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Table of Contents for the Handbook of Knot Theory

William W. Menasco and Morwen B. Thistlethwaite, Editors

Knot theory of complex plane curves

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Abstract

The primary objects of study in the "knot theory of complex plane curves" are $\mathbb{C}$-*links*: links (or knots) cut out of a 3-sphere in $\mathbb{C}^2$ by complex plane curves. There are two very different classes of $\mathbb{C}$-links, *transverse* and *totally tangential*. Transverse $\mathbb{C}$-links are naturally oriented. There are many natural classes of examples: links of singularities; links at infinity; links of divides, free divides, tree divides, and graph divides; and—most generally—quasipositive links. Totally tangential $\mathbb{C}$-links are unoriented but naturally framed; they turn out to be precisely the real-analytic Legendrian links, and can profitably be investigated in terms of certain closely associated transverse $\mathbb{C}$-links.

The knot theory of complex plane curves is attractive not only for its own internal results, but also for its intriguing relationships and interesting contributions elsewhere in mathematics. Within low-dimensional topology, related subjects include braids, concordance, polynomial invariants, contact geometry, fibered links and open books, and Lefschetz pencils. Within low-dimensional algebraic and analytic geometry, related subjects include embeddings and injections of the complex line in the complex plane, line arrangements, Stein surfaces, and Hilbert's 16th problem. There is even some experimental evidence that nature favors quasipositive knots.

Key words: Arrangement, braid, braid monodromy, braided surface, Chisini's statement, $\mathbb{C}$-link, Hilbert's 16th problem, Jacobian conjecture, knot, labyrinth, link, link at infinity, link of indeterminacy, Milnor map, Milnor's question, monodromy, plane curve, quasipositivity, singularity, Thom conjecture, Thurston–Bennequin invariant, unknotting number, Zariski conjecture
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Foreword

In the past two decades, knot theory in general has seen much progress and many changes. "Classical knot theory"—the study of knots as objects in their own right—has taken great strides, documented throughout this Handbook (see the contributions by Birman and Brendle, Hoste, Kauffman, Livingston, and Scharlemann). Simultaneously, there have been extraordinarily wide and deep developments in what might be called "modern knot theory": the study of knots and links in the presence of extra structure,[2] for instance, a hyperbolic metric on the knot complement (as in the articles by Adams and Weeks) or a contact structure on the knot's ambient 3-sphere (as in the article by Etnyre). In these terms, the knot theory of complex plane curves is solidly part of modern knot theory—the knots and links in question are $\mathbb{C}$-links, and the extra structures variously algebraic, analytic, and geometric.

"Some knot theory of complex plane curves" (Rudolph, 1983d) was a broad view of the state of the art in 1982. Here I propose to look at the subject through a narrower lens, that of quasipositivity. §1 is devoted to general notations and definitions. §2 is a treatment of braids and braided surfaces tailored to quasipositivity.[3] General transverse $\mathbb{C}$-links are constructed and described in §3, while §4 is a brief look at the special transverse $\mathbb{C}$-links that arise from complex algebraic geometry "in the small" and "in the large"—to wit, links of singularities and links at infinity.[4] Totally tangential $\mathbb{C}$-links are constructed and described in §5. The material in §§3–5 is related to other research areas in §6. In §7 I give some fairly explicit, somewhat programmatic suggestions of directions for future research in the knot theory of complex plane curves.

Original texts of some motivating problems in the knot theory of complex plane curves are collected in an Appendix. This survey concludes with an index of definitions and notations and a bibliography. Open questions are distributed throughout.

[2] Some observers have also detected "postmodern knot theory": the study of extra structure in the absence of knots.
[3] Consult Birman and Brendle for a deeper and broader account of braid theory.
[4] One consequence of this survey's bias towards quasipositivity is a de-emphasis of other aspects of the knot theory of links of singularities and links at infinity; the reader is referred to Boileau and Fourrier (1998) (who include sections on both these topics), to the discussions of singularities and their higher-dimensional analogues by Durfee (1999, §2) and Neumann (2001, §1), and of course to the extensive literature on both subjects—particularly links of singularities—referenced in those articles.

