Table of Contents for the Handbook of Knot Theory

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HYPERBOLIC KNOTS

COLIN ADAMS

1. Introduction

In 1978, Thurston revolutionized low dimensional topology when he demonstrated that many 3-manifolds had hyperbolic metrics, or decomposed into pieces, many of which had hyperbolic metrics upon them. In view of the Mostow Rigidity theorem([gM73]), when the volume associated with the manifold is finite, these hyperbolic metrics are unique. Hence geometric invariants coming out of the hyperbolic structure can be utilized to potentially distinguish between manifolds. One can either think of a hyperbolic 3-manifold as having a Riemannian metric of constant curvature -1, or equivalently, of there being a lift of the manifold to its universal cover, which is hyperbolic 3-space H^3, with the covering transformations acting as a discrete group of fixed point free isometries Γ. The manifold M is then homeomorphic to the quotient H^3/Γ.

A hyperbolic knot K in the 3-sphere S^3 is defined to be a knot such that $S^3 - K$ is a hyperbolic 3-manifold. Note that the complement is a finite volume but noncompact hyperbolic 3-manifold.

Given a hyperbolic knot, the SNAPPEA program, written by Jeffrey Weeks, can be utilized to determine the hyperbolic structure. (See [jW03].)In particular, the program yields the volume of the manifold, the symmetry group, and a variety of invariants that are associated to the cusps of the manifold and that will be discussed in Section 5. Details of how the program determines the hyperbolic structure appear in this volume ([jW04]).

In addition, the program gives the option of determining whether or not two hyperbolic manifolds are isometric. Hence, since knots are known to be determined by their complements, (cf. [GL89]), we can use SNAPPEA to decide if two given knots are the same, assuming first of all that they are both hyperbolic and second of all that SNAPPEA successfully finds their hyperbolic structure. In practice, this is one of the fastest means for determining if two hyperbolic knots are identical. It was utilized in the tabulation of the prime knots of 16 and fewer crossings(cf. [HTW98]).

Many knot and link complements in the 3-sphere are in fact hyperbolic. Moreover, knot and link complements in other 3-manifolds are often hyperbolic as well. In this paper, we will discuss the prevalence of hyperbolic knot and link complements and the various invariants associated with the hyperbolic metric carried by the complement. We will focus on the geometric invariants. For an excellent overview of hyperbolic knots with emphasis on algebraic invariants, see [CR98]. That paper also contains a description of some of the applications, including the Smith Conjecture and the determination of symmetries of a knot.

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Thanks to Eric Schoenfeld for help with the figures.

We will not cover certain topics that are surveyed elsewhere. See [cG04] or [sB02] for details on Dehn surgery on hyperbolic knots in S^3. The paper [mS98] covers unknotting tunnels and their connections with the hyperbolic structure on a knot complement. Branched coverings, and their relationship with hyperbolicity are covered in [CRV01] and [RZ01].

2. What knot and link complements are known to be hyperbolic?

The seminal work of Thurston in the 1970's and 80's demonstrated that every knot in S^3 is either a torus knot, a satellite knot or a hyperbolic knot. (We include composite knots as satellite knots.) These three categories are mutually exclusive. Torus knots are well understood. Their fundamental groups have presentations of the form $< a, b : a^p = b^q >$. They are exactly the knots such that their fundamental groups have nontrivial center. If the standardly embedded torus upon which a torus knot projects without crossings is cut open along the knot, the result is an essential annulus in the complement of the knot, which precludes a hyperbolic structure by fundamental group considerations.

Satellite knots have an incompressible non-boundary parallel torus in their complement, which also precludes a hyperbolic metric. This torus can be used to decompose the knot complement into simpler pieces, each of which may then be hyperbolic.

The remaining case is when the knot complement has neither an essential annulus nor a non-boundary parallel incompressible torus. Thurston's remarkable theorem demonstrates that such a knot must have a hyperbolic complement.

One might ask if it is likely that a randomly chosen knot is hyperbolic. If one uses a Gaussian distribution to select a cyclic sequence of n sticks glued end-to-end, forming a knot, then as n increases, the probability that the result is composite and hence non-hyperbolic, goes to one(cf. [DYS94]). If one restricts to prime knots, it is still the case that as n increases, the probability that the result is a satellite knot and hence non-hyperbolic, goes to one(cf. [dJ94]). So in some sense, hyperbolic knots are substantially less prevalent than non-hyperbolic knots.

But on the other hand, when these non-hyperbolic knot complements are cut open along essential annuli and tori, the resultant pieces have a high probability of being hyperbolic. And for small crossing number, the hyperbolic knots predominate. In fact, for the 2977 nontrivial prime knots through twelve crossings, the only non-hyperbolic knots are seven torus knots.

In addition, although many 3-manifolds are not hyperbolic, it was proved in [rM82] that every compact orientable 3-manifold contains a knot such that its complement is hyperbolic. In other words, every closed orientable 3-maifold is obtained by Dehn filling some hyperbolic 3-manifold. In this sense, hyperbolic knot complements are ubiquitous.

Although the decomposition of the set of knots into the three classes of torus, satellite and hyperbolic knots is fundamental, it does not necessarily allow us to easily determine whether a given knot is hyperbolic. It is often difficult to decide whether or not a given knot is a torus or satellite knot.

One approach is to input the knot into Jeff Weeks SNAPPEA program, which attempts to find a hyperbolic metric on the complement. If the knot is hyperbolic and of reasonable crossing number, the program will almost always find the hyperbolic structure, thereby verifying that the knot or link is hyperbolic. However, if the program fails to find a hyperbolic structure, it could be that the knot is a torus knot or a satellite knot. Or it could be that the computations for

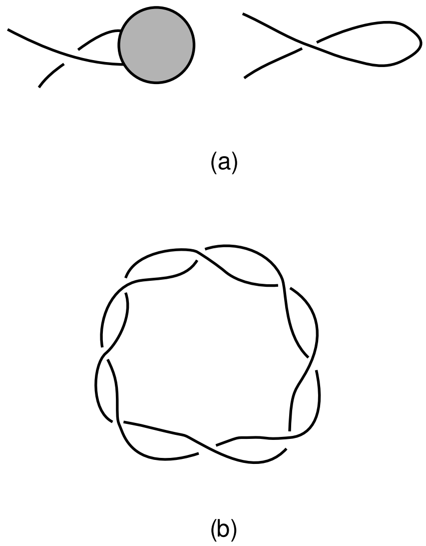

FIGURE 1. Unreduced diagrams and a 2-braid.

determining a hyperbolic structure are too complex for the computer to handle with its limited memory.

There are certain categories of knots and links in S^3 that are known to be hyperbolic. It has been proved that their complements contain no essential spheres, annuli or tori. We list some of these categories below:

(1) Prime non-splittable alternating links that are not 2-braids are hyperbolic.

This was proved in [wM84]. This particular category of link is exceptionally easy to recognize as Menasco proved that an alternating link is splittable if and only if any and every alternating projection is disconnected. He also proved that an alternating link is composite if and only if any and every reduced alternating projection has a circle that crosses the link twice and that contains crossings to either side. A projection is *reduced* if there are no crossings as in Figure 1a.

By a *2-braid*, we mean two strands that twist around one another as in Figure 1b. If an alternating projection shows that the link is either splittable, composite or a 2-braid, then the complement of the link is not hyperbolic. Therefore, given an alternating projection of a link, we can immediately determine by examination whether or not it is hyperbolic. Note that the only reduced alternating projection of a 2-braid is the standard one, by the Flyping Theorem of Menasco and Thistlethwaite(cf. [MT91] and [MT93]).

Two-bridge links are all known to be prime and alternating. Hence, assuming that we do not have a 2-braid (again, immediately obvious in an alternating 2-bridge representation), such a link is hyperbolic.

(2) The nontrivial prime non-torus almost alternating knots are hyperbolic. (Not true for links.)

An almost alternating link is a non-alternating link with a projection such that one crossing change will make the projection alternating. This category of knot was introduced in [Aetal92], where it was proved that of

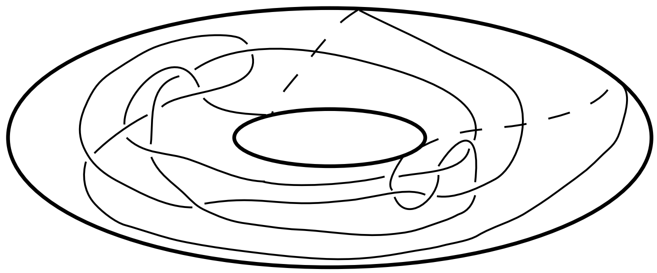

FIGURE 2. A toroidally alternating knot.

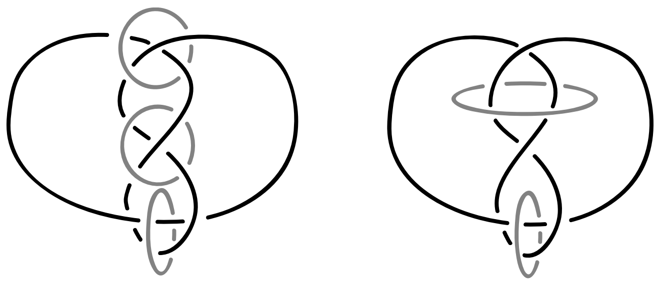

FIGURE 3. Two augmented alternating links from the figure-eight knot.

the non-alternating knots of 11 or fewer crossings, all but at most three are almost alternating. Since then, one of those knots was shown to be almost alternating in [GHY01]. In [Aetal92], it was also proved that nontrivial prime non-torus almost alternating knots are hyperbolic. However, determining whether or not an almost alternating knot is prime, torus or even nontrivial remains difficult.

(3) Toroidally alternating knots that are prime and non-torus are hyperbolic.

A toroidally alternating link is a link such that it can be projected onto the surface of a standardly embedded torus so that the crossings alternate over and under as we travel around each component on the surface of the torus and any nontrivial closed curve on the torus intersects the projection. An example appears in Figure 2. Almost alternating knots are all toroidally alternating. In [cA94], it was proved that nontrivial prime non-torus toroidally alternating knots are hyperbolic. But again, showing that a toroidally alternating knot is not trivial, composite or torus remains a difficult question.

(4) Augmented alternating links are almost all hyperbolic.

Given a prime alternating link projection, one can add trivial components that bound disjoint disks, each of which is perpendicular to the projection plane and intersects the knot at exactly two points. If the resulting link is nonsplittable, and no two of these added components are parallel in the resulting link complement, we call the result an augmented alternating link. See Figure 3.

In [cA86], it was proved that if the initial link is not a 2-braid, then an augmented alternating link is hyperbolic. Note that in [mL03], Lackenby proved that the closure in the geometric topology of the set of all hyperbolic alternating links is the set of all hyperbolic alternating links and augmented alternating links.

In addition, one can cut open along a twice-punctured disk, twist any number of half-twists and reglue. If the number of half-twists is even, we simply obtain a new link whose complement is homeomorphic to the original. If the number of half-twists is odd, we obtain a new link complement. In [cA85], it was proved that if the original link complement is hyperbolic, so is the new link complement, and it will have the same volume.

So, given a minimal projection of any nontrivial nonsplit link complement other than the standard projection of the two-braid or an obviously composite projection, one can add these trivial components around crossings, except where doing so yields parallel copies, to obtain a link, which is hyperbolic, since we can switch the crossings on the original link to make it alternating. We call such a link fully augmented. See [eS03] for more details.

(5) Arithmetic Links

Some of the first link complements known to be hyperbolic were arithmetic, meaning that their fundamental groups are commensurable (up to conjugacy in $\mathrm{PSL}(2, \mathbf{C})$) with a Bianchi group $\mathrm{PSL}(2, O_d)$, where d is a square-free positive integer, and O_d is the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. See for instance, [rR75, rR79, wT78, nW78, aH83, mB92, jS96, jS99, mB01]. Although it appears that arithmetic links are a small subset of the set of all links, Baker proved in [mB02] that every link in S^3 is a sub-link of an arithmetic link in S^3. In the case of knots, Reid proved that the only arithmetic knot is the figure-eight knot(cf. [aR191]).

(6) Montesinos links

A Montesinos link (also called a star link) is obtained by conecting n rational tangles in a simple cyclic fashion. Such a knot or link is denoted $K(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n})$, where $\frac{p_i}{q_i}$ denotes the ith rational tangle. In [uO84], it was proved that a Montesinos link $K(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n})$ with $q_i \geq 2$, is hyperbolic if it is not a torus link and not equivalent to $K(1/2, 1/2, -1/2, -1/2)$, $K(2/3, -1/3, -1/3)$, $K(1/2, -1/4, -1/4)$, $K(1/2, -1/3, -1/6)$ or the mirror image of these links. In [BS80] the torus links that are Montesinos links are identified.

(7) Mutants of a hyperbolic link.

Given a knot or link in a projection and a circle in the projection plane that intersects the knot at four points and separates the knot into two tangles, one can perform the following operation. Cut the knot open at these four points and flip the interior tangle either around a vertical or horizontal axis, or rotate it 180 degrees about an axis perpendicular to the projection plane before reattaching it to the outside tangle at the four points. This operation is called *mutation* and the resulting knots are called *mutants* of the original. In [dR87], Ruberman demonstrated that a mutant of a hyperbolic knot or link is also hyperbolic and it has the same volume.

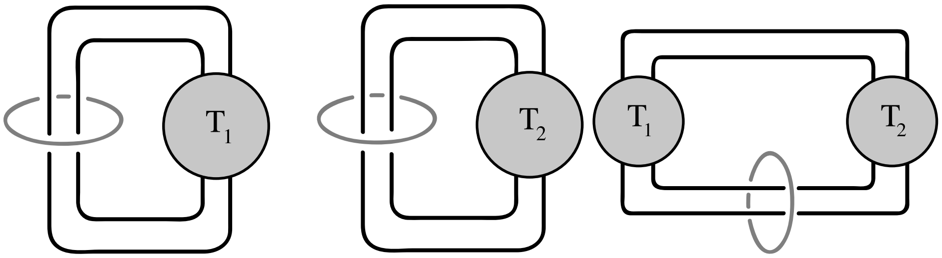

FIGURE 4. The belted sum of two links.

(8) Belted sums of hyperbolic links.

Given two links, each with a trivial component bounding a disk punctured twice by the link, one can cut them open along the disks in each, glue the resultant disks from the one complement to the disks from the other appropriately, to obtain a link complement as in Figure 4. In [cA85], it was proved that the belted sum of hyperbolic links is always hyperbolic with volume equal to the sum of the volumes of the original links.

(9) Hyperbolic Tangles.

In [yW96], Wu determined all of the non-hyperbolic alegbraic tangles, finding them to be a small subset of the set of all algebraic tangles. In particular, the hyperbolic algebraic tangles can be closed off to obtain algebraic knots and links that are hyperbolic. This gives a broad set of examples. Note that the tangles themselves can be thought of as complements of genus two handlebodies, and as such, are realized as hyperbolic sub-manifolds of S^3 with totally geodesic boundary.

In [kS99], it is shown that certain n-string alternating tangles are hyperbolic. These tangles can be utilized to show that prime semi-alternating links are hyperbolic. (See [LT88] for the definition of semi-alternating links.)

3. Volumes of Knots

The most natural invariant associated to hyperbolic knots is the volume of the complement. Work of Thurston and Jørgensen shows that knot volumes are a well-ordered subset of $\mathbb{R}$. In [AHW91], lists of volumes of knots through ten crossings were calculated. Cao and Meyerhoff (cf. [CM01]) proved that the figure-eight knot has the smallest possible volume for a knot complement, the volume of which is $2.02988\dots = 2v_3$, where $v_3 = 1.01494\dots$ is the volume of an ideal regular tetrahedron. In fact, they proved that this is the smallest volume of any noncompact orientable hyperbolic 3-manifold, the volume being shared by the figure-eight knot complement and one other manifold known as the sibling of the figure-eight knot complement.
A hyperbolic n-component link is known to have volume at least nv_3 (cf. [cA88]), and a 2-component link is known to have volume at least $2.3952v_3$(cf. [hY01]), although the expectation is that this bound is not sharp. The smallest known 2-component link is the Whitehead link, with volume $3.6638\dots$.
Results of Thurston and Jørgensen demonstrate that if one does (p, q)-Dehn filling on a hyperbolic knot or link complement, with $p^2 + q^2$ large enough, the resulting

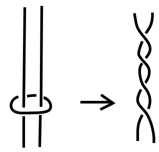

FIGURE 5. Twisting by surgery.

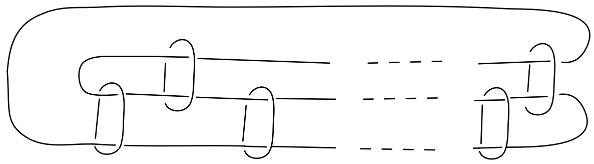

FIGURE 6. Two-bridge knots come from surgery on these links.

manifold will also be hyperbolic with volume less than the volume of the original manifold but approaching the volume of the original manifold as $p^2 + q^2$ approaches ∞. (See [wT78].)

In particular, we can take a component as in Figure 5, and do $(1, p)$-surgery to obtain a knot with arbitrarily many crossings but volume bounded by the original manifold.

On the other hand, there are knot complements with arbitrarily large volume. We will show this is true for the two-bridge knots. Two-bridge knots are obtained from surgery on the smaller components in the links depicted in Figure 6. Each of these links is a belted sum (see the previous section for the definition and properties) of Borromean rings.

The belted sum of n copies of the Borromean rings has volume equal to $n(7.3276\ldots)$. When we do high surgery on the augmenting components, we obtain two-bridge knots with volume arbitrarily close to that of the belted sum. Hence, volumes of knot complements can be arbitrarily high.

How effective an invariant is volume for distinguishing between knots? In general, it is very good. The first example of two knots with the same volume doesn't occur until we consider knots of up to twelve crossings. The 5_2 knot and a 12-crossing knot do have the same volume.

However, the operation of mutation preserves volume. Hence, many knots do share their volumes with other knots. Moreover, the operation of twisting along a twice-punctured disk preserves volume as well, yielding a variety of link complements with the same volume.

Can we say anything about the volume of a particular knot or link complement simply by looking at its projection? In [cA83], it was proved that the volume of the complement of an n-crossing hyperbolic knot other than the figure-eight knot is bounded above by $(4n - 16)v_3$.

In [mL03], Lackenby defines a twist in an alternating projection to be a maximal chain of adjacent bigon regions(as in the second part of Figure 5), or to be a single crossing that is not adjacent to a bigon region. The twist number of a projection is the number of twists within it. He proves(with an improvement of his upper

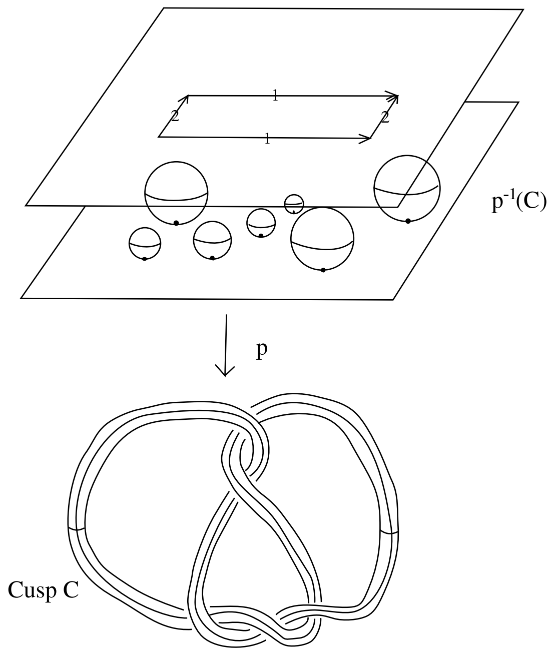

FIGURE 7. The cusp of a knot.

bound due to Ian Agol and Dylan Thurston) that a hyperbolic alternating knot or link in a prime alternating projection of twist number t satisfies

$$v_3(\frac{t-2}{2}) \leq Volume(S^3 - K) < v_3(10t - 16)$$

By choosing a twist reduced projection, which always exists, the lower bound can be improved to $v_3(t-2)$.
Is hyperbolic volume related to the more recently defined quantum invariants? In [rK97], Kashaev conjectured that the hyperbolic volume of a knot complement is determined by the asymptotic behavior of a link invariant that depends on the quantum dilogarithm and that was introduced by Kashaev in [rK95]. This is known as the Kashaev Conjecture. It has been verified for a handful of knots.

4. Cusps

Given a hyperbolic knot in S^3, one can define a cusp C for the knot to be a neighborhood of the missing knot such that it lifts to a set of horoballs with disjoint interiors in the universal cover H^3. See Figure 7.
Topologically, a cusp is homeomorphic to $T^2 \times [0,1)$ where $T^2 \times 0$ corresponds to the boundary of the neighborhood and the missing $T^2 \times 1$ corresponding to the knot itself. Choosing one of the covering horoballs to be the horoball centered at ∞ in the upper-half-space model of H^3, the subset of covering translations sending this horoball back to itself are all Euclidean translations, generated by two translations. A fundamental domain for their action is a parallelogram P in the horizontal boundary plane and the collection of horizontal parallelograms directly above this one. The parallelogram P projects to the torus boundary of the cusp. As we move up the z-axis in hyperbolic space, each horizontal parallelogram projects to a concentric torus around the missing knot, the set of them shrinking in toward the missing knot.

In $H^3 \cup \partial H^3$, the Euclidean translations fix a single point on the boundary, that being the point at ∞. We call an isometry that fixes a single point on the boundary a *parabolic* isometry. All such isometries will send the horoballs tangent to the boundary at that fixed point back to themselves. All other isometries in the group Γ of covering translations have two fixed points on the boundary. These are called *hyperbolic* isometries. They correspond to translations along the geodesic with the two fixed points as endpoints, together with a possible rotation about the geodesic.

The third type of orientation preserving isometry is a pure rotation about a geodesic. But as it is not fixed point free in H^3, it cannot appear as a covering translation in Γ. In the case of a knot, we have a single cusp in the complement. We can expand that cusp until it touches itself. Then, the set of horoballs that are the pre-image of C in H^3 will expand until two first touch. Since every horoball in $p^{-1}(C)$ is identified to every other one, the horizontal plane that is the boundary of the horoball centered at ∞ will touch other horoballs. We call this cusp a *maximal cusp*. In the case we have a link, we can expand the cusps until they touch each other or themselves, to obtain a maximal set of cusps. However, in different situations, it may be appropriate to do the expansions in different ways. For instance, we may insist that all the cusps have the same volume or the same length of shortest curve in their boundaries while we expand the set. Or we may choose to put the largest possible volume in the expanded set of cusps.

We define the *cusp density* of a hyperbolic knot or link complement to be the ratio of the largest possible volume in a maximal set of cusps divided by the total volume. Meyerhoff noted in [rM86] that this number is at most $0.853 = \frac{\sqrt{3}}{2v_3}$ where again, $v_3 = 1.01494\ldots$ is the volume of an ideal regular tetrahedron. The cusp density of the figure-eight knot complement realizes this upper bound.

In [Aetal02], it was proved that there are knots of arbitrarily small cusp density, although other methods for generating such knots were previously known, but had not appeared in the literature. Although it is known that the set of cusp densities for hyperbolic 3-manifolds are dense in the interval $[0, .853\ldots]$, it is not known whether the same holds true for cusp densities of hyperbolic link complements or perhaps, even for cusp densities of hyperbolic knot complements.

5. Meridians and other cusp invariants

The Thurston-Gromov 2π theorem implies that for any Dehn filling along a curve c in the cusp boundary of a hyperbolic 3-manifold, the resultant manifold will be negatively curved if c has length at least 2π. It is conjectured that in fact the resultant manifolds are hyperbolic with constant negative curvature, but this question remains open. (See [BH96] for more details.) If we define the length $|m|$ of a meridian for a hyperbolic knot K to be the length of its shortest representative in the maximal cusp boundary, then $|m| < 2\pi$, since Dehn filling along the meridian yields S^3, which is not negatively curved.

In [mL00] and [iA00], a variation on the 2π Theorem, known as the 6-Theorem, was proved. It shows that a manifold obtained by Dehn filling along a minimal length simple closed curve of length greater than 6 in the cusp boundary of a hyperbolic manifold is *hyperbolike*, which is to say that it is irreducible, boundary-irreducible, and has infinite fundamental group that is word hyperbolic. In other words, it has these attributes that one would expect of a negatively curved manifold. Agol also gave an example of a cusped manifold such that there was a curve of length 6 in the cusp boundary such that surgery on that curve yielded a non-hyperbolike manifold. So the 6 bound is sharp. In [Aetal03], we

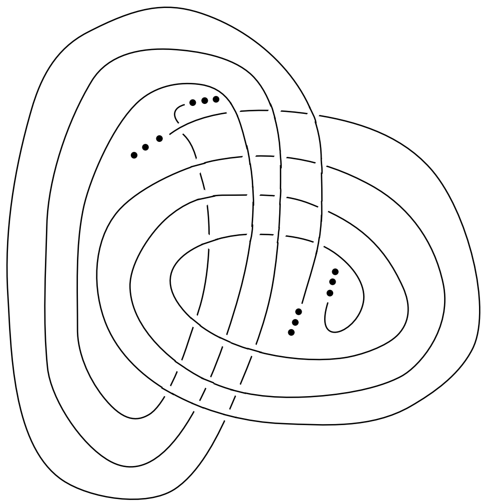

FIGURE 8. These knots have meridian length approaching 4 from below.

produce examples of knot complements with longitude of length exactly 6 such that surgery on the longitude yields non-hyperbolike manifolds. So the 6-theorem is sharp for knot complements in S^3.

Since Dehn filling along the meridian of a knot yields the 3-sphere, which has trivial fundamental group, the 6-Theorem implies every meridian of a knot must have length no greater than 6. Moreover, any nontrivial curve in the cusp boundary must have length at least 1. This is because a maximal cusp when lifted to the horoball centered at ∞, will have other horoballs tangent to it. The shortest parabolic isometry must shift the horoballs at least a distance 1 so that they do not overlap with one another.

So all meridian lengths fall in the interval [1, 6]. In [cA202], it was proved that there is only one knot that realizes the lower bound. The length of the meridian in the figure-eight knot complement is exactly 1, and no other hyperbolic manifold has a nontrivial curve in its maximal cusp boundary this small. Moreover, in [cA03], it is proved that the next shortest meridian is that of the 5_2 knot, with a length of $1.150964\ldots$, and that there are no other knots of meridian length less than $\sqrt[4]{2} = 1.189207\ldots$.

What about the upper bound? In [iA00], Agol gives an example of a sequence of knots with meridian lengths approaching 4 from below. We show how to construct such a sequence of knots in Figure 8. As the number of times that the knot wraps around itself both vertically and horizontally increases, the meridian length approaches 4 from below. These are the largest known meridian lengths to date. Utilizing ideas from [zH98], it was proved in [Aetal02] that the meridian length of a knot K is always at most $6 - \frac{7}{c}$, where c is the crossing number of K. The idea is to take the singular punctured surface obtained by coning the knot to a point below the projection plane. If it is not incompressible or boundary-incompressible, then we can do compressions and boundary-compressions that will only improve the ultimate bounds. Then we pleat the surface and it inherits a 2-dimensional hyperbolic metric from the 3-dimensional hyperbolic metric on the manifold. The length of the boundary curves of the surface intersected with the cusp is equal to the area of the intersection of the cusp with the surface. This is bounded above

by $\frac{3}{\pi}$ times the area of the surface, which is given by $2\pi|\chi(S)|$. From these inequalities, we can obtain bounds on meridian length and cusp volume.
One can obtain substantially better bounds for certain classes of knots. In [cA96], it was proved that the two-bridge knots have meridian lengths in the interval [1,2), with sequences of meridians approaching 2.
In [Aetal02], upper bounds for alternating knots were determined. In particular, it was shown that the meridian of an alternating knot with crossing number c is bounded above by $3 - \frac{6}{c}$. The proof utilizes the checkerboard surfaces that come from any given projection of a knot. In the case of alternating knots, it was proved in [MT93] that these surfaces are incompressible and boundary-incompressible. Pleating these surfaces allows one to to prove the result.
The expectation is that the actual upper bound for the meridian length of alternating knots is substantially smaller. For example, we say a property J is true for almost all hyperbolic 3-manifolds if for every fixed volume $V > 0$, J is true for all but finitely many hyperbolic 3-manifolds with volume less than V.
In [eS03],it is proved that almost all alternating hyperbolic knots have meridian length bounded above by 2. The expectation is that 2 is the correct bound for all alternating hyperbolic knots.
In addition, in [eS03], Schoenfeld proves that if we fully augment any reduced projection of a nontrivial knot other than a 2-braid, as in Section 2, the meridian of the original knot becomes 2 in the resultant link complement. In particular, this means that if we take any reduced projection of any non-2-braid knot and twist about each of its crossings to add bigons, the resultant knots will have meridian approaching 2.
In [zH98], an application of meridian length to the determination of crossing number of a knot is given. Specifically, it is shown that if C is a maximal cusp for a hyperbolic knot K, then the crossing number $\mathrm{cr}(K)$ satisfies

$$\mathrm{cr}(K) \geq \mathrm{area}(\partial C)/(|m|(2\pi - |m|)).$$

Moreover, if K' is a satellite of the hyperbolic knot K of degree p, then $\mathrm{cr}(K') \geq p^2 \mathrm{area}(\partial C)/(|m|(2\pi - |m|))$.

6. Geodesics and Totally Geodesic Surfaces

Geodesics in hyperbolic knot complements may or may not be closed. If a geodesic is closed, it might intersect itself or not. However, every hyperbolic knot or link complement contains a simple closed geodesic([AHS99]).
A *systole* is a shortest closed geodesic, and the *systole length* is the length of the shortest closed geodesic in a manifold. If we take two strands in a knot complement, and obtain a sequence of knots by twisting these two strands around one another as in Figure 5, then the length of the geodesic that wraps around the two strands will be approaching 0. Hence, this yields a sequence of knot complements with systole length shrinking to 0. So there is no lower bound for systole length in knot and link complements. On the other hand, there is an upper bound. Although the systole length can be arbitrarily large for hyperbolic 3-manifolds in general, it was proved in [AR02] that systole length for hyperbolic knots and links in S^3 is bounded above by 7.35534. For hyperbolic alternating knots, this was improved to 4.5 in [Aetal02]. It would be interesting to obtain better bounds on systole length for a variety of categories of knots and links.
Closed geodesics in knot complements can themselves be either unknotted or knotted as curves in S^3. See for instance, [sM01], where a variety of knotted geodesics in the figure-eight knot complement are displayed.

In [tS91] (see also [sK88]), it was proved that the complement of a simple closed geodesic or a set of disjoint simple closed geodesics in a hyperbolic manifold will itself yield a hyperbolic manifold. One can look at the link complements obtained from removing certain geodesics from a knot or link complement, in the hopes of decomposing the resultant manifolds into certain canonical pieces.
One might hope that the complement of a simple closed geodesic in a hyperbolic knot complement has minimal volume among all the complements of simple closed curves in the same free homotopy class. But counterexamples to this conjecture are given in the figure-eight knot complement in [sM01].

We turn now to surfaces in knot complements. A closed embedded incompressible surface in a hyperbolic knot complement can come in one of two varieties. The first possibility is that it is quasi-Fuchsian. This means that it lifts to the disjoint union of topological planes in H^3, each with a limit set that is a quasi-circle. In the case that the limit set is an actual circle, and the planes are geodesic, and we say that the surface is *Fuchsian* or *totally geodesic*.
The second possibility for a closed embedded incompressible surface is that there are simple closed curves on the surface that can be homotoped through the knot complement into a neighborhood of the missing knot. This means that the corresponding isometry, when we lift to hyperbolic space, is a parabolic isometry. We call such a curve an *accidental parabolic curve* and we call such a surface an *accidental surface*. In [IO200], it was proved that if an accidental parabolic curve exists for a given surface S, then it is isotopic to a unique curve on the boundary torus of the cusp. In particular, there is a well-defined accidental slope for each accidental surface. However, there can be more than one accidental slope for a given knot. It follows from [CGLS87] that accidental slopes must be meridional or integer.
In [AR93], explicit examples were given of closed quasi-Fuchsian surfaces in knot complements. These surfaces were shown not to be totally geodesic. In fact, in [MR92], it was conjectured that hyperbolic knot complements in S^3 do not contain any closed embedded totally geodesic surfaces. This has been proved for hyperbolic knots that are alternating knots, tunnel number one knots, 2-generator knots and knots of braid index three [MR92], almost alternating knots [Aetal92], toroidally alternating knots [cA94], Montesinos knots [uO84], 3-bridge knots and double torus knots [IO200]) and knots of braid index three [LP85] and four [hM02]. Note that the conjecture does not hold for links. An explicit counterexample is given in [MR92].
There are closed totally geodesic surfaces immersed in hyperbolic knot complements. In [aR291], Reid shows that the figure-eight knot complement contains infinitely many such non-homotopic surfaces. See [AR97] for two additional knots with immersed totally geodesic surfaces in their complements.
An incompressible boundary incompressible surface S with boundary properly embedded in a hyperbolic knot exterior can have one of three possible behaviors:

(1) S can be quasi-Fuchsian.
(2) S can be accidental.
(3) S can be a virtual fiber in a fibered knot, with limit set the entire boundary of H^3.

Specific examples of incompressible boundary-incompressible surfaces are afforded by minimal genus Seifert surfaces. In [sF98], it is proved that a minimal genus Seifert surface in a non-fibered hyperbolic knot complement must be quasi-Fuchsian. It can never be accidental.

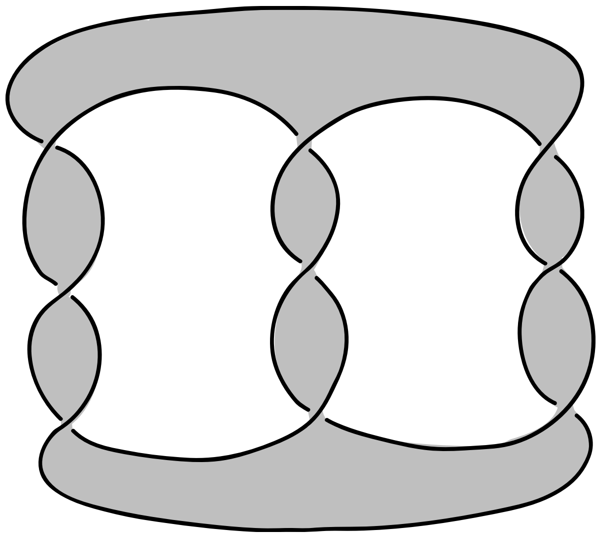

FIGURE 9. A (3,3,3)-pretzel knot has a totally geodesic Seifert surface.

Knots can have totally geodesic minimal genus Seifert surfaces. In [AS03], examples such as a (p,p,p) pretzel knot(Montesinos knot $K(1/p, 1/p, 1/p)$) are shown to have totally geodesic Seifert surfaces.(See Figure 9.) There are examples of hyperbolic knots with totally geodesic Seifert surfaces, where the Seifert surfaces are both free (the complement of a neighborhood of the Seifert surface in the knot complement is a handlebody) and non-free.

But the expectation is that knots with totally geodesic Seifert surfaces are the exception. For instance, in [AS03], it is proved that 2-bridge knots never have totally geodesic Seifert surfaces.

An interesting question is whether there are totally geodesic surfaces in knot complements other than such Seifert surfaces. This is one of many open questions that still remain in the theory of hyperbolic knots.

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