\alpha}{2} = 2i\ \arctan\frac{\overline{u}}{B} - 2i\ \arctan\frac{u}{B} = 2i\ \arctan\frac{A}{\zeta} - 2i\ \arctan\frac{A}{\overline{\zeta}},$$

where $\zeta = AB/\overline{u}, \Im(\zeta) > 0$ satisfy the following equation

$$Q(z) = (A^2B^2{+}A^2{+}B^2{-}3)z^3{-}(3A^2B^2{+}3A^2{+}3B^2{-}1)z^2{+}4A^2B^2z{-}4A^2B^2 = 0.$$

To finish the proof we note that

$$(z+1)Q(z) = -4(z^2 + A^2)(z^2 + B^2) + (1 + A^2)(1 + B^2)(z - z^2)^2.$$

■

In the next section we will apply this result to calculate the volume of $T_2(\alpha, \beta)$ via Schläfli formula.

We remark that formulae (12.1) and (12,2), as a consequence of Tangent Rule, also hold for all twist links $T_p$, with $\zeta = AB/\bar{u}$, where $u = \cosh\rho$.

For example, an algebraic equation for $\zeta$ in the case of twist link $T_3$ can be easily obtained from Proposition 4. But in this case the equation became too complicated and Schläfli formula can not be applied in explicit way.

## 3.2 Volume of twist link cone–manifolds

The case of Whitehead link cone manifolds $T_1(\alpha, \beta)$ has already been solved (see [Me1] and [MV2]).

**Theorem 13** *[Me1, MV2] Let $T_1(\alpha, \beta)$ be a hyperbolic Whitehead link cone–manifold with cone angles $\alpha$ and $\beta$. Then the volume of $T_1(\alpha, \beta)$ is given by the formula*

$$\mathrm{Vol}\, T_1(\alpha, \beta) = i\int_{\zeta_1}^{\zeta_2} \log\left[\frac{2(\zeta^2 + A^2)(\zeta^2 + B^2)}{(1 + A^2)(1 + B^2)(\zeta^2 - \zeta^3)}\right]\frac{d\zeta}{\zeta^2 - 1}.$$

*where $A = \cot\frac{\alpha}{2}$, $B = \cot\frac{\beta}{2}, \zeta_1 = \overline{z}, \zeta_2 = z$ and $z$ is a non-real root, with $\Im(z) > 0$, of the equation*

$$2(z^2 + A^2)(z^2 + B^2) = (1 + A^2)(1 + B^2)(z^2 - z^3).$$

The case of $T_2(\alpha,\beta)$ is discussed below.

**Theorem 14** *Let $T_2(\alpha,\beta)$ be a hyperbolic twist link cone–manifold with cone angles $\alpha$ and $\beta$. Then the volume of $T_2(\alpha,\beta)$ is given by the formula*

$$\operatorname{Vol} T_2(\alpha,\beta) = i\int_{\zeta_1}^{\zeta_2} \log\left[\frac{4(\zeta^2+A^2)(\zeta^2+B^2)}{(1+A^2)(1+B^2)(\zeta-\zeta^2)^2}\right]\frac{d\zeta}{\zeta^2-1}.$$

*where $A=\cot\frac{\alpha}{2}$, $B=\cot\frac{\beta}{2}, \zeta_1=\overline{z}, \zeta_2=z$ and $z$ is a non-real root, with $\Im(z)>0$, of the equation*

$$4(z^2+A^2)(z^2+B^2)=(1+A^2)(1+B^2)(z-z^2)^2.$$

**Proof.** Denote by $V=Vol\,T_3(\alpha,\beta)$ the hyperbolic volume of $T_3(\alpha,\beta)$. Then by virtue of the Schlänfli formula we have

$$\frac{\partial V}{\partial\alpha}=-\frac{l_\alpha}{2},\ \ \frac{\partial V}{\partial\beta}=-\frac{l_\beta}{2}, \tag{2.2.4}$$

where $l_\alpha$ and $l_\alpha$ are lengths of singular geodesics corresponding to cone angles $\alpha$ and $\beta$ respectively.

We note that the geometrical limit $T_2(\pi,0)$ of the cone-manifold $T_2(\alpha,0)$ as $\alpha\to\pi-0$ is not hyperbolic, since its 2-fold covering branched over the $\pi$-component is the complement of the two-bridge link of type $(6,1)$, which is toric. Moreover, $T_2(\pi,0)$ does not contain 2-dimensional suborbifolds of type $S^2(\pi,\pi,\pi)$. Hence, by Theorem 7.1.2 of [Kj], we have

$$V\to 0 \ \text{ as }\ \alpha\to\pi \ \text{ and }\ \beta\to 0. \tag{2.2.5}$$

We set $W=\int_{\zeta_1}^{\zeta_2}F(\zeta,A,B)d\zeta$, where

$$F(\zeta,A,B)=\frac{i}{\zeta^2-1}\log\frac{4(\zeta^2+A^2)(\zeta^2+B^2)}{(1+A^2)(1+B^2)(\zeta-\zeta^2)^2}.$$

and show that $W$ satisfies conditions (2.2.4) and (2.2.5). Then $W=V$ and the theorem is proven.

By the Leibnitz formula we get

$$\frac{\partial W}{\partial\alpha}=F(\zeta_2,A,B)\frac{\partial\zeta_2}{\partial\alpha}-F(\zeta_1,A,B)\frac{\partial\zeta_1}{\partial\alpha}+\int_{\zeta_1}^{\zeta_2}\frac{\partial F(\zeta,A,B)}{\partial A}\frac{\partial A}{\partial\alpha}\,d\zeta \tag{2.2.6}$$

We note that $F(\zeta_1, A, B) = F(\zeta_2, A, B) = 0$ if $\zeta_1, \zeta_2, A$ and $B$ are the same as in the statement of the theorem. Moreover, since $\alpha = 2 \operatorname{arccot} \mathrm{A}$ we have $\frac{\partial A}{\partial \alpha} = -\frac{1+A^2}{2}$ and

$$\frac{\partial F(\zeta, A, B)}{\partial A} \frac{\partial A}{\partial \alpha} = \frac{i}{\zeta^2+1}.$$

Hence, by Proposition 1.2.3 we obtain from (2.2.6)

$$\frac{\partial W}{\partial \alpha} = i \int_{\zeta_1}^{\zeta_2} \frac{d\zeta}{\zeta^2+1} = i \text{ arctan} \frac{A}{\zeta_2} - i \text{ arctan} \frac{A}{\zeta_1} = -\frac{l_\alpha}{2}.$$

The equation $\frac{\partial W}{\partial \beta} = -\frac{l_\beta}{2}$ can be obtained in the same way. The boundary condition (2.2.5) for the function $W$ follows from the integral formula. ■

## 3.3 Particular cases and examples

1. Case $\alpha = \beta$.

   In this case the equation $Q(z) = ...$ splits into two quadratic equations: $(1+A^2)(z-z^2)+2(z^2+A^2) = 0$ and $(1+A^2)(z-z^2)-2(z^2+A^2) = 0$. The first one has two real roots $z = -1$ and $z = 2A^2/(A^2-1)$. The second has two non-real roots

   $$z_{1,2} = \frac{1+A^2 \pm \sqrt{1-22A^2-7A^4}}{2(3+A^2)}.$$

   By [HLM4], $\Delta = 1-22A^2-7A^4$ is $< 0$ in the hyperbolic case, $= 0$ in the Euclidean case and $> 0$ in the spherical case. In the Euclidean case we obtain $A^2 = \cot^2(\alpha_0/2) = (\sqrt{128}-11)/7 = 0.0448....$ $A = A_0 = \cot(\alpha_0/2) = 0.21169....$ So cone-manifold is hyperbolic for $0 \leq \alpha < \alpha_0 =??$ and is Euclidean for $\alpha = \alpha_0$.

   This gives

   $$\operatorname{Vol} T_2(\alpha, \alpha) = \int_{A_0}^{A} \operatorname{arctanh} \frac{\sqrt{7t^2+22t-1}}{t(5+t^2)} \frac{\mathrm{d}\,t}{t^2+1}.$$

   Since the integrant is pure imaginary for $0 \leq t < A_0$ we are able to compute the volume in a more convenient way

   $$\operatorname{Vol} T_2(\alpha, \alpha) = 4\Re \int_{0}^{A} \operatorname{arctanh} \frac{\sqrt{7t^2+22t-1}}{t(5+t^2)} \frac{\mathrm{d}\,t}{t^2+1}.$$

2.
$$\operatorname{Vol} T_2(0,0) = 2i \int_{\frac{1-i\sqrt{7}}{2}}^{\frac{1+i\sqrt{7}}{2}} \log \frac{2}{z-z^2} \frac{\mathrm{d}\, z}{z^2-1} = 5.333489...$$

3.
$$\operatorname{Vol} T_2(\pi/2,\pi/2) = 2i \int_{\frac{1-i\sqrt{7}}{4}}^{\frac{1+i\sqrt{7}}{4}} \log \frac{z^2+1}{z-z^2} \frac{\mathrm{d}\, z}{z^2-1} = 2.66674...$$

Note that $\operatorname{Vol} T_2(0,0) = 2 \operatorname{Vol} T_2(\pi/2,\pi/2)$.

4.
$$\operatorname{Vol} T_2(0,\pi/3) = i \int_{1-\frac{1+i\sqrt{3}}{\sqrt[3]{4}}}^{1-\frac{1-i\sqrt{3}}{\sqrt[3]{4}}} \log \frac{z^2+3}{(z-z^2)^2} \frac{\mathrm{d}\, z}{z^2-1} = 4.61656...$$

The results of above numerical calculation coincide with the correspondent results obtained by Weeks's SnapPea program [We].

# References


[BZ] Yu. D. Burago, V. A. Zalgaller, *Geometric inequalities,* Grundlehren der Mathematischen Wissenschaften, 285, Berlin ets., Springer-Verlag, 1988.

[BZie] G. Burde, H. Zieschang, *Knots,* De Gruyter Studies in Mathematics, Berlin - New-York, 1985.

[CCGLS] D. Cooper, M. Culler, H. Gillet, D. D. Long and P. B. Shalen, *Plane curves associated to character varieties of 3-manifolds,* Invent. Math., vol. 118 (1994), 47-84.

[DM] D. Derevnin, A. Mednykh, *On the volume of spherical Lambert cube,* Preprint, 2001.

[D] R. Diaz, *A Characterization of Gram Matrices of Polytopes,* Discrete Comput. Geom., Vol. 21 (1999), 551-601.

[Fench] W. Fenchel, *Elementary Geometry in Hyperbolic Space,* Walter de Gruyter, Berlin–New York 1989.

[GM] F. González-Acuña, J. M. Montesinos-Amilibia, *On the character varieties of group representations in $SL(2,\mathbf{C})$ and $PSL(2,\mathbf{C})$,* Math. Z. 214, 1993, 627–652.

[G] H. Grötsch, *Extremalprobleme der Konformen Abbildung I,* Berichte Akad. Leipzig, 80, 1928, 367-376.

[HKW] P. de la Harpe, M. Kervaire and C. Weber, *On the Jones Polynomial,* L'Enseignement Mathématique, Vol.32 (1986), 271–335.

[HKM] H. Helling, A. C. Kim, J. L. Mennicke, *Some Honey-combs in the hyperbolic 3-space,* Communications in Algebra, Vol.23 (1995), 5221-5242.

[HLM1] H. M. Hilden, M. T. Lozano, J. M. Montesinos-Amilibia, *On the Borromean orbifolds: geometry and arithmetic,* in: Topology'90, eds. B. A. Apanasov, W. Newmann, A. Reid, L. Siebenmann, de Gruyter, Berlin, 1992, 133–167.

[HLM2] H. M. Hilden, M. T. Lozano, J. M. Montesinos-Amilibia, *On the arithmetic 2-bridge knots and link orbifolds and a new knot invariant,* Journal of Knots and its Ramification, vol.4., no.1, 1995, 81-114.

[HLM3] H. M. Hilden, M. T. Lozano, J. M. Montesinos-Amilibia, *On a remarkable polyhedron geometrizing the figure eight cone manifolds,* J. Math. Sci. Univ. Tokyo, vol. 2, 1995, 501–561.

[HLM4] H. M. Hilden, M. T. Lozano, J. M. Montesinos-Amilibia, *Volumes and Chern-Simons invariants of cyclic covering over rational knots,*in: Proceedings of the 37th Taniguchi Symposium on Topology and Teichmüller Spaces held in Finland, July 1995, ed. by Sadayoshi KOJIMA et al., 1996, 31–35.

[Hds] C. D. Hodgson, *Schläfli revisited: Variation of volume in constant curvature spaces,* Preprint.

[Kauf] L. Kauffman, *New Invariants in the Theory of Knots,* Amer. Math. Monthly, vol. 95, no. 3, 195-242.

[K] R. Kellerhals, *On the volume of hyperbolic polyhedra,* Math. Ann., vol. 28 (1989), 245–270.

[KM] A.-Ch. Kim, A. Mednykh, *On the hyperbolic structure on the Whitehead link cone–manifold,* Preprint, 1999.

[Kn] C. G. Knott, *Life and Scientific Work of P.G.Tait,* Cambridge University Press (1911).

[Kj] S. Kojima, *Deformation of hyperbolic 3-cone-manifolds,* J. Differential Geometry, Vol.48 (1999) 469-516.

[M] A. Mednykh, *On the the Remarkable Properties of the Hyperbolic Whitehead Link Cone–Manifolds,* in: Knots in Hellas '98 (Gordon, Jones, Kauffman, Lambropoulou, Przytycki Editors), World Scientific, 2000, pp. 290–305.

[Me] A. Mednykh, *Trigonometrical identities and geometrical inequalities for links and knots,* to appear in the Proceedings of the Asian Mathematical Conference, Manila, 2000???.

[Me1] A. Mednykh, *On hyperbolic and spherical volumes for knot and link come–manifolds,* to appear in the Proceedings of the conference "Kleinian Groups and Hyperbolic 3–manifolds" held at Warwick University, September 11–15th 2001. ???

[MR1] A. Mednykh, A. Rasskazov, *On the structure of the canonical fundamental set for the 2-bridge link orbifolds,* Universität Bielefeld, Sonderforschungsbereich 343, "Discrete Structuren in der Mathematik", Preprint, 98–062.

[MR2] A. Mednykh, A. Rasskazov, *Volumes and degeneration of cone-structures on the figure-eight knot,*, Preprint, 1999.

[MV1] A. Mednykh, A. Vesnin, *Hyperbolic volumes of Fibonacci manifolds,* Siberian Math. J.36, 1995, 235–245.

[MV2] A. Mednykh, A. Vesnin *On the volume of hyperbolic Whitehead link cone–manifolds,* Scientia, Series A: Mathemetical Scincies, Vol. 8 (2001-2002), 1–11.

[R] R. Riley, *An elliptical path from parabolic representations to hyperbolic structure,* in: Topology of Low-Dimension manifolds, LNM, 722, Springer-Verlag, 1979, 99-133.

[Rolf] D. Rolfsen, *Knots and links,* Publish of Perish Inc., Berkeley Ca., 1976.

[Sch] R. N. Schmatkov *Euclidean structure on the Whitehead link cone–manifold,* Manuskripte der Forschergruppe Atithmetik, Universität Heidelberg, Preprint Nr. 5(2001), 1-26.

[T] W. P. Thurston, *The geometry and topology of three-manifolds,* Princeton University Press (1979).

[V] E. B. Vinberg, *Geometry II,* Encyclopeadia of Mathematical Sciences, Vol.29, Springer Verlag, 1992.

[We] J. Weeks, *SnapPea,* Computer program for work with hyperbolic structures, available by http//www.northnet.org/weeks/index/SnapPea3Doc.

[Zhou] Qing Zhou, *The Moduli Space of Hyperbolic Cone Structures,* J. Diferential Geometry, Vol. 51 (1999), 517-550.

DMITRY DEREVNIN, Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia. E-mail: derevnin@mail.ru

ALEXANDER D. MEDNYKH, Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia. E-mail: mednykh@math.nsc.ru

MICHELE MULAZZANI, Department of Mathematics and C.I.R.A.M., University of Bologna, ITALY. E-mail: mulazza@dm.unibo.it