Contents

1 Preliminaries

Terms being defined are set in *this* typeface; mere emphasis is indicated *thus*. A definition labelled as such is either of greater (local) significance, or non-standard to an extent which might lead to confusion; labelled or not, potentially startling definitions and notations are flagged with the symbol ☡ in the margins of both the text and the index. The end or omission of a proof is signalled by $\square$. Both $A := B$ and $B =: A$ mean "A is defined as B". The symbol $\cong$ is reserved for a *natural isomorphism* (in an appropriate category). ☡

1.1 Sets and groups

The set of real (resp., complex) numbers is $\mathbb{R}$ (resp., $\mathbb{C}$); write $z \mapsto \bar{z}$ for *complex conjugation* $\mathbb{C} \to \mathbb{C}$, and Re (resp., Im) for *real part* (resp., *imaginary part*) $\mathbb{C} \to \mathbb{R}$. For $x \in \mathbb{R}$, let $\mathbb{R}_{\geq x} := \{t \in \mathbb{R} \,:\, t \geq x\}$, $\mathbb{R}_{>x} := \{t \in \mathbb{R} \,:\, t > x\}$, $\mathbb{R}_{\leq x} := \{t \in \mathbb{R} \,:\, t \leq x\}$, $\mathbb{R}_{<x} := \{t \in \mathbb{R} \,:\, t < x\}$. Let $\mathbb{C}_{\pm} := \{w \in \mathbb{C} \,:\, \pm \operatorname{Im} w \geq 0\}$. Let $\mathbb{N} := \mathbb{Z} \cap \mathbb{R}_{\geq 0}$ and $\mathbb{N}_{>0} := \mathbb{N} \cap \mathbb{R}_{>0}$. For $n \in \mathbb{N}$, let $\mathbf{n} := \{k \in \mathbb{N}_{>0} \,:\, k \leq n\}$. Denote projection on the ith factor of a cartesian product by pr_i. The Euclidian norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ is $\|\cdot\|$. For $\mathbf{u}, \mathbf{v}$ in a real vectorspace, $[\mathbf{u}, \mathbf{v}]$ is $\{(1-t)\mathbf{u} + t\mathbf{v} \,:\, 0 \leq t \leq 1\}$.

Let X be a set. Denote the identity map $X \to X$ by id_X, and the cardinality of X by $\mathrm{card}(X)$. A *characteristic function* on X is an element of $\{0,1\}^X$; for $Y \subset X$, let $c_{Y \hookrightarrow X} \colon X \to \{0,1\}$ denote *the characteristic function of Y in X*, so $c_{Y \hookrightarrow X}(x) = 1$ for $x \in Y$, $c_{Y \hookrightarrow X}(x) = 0$ for $x \in X \setminus Y$. A *multicharacteristic function* on X is an element of $\mathbb{N}^X$. Let $c \in \mathbb{N}^X$. The *total multiplicity* of c is $\sum_{x \in X} c(x) \in \mathbb{N} \cup \{\infty\}$. For $n \in \mathbb{N}$, an *n-multisubset* of X is a pair $(\mathrm{supp}(c), c|\mathrm{supp}(c))$ where c is a multicharacteristic function on X of total multiplicity n. Call the set of all n-multisubsets of X the nth *multipower set* of X, denoted $\mathcal{MP}^{[n]}(X)$. (Note that if $X \neq X'$ then no multicharacteristic function on X is a multicharacteristic function on X', whereas $\mathcal{MP}^{[n]}(X) \cap \mathcal{MP}^{[n]}(X') \neq \varnothing$ if and only if $X \cap X' \neq \varnothing$.) For any A, any $f \colon A \to \mathcal{MP}^{[n]}(X)$ can be construed as a *multivalued* (specifically, an *n-valued*) *function from A to X*; typically $\mathrm{gr}(f) := \{(a,x) \in A \times X \,:\, x \in f(a)\}$, the *multigraph of f*, determines neither n nor f (unless n is—*and is known to be*—equal to 1), but the notation is still useful. The *type* of $c \in \mathbb{N}^X$ is $\tau(c) := \mathrm{card} \circ (c^{-1}|\mathbb{N}_{>0}) \in \mathbb{N}^{\mathbb{N}_{>0}}$ (a multicharacteristic function on $\mathbb{N}$). Identify an *n-subset* of X (i.e., $Y \subset X$ with $\mathrm{card}(Y) = n$) with the n-multisubset $(Y, 1)$ of type $nc_{\{1\} \hookrightarrow \mathbb{N}}$, and the set of n-subsets of X with the nth *configuration set*

$$\binom{X}{n} := \{(\mathrm{supp}(c), c|\mathrm{supp}(c)) \in \mathcal{MP}^{[n]}(X) \,:\, \tau(c) = nc_{\{1\} \hookrightarrow \mathbb{N}}\}$$

of X. Call $\Delta_n(X) := \mathcal{MP}^{[n]}(X) \setminus \binom{X}{n}$ the nth *discriminant set* of X.

Let G be a group. For $g, h \in G$, let ${}^g h$ (resp., $[g,h]$;$\lll g, h \ggg$) denote the *conjugate* (resp., *commutator*; *yangbaxter*) ghg^{-1} (resp., $ghg^{-1}h^{-1} = {}^g hh^{-1}$; $ghgh^{-1}g^{-1}h^{-1} = {}^{gh}gh^{-1}$). For $A \subset G$, let $\langle A \rangle_G$ be the subgroup generated by A, i.e., $\bigcap\{H : A \subset H$ and H is a subgroup of $G\}$, and let $\langle a \rangle_G := \langle \{a\} \rangle_G$ (when G is understood, it may be dropped from these notations). The *normal closure* of A in G is $\langle \{{}^g a : g \in G, a \in A\} \rangle_G$. A *presentation* of G, denoted

$$(\mathbf{A}) \qquad G = \mathrm{gp}\Big(\mathbf{g}_i\,(i \in I)\colon \mathbf{r}_j\,(j \in J)\Big),$$

consists of: (1) a short exact sequence $R \subset F \overset{\pi}{\twoheadrightarrow} G$ in which F is a free group and R is a subgroup of F; (2) a family $\{\mathbf{g}_i : i \in I\} \subset F$ of free generators of F, the *generators of* $(\mathbf{A})$; and (3) a family $\{\mathbf{r}_j : j \in J\} \subset R$ with normal closure R, the *relators of* $(\mathbf{A})$. (Sometimes the elements $\pi(\mathbf{g}_i)$ of G are also, abusively, called *the generators of* G with respect to $(\mathbf{A})$.) The presentation $(\mathbf{A})$ is *Wirtinger* in case every relator $\mathbf{r}_j$ is of the form ${}^{w(j)}\mathbf{g}_{s(j)}\mathbf{g}_{t(j)}{}^{-1}$ for some $s, t\colon J \to I$ and $w\colon J \to F$. Denote the *free product* of groups G_0 and G_1 by $G_0 * G_1$.

A *partition* of a set X is the quotient set $X/{\equiv}$ of X by an equivalence relation $\equiv$ on X. Call $X/{\approx}$ a *refinement* of $X/{\equiv}$ in case each $\approx$-class is a union of $\equiv$-classes. Given $f\colon X \to Y$, write X/f for $X/{\equiv_f}$, where $x_0 \equiv_f x_1$ iff $f(x_0) = f(x_1)$. Given a (right) action of a group G on X, write X/G for $X/{\equiv_G}$, where $x_0 \equiv_G x_1$ iff $x_1 = x_0 g$ for some $g \in G$; as usual, xG stands for the $\equiv_G$-class (i.e., the G-orbit) of x. The group $\mathfrak{S}_n$ of bijections $\mathbf{n} \to \mathbf{n}$ acts in a standard way on $X^n = X^{\mathbf{n}} = \{f : f\colon \mathbf{n} \to X\}$. An *unordered n-tuple* in X is an element $U(x_1, \ldots, x_n) := (x_1, \ldots, x_n)\mathfrak{S}_n$ of the nth *symmetric power* $X^n/\mathfrak{S}_n$ of X, where $U = U_{X,n}\colon X^n \to X^n/\mathfrak{S}_n$ is the *unordering map*. The map

$$(\mathbf{B}) \qquad \begin{aligned} X^n/\mathfrak{S}_n \to \mathcal{MP}^{[n]}(X) &: U(\overbrace{x_1, \ldots, x_1}^{n_1}, \overbrace{x_2, \ldots, x_2}^{n_2}, \ldots, \overbrace{x_k, \ldots, x_k}^{n_k}) \\ &\mapsto (\{x_1, x_2, \ldots, x_k\}, (\{x_1, x_2, \ldots, x_k\} \to \mathbb{N}_{>0} : x_i \mapsto n_i)) \end{aligned}$$

(where $x_i \neq x_j$ for $i \neq j$, $n_i > 0$, and $n = \sum_{i=1}^k n_i$) is a bijection. *Except* in $(\mathbf{B})$, notation is abused in the standard way, so that $\{x_1, \ldots, x_n\}$ denotes $U(x_1, \ldots, x_n)$. In case X is ordered by $\preccurlyeq$, notations like $\{x_1 \preccurlyeq \cdots \preccurlyeq x_k\} \in \binom{X}{k}$, $\{i \preccurlyeq j\}\colon C \to \binom{X}{2}$, and so on, mean "$\{x_1, \ldots x_k\} \in \binom{X}{k}$ and $x_1 \preccurlyeq \cdots \preccurlyeq x_k$", "$\{i(c), j(c)\} \in \binom{X}{2}$ and $i(c) \preccurlyeq j(c)$ for all $c \in C$", and so on. In case X is totally ordered by $\prec$, call $\{s \prec t\}, \{s' \prec t'\} \in \binom{X}{2}$ *linked* (resp., *unlinked*; *in touch at* u) iff either $s \prec s' \prec t \prec t'$ or $s' \prec s \prec t' \prec t$ (resp., either $s \prec s' \prec t' \prec t$ or $s' \prec s \prec t \prec t'$; $\{u\} = \{s,t\} \cap \{s', t'\}$).

The bijection $(\mathbf{B})$ induces a definition of $\tau(\{x_1, \ldots, x_n\})$ as itself an unordered n-tuple, such that, e.g., $\tau(\{1,1,1,1\}) = \{4\}$, $\tau(\{1,1,2,3\}) = \{1,1,2\}$, etc.

A simplicial complex is not necessarily finite. A geometric realization of a simplicial complex $\mathcal{K}$ is denoted $\|\mathcal{K}\|$. A *triangulation* of a topological space X is a homeomorphism $\|\mathcal{K}\| \to X$ for some $\mathcal{K}$. A *polyhedron* is a topological space X which is the target of some triangulation. Let X be a polyhedron. The set of components of X is denoted $\pi_0(X)$. The Euler characteristic of X is denoted $\chi(X)$. The fundamental group of X with *base point* $*$ is denoted by $\pi_1(X;*)$, or simply $\pi_1(X)$ in contexts where $*$ can be safely suppressed. Call $\pi_1(X \setminus K)$ the *knotgroup* of K in case X is connected and $K \subset X$.

Manifolds are smooth ($\mathcal{C}^\infty$) unless otherwise stated. A manifold M may have boundary, but corners (possibly cuspidal) only when so noted. Denote by ∂M (resp., $\operatorname{Int} M$; $T(M)$) the interior (resp., boundary; tangent bundle) of M. Call M *closed* (resp., *open*) in case M is compact (resp., M has no compact component) and $\partial M = \varnothing$. Manifolds are (not only orientable, but) oriented, unless otherwise stated: in particular, a complex manifold (e.g., $\mathbb{C}^n$ or $\mathbb{P}_n(\mathbb{C})$) has a *natural orientation*, and $\mathbb{R}$, $D^{2n} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \|(z_1, \ldots, z_n)\| \leq 1\}$, and $S^{2n-1} := \partial D^{2n}$ have *standard orientations*, as do S^2 (identified with $\mathbb{P}_1(\mathbb{C}) := \mathbb{C} \cup \{\infty\}$), $\mathbb{R}^3$ (identified with $S^3 \setminus (0,1)$) and the *bidisk* $D^2 \times D^2$ (with corners $S^1 \times S^2$). The tangent space $T_x(M)$ to M at $x \in M$ is an oriented vectorspace. Let $-M$ (resp., $+M$; $|M|$) denote M with orientation reversed (resp., preserved; forgotten); in case $\mathsf{M}\colon M \to M$ is a diffeomorphism reversing orientation, $\operatorname{Mir} Q := \mathsf{M}(Q)$ is a *mirror image* of $Q \subset M$.

Call $Q \subset M$ *interior* (resp., *boundary*) in case $Q \subset \operatorname{Int} M$ (resp., $Q \subset \partial M$). In case Q has a closed regular neighborhood $\operatorname{Nb}(Q \subset M)$ in $(M, \partial M)$, the *exterior* of Q in $(M, \partial M)$ is $\operatorname{Ext}(Q \subset M) := M \setminus \operatorname{Int} \operatorname{Nb}(Q \subset M)$.

A *connected sum* (resp., *boundary-connected sum*) of n–manifolds M_1 and M_2 is denoted $M_1 \mathbin{\#} M_2$ (resp., $M_1 \natural M_2$); notations like $M_1 \mathbin{\#}_{D^n} M_2$, $M_1 \natural_{D^{n-1}} M_2$, and so on, can be used for greater precision.

1. Definitions. A *stratification* of a topological space X is a locally finite partition $X/{\equiv}$ such that (1) each $\equiv$-equivalence class, equipped with the topology induced from X, is a connected, not necessarily oriented, manifold (called a $\equiv$-*stratum* of X, or simply a *stratum* of X when $\equiv$ is understood), and (2) for every stratum M, the closure of M in X is a union of strata. A *vertex* of $X/{\equiv}$ is a point x such that $\{x\}$ is a stratum; let $\mathsf{V}(X/{\equiv})$ denote the set of vertices of $X/{\equiv}$. An *edge* of $X/{\equiv}$ is the closure of a stratum of dimension 1; let $\mathsf{E}(X/{\equiv})$ denote the set of edges of $X/{\equiv}$. A *cellulation* is a stratification such that each stratum is diffeomorphic to some $\mathbb{R}^k$. The cellulation *associated to a triangulation* $h\colon \|\mathcal{K}\| \to P$ is that with strata the h-images of open simplices of $\|\mathcal{K}\|$; a fixed or understood triangulation h of a polyhedron P determines

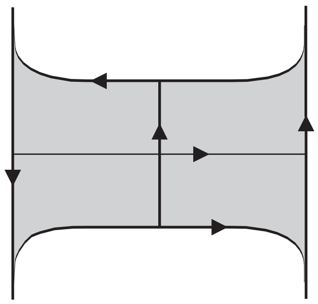

Figure 1. The standard 1-handle and associated arcs.

edges and vertices of P and gives sense to the notations $\mathsf{V}(P)$ and $\mathsf{E}(P)$.

An *arc* is a manifold diffeomorphic to $[0,1]$; in particular, in a real vectorspace if $[\mathbf{u},\mathbf{v}] \neq \{\mathbf{u}\}$ then $[\mathbf{u},\mathbf{v}]$ is an arc oriented from $\mathbf{u}$ to $\mathbf{v}$. An *edge* is an unoriented arc (this is mildly inconsistent with the definition above). A *simple* (*collection of*) *closed curve*(*s*) is a manifold diffeomorphic to (a disjoint union of copies of) S^1. A *graph* is a polyhedron G of dimension ≤ 1 equipped with a cellulation $G/{\equiv}$ (which need not be associated to a triangulation of G). For every graph G, there exists $\mathrm{val}_G\colon \mathsf{V}(G) \to \mathbb{N}$ such that, for every triangulation of G, $\mathrm{val}_G(x) = \mathrm{card}(\{\mathbf{e} \in \mathsf{E}(G) : x \in \mathbf{e}\})$; $\mathrm{val}_G(x)$ is the *valence of* $x \in \mathsf{V}(G)$ *in* G. Call $x \in \mathsf{V}(G)$ an *endpoint* (resp., *isolated point*; *intrinsic vertex*) of G in case $\mathrm{val}_G(x) = 1$ (resp., $\mathrm{val}_G(x) = 0$; $\mathrm{val}_G(x) > 2$). Call $x \in G$ an *ordinary point* in case either $x \in \mathsf{V}(G)$ and $\mathrm{val}_G(x) = 2$ or $x \notin \mathsf{V}(G)$. A graph embedded in $\mathbb{C}$ is *planar*. A *tree* is a finite, connected, acyclic graph. Let $n \in \mathbb{N}_{>0}$. An n-star is a tree with $n+1$ vertices of which at least n are endpoints; an n-*gon* is a 2–disk P equipped with a cellulation having exactly n edges, all in ∂P.

2. Definitions. A *surface* is a compact 2–manifold no component of which has empty boundary. The *genus* of a connected surface S is denoted $g(S)$. A surface is *annular* in case each component is an annulus. A subset X of a surface S is *full* provided that no component of $S \setminus X$ is contractible.

The *standard* (2-dimensional) 0-*handle* is $h^{(0)} := D^2$. Fix some continuous function $H\colon [-2,2] \to [1,2]$ such that: (1) H is even; (2) $H(x) = 1$ for $|x| \leq 1$, $H(2) = 2$, and $H|[1,2]$ is strictly increasing; (3) $H|]-2,2[$ and $(H|]1,2])^{-1}$ are smooth; (4) for $n \in \mathbb{N}_{>0}$, $D^n((H|]1,2])^{-1})(2) = 0$. The *standard* (2-dimensional) 1-*handle* is $h^{(1)} := \{z \in \mathbb{C} : |\operatorname{Im} z| \leq H(\operatorname{Re} z), |\operatorname{Re} z| \leq 2\}$, a 2–manifold with cuspidal corners. The *attaching* (resp., *free*) *arcs* of $h^{(1)}$ are the intervals $[\pm(2-2i), \pm(2+2i)] \subset \mathbb{C}$ (resp., the arcs $\mp\{z \in \mathbb{C} : \operatorname{Im} z = \pm H(\operatorname{Re} z), |\operatorname{Re} z| \leq 2\}$). The union of the attaching arcs of $h^{(1)}$ is the *attaching region* $A(h^{(1)})$ of $h^{(1)}$. The *standard core* (resp., *transverse*) *arc* of $h^{(1)}$ is $\kappa(h^{(1)}) := [-2,2]$ (resp., $\tau(h^{(1)}) := [-i,i]$); a *core arc* (resp., *transverse arc*) of $h^{(1)}$ is any arc isotopic in $h^{(1)}$ to $\kappa(h^{(1)})$ (resp., $\tau(h^{(1)})$). (See Figure 1.)

3. Definitions. Fix some smooth function $g\colon \mathbb{R} \to [0,\pi]$ such that: (1) g is

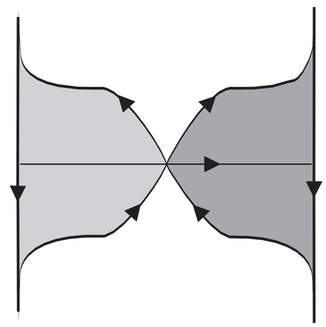

Figure 2. The standard bowtie and associated arcs.

odd, and periodic with period 8; (2) $g'(x) > 0$ for $x \in \,]0,1[$, $g(x) = \pi$ for $x \in [1,3]$, $g'(x) < 0$ for $x \in \,]3,4[$, and $g(4) = 0$. The *standard bowtie* is $\bowtie\, := \Theta(h^{(1)})$, where $\Theta\colon \mathbb{C} \to \mathbb{C} : \; z \mapsto \operatorname{Re} z + i\cos(\pi + g(\operatorname{Re} z)) \operatorname{Im} x$. Give each of the two halves $\bowtie \cap\, i\mathbb{C}_\pm$ of $\bowtie$ the orientation induced on it from $h_0^{(1)}$ by Θ. The *attaching region* of $\bowtie$ is $A(\bowtie) := \Theta(A(h^{(1)}))$; the *attaching arcs* of $\bowtie$ are the components of $A(\bowtie)$. The *standard core arc* of $\bowtie$ is $\Theta(\kappa(h^{(1)}))$. The *crossed arcs* of $\bowtie$ are the Θ-images of the free arcs of $h_0^{(1)}$, and the *crossing* of $\bowtie$ is their point of intersection. (See Figure 2.)

Let S be a 2–manifold. Call

$$(\mathbf{C}) \qquad S = \bigcup_{x\in X} h_x^{(0)} \cup \bigcup_{t\in T} h_t^{(1)}$$

a $(0,1)$-*handle decomposition* of S provided that: (1) X and T are finite; (2) each 0-*handle* $h_x^{(0)}$ is diffeomorphic to the standard 0-handle D^2; (3) each 1-*handle* $h_t^{(1)}$ is diffeomorphic, as a manifold with corners, to the standard 1-handle $h^{(1)}$, and thereby equipped with an attaching region $A(h_t^{(1)})$, attaching arcs, free arcs, a standard core arc $\kappa(h_t^{(1)})$ and other core arcs, a standard transverse arc $\tau(h_t^{(1)})$ and other transverse arcs; (4) the 0-handles are pairwise disjoint, as are the 1-handles; (5) $h_t^{(1)} \cap \bigcup_{x\in X} h_x^{(0)} = h_t^{(1)} \cap \partial \bigcup_{x\in X} h_x^{(0)} = A(h_t^{(1)})$ for each $t \in T$; and (6) the orientations of S and all the 0- and 1-handles are compatible. A 2–manifold S has a $(0,1)$-handle decomposition if and only if S is a surface.

1.3 Smooth maps

Maps between manifolds are smooth ($\mathcal{C}^\infty$), and isotopies are ambient, unless otherwise stated. Given manifolds M and N, let $\operatorname{Diff}(M,N)$ denote the set of maps $M \to N$, with a suitable topology. Let $f \in \operatorname{Diff}(M,N)$. The *$f$-multiplicity* $x \in M$ is $\operatorname{card}(f^{-1}(x))$. A point of f-multiplicity 1 (resp., 2; at least 2) is a *simple point* (resp., *double point*; *multiple point*) of f; the image by f of a simple (resp., double; multiple) point of f is a *simple* (resp., *double*;

multiple) *value* of f. Let $\operatorname{simp}(f) := \{x \in M : x \text{ is a simple point of } f\}$, $\operatorname{doub}(f) := \{x \in M : x \text{ is a double point of } f\}$, $\operatorname{mult}(f) := \{x \in M : x \text{ is a multiple point of } f\}$. Denote by $Df(x)\colon T_x(M) \to T_{f(x)}(N)$ the derivative of f at $x \in M$, and by $Df\colon T(M) \to T(N)$ the map induced by f on tangent bundles. A *critical point* of f is any $x \in M$ with $\operatorname{rank}(Df(x)) < \dim T_{f(x)}(N)$. The set of critical points of f is denoted $\operatorname{crit}(f)$, so $f(\operatorname{crit}(f))$ is the set of *critical values* of f. As usual, $y \in N \setminus f(\operatorname{crit}(f))$ is called a *regular value* of f (even if $y \notin f(M)$). For $\dim(N) = 1$, the index of f at $x \in \operatorname{crit}(f)$ is denoted $\operatorname{ind}(f; x)$. Call f a *Morse function* (resp., *Morse map*) in case $N = \mathbb{R}$ (resp., $N = S^1$), f is constant on ∂M, and every $x \in \operatorname{crit}(f)$ is non-degenerate; in case also $\operatorname{ind}(f; x) < \dim M$ for all $x \in \operatorname{crit}(f)$, call f *topless*. Call $f \in \operatorname{Diff}(M, N)$ an *immersion* in case $Df(x)\colon T_x(M) \to T_{f(x)}(N)$ is injective for every $x \in M$. An *embedding* is an immersion that is a homeomorphism onto its image. Write $f\colon M \looparrowright N$ (resp., $f\colon M \hookrightarrow N$) to indicate that f is an immersion (resp., embedding). The *normal bundle* of $f\colon M \looparrowright \mathbb{N}$ is denoted $\nu(f)$; given a submanifold $M \subset N$, the *normal bundle of M in N* is $\nu(\mathrm{i}_{M\hookrightarrow N})$, where $\mathrm{i}_{M\hookrightarrow N}$ denotes inclusion.

Here are various constructions with normal bundles, in the course of which assorted notations and definitions are established.

4. Definitions. Let M be a manifold of dimension m.

4.1. Let $Q \subset M$ be a submanifold of dimension $m-1$ with trivial normal bundle. A *collaring* of Q in M is an orientation-preserving embedding $\operatorname{col}_{Q\subset M}\colon Q \times [0,1] \hookrightarrow \operatorname{Nb}(Q \subset M)$ with $\operatorname{col}_{Q\subset M}(q, 0) = q$ for all q; a *collar* of Q in M is the image $\operatorname{Col}(Q \subset M) := \operatorname{col}_{Q\subset M}(Q \times [0,1])$ of a collaring. Let $M \wr Y := M \setminus \operatorname{col}_{Q\subset M}(Q \times \,]0,1[)$ be called M *cut* along Y. The *push-off map* of Q is $Q \hookrightarrow M \setminus Q\colon q \mapsto \operatorname{col}_{Q\subset M}(q, 1)$. Call the image Y^+ of $Y \subset Q$ by the push-off map the *push-off* of Y. (Note that $Q^+ \subset \partial \operatorname{Col}(Q \subset M)$ has the "outward normal" orientation, whereas $Q \hookrightarrow \partial \operatorname{Col}(Q \subset M)$ reverses orientation.) It is convenient to define various *standard* collarings, thus. (a) Given $Q \subset S^n = \partial D^{n+1}$, let $\operatorname{col}_{Q\subset D^{n+1}} : (x, t) \mapsto \epsilon(1-t)x$ for a suitable $\epsilon \in \,]0,1[$. (b) Given a manifold W, $u \in \mathbb{R}_{>0} \cup \{\infty\}$, and $Q \subset \operatorname{Int} W \times \{0\} \subset \partial W \times [0, \pm u[$, let $\operatorname{col}_{Q\subset \partial W\times[0,\pm u[} : (x, t) \mapsto (x, \pm\epsilon t)$ for a suitable $\epsilon \in \mathbb{R}_{>0}$. (c) For a suitable $\epsilon\colon\,]-2, 2[\, \to \,]0,1[$, $(\pm 2 + iy, t) \mapsto \pm 2 + i\epsilon(t)y$ is a collaring $\operatorname{col}_{\operatorname{Int} A(h^{(1)})\subset h^{(1)}}$. (The cusps of $h^{(1)}$ prevent the existence of a collaring of $A(h^{(1)})$ in $h^{(1)}$.)

4.2. Let Q be a manifold of dimension $q < m$, $j\colon Q \looparrowright M$ an immersion, $B := j(Q)$. For $x \in B \setminus j(\partial Q \cup \operatorname{mult}(j))$, a *meridional* $(m-q)$*–disk of B at* x is the image $\pitchfork(B; x)$ of an embedding $f\colon D^{m-q} \hookrightarrow M$ such that $f^{-1}(B) = f^{-1}(x) = 0$, f is transverse to j, and $\pitchfork(B; x)$ intersects B positively (with respect to given orientations of M and Q). In case M is connected and $q = m-2$, any element of $\pi_1(M \setminus B)$ represented by a loop freely homotopic to $\partial\pitchfork(B; x)$ in $M \setminus B$ is called a *meridian* in that knotgroup.

4.3. Let $B \subset M$ be a submanifold of dimension $m-2$ with trivial normal bundle. Let $n\colon\ B\times\mathbb{C} \to \nu(B)$ be a fixed trivialization of $\nu(B)$. In the standard way, using (inexplicit) metrics, etc., identify $\mathrm{Nb}(B \subset M)$ to $n(B \times D^2)$ in such a way that if $x \in B$, then the image of $D^2 \to M\ :\ z \mapsto n(x,z)$ is a meridional 2–disk $⫛(B;x)$. Call $f\colon\ M \setminus B \to S^1$ *weakly adapted* to n in case for every $Q \in \pi_0(B)$ there is an integer $d(Q)$ such that, if $d(Q) \neq 0$, then $f(n(x,z)) = (z/|z|)^{d(Q)}$ for all $x \in Q$, $z \in D^2 \setminus \{0\}$. Call $f\colon\ M \setminus B \to S^1$ *adapted* in case f is weakly adapted and, in addition, if $d(Q) = 0$, then f extends to $f_Q \in \mathrm{Diff}(M \setminus B \setminus Q, S^1)$, $f_Q|Q\colon\ Q \to S^1$ is an immersion, and $f_Q|⫛(B;x)$ is constant for each $x \in Q$.

5. Definitions. Let $f\colon\ M \to N$ be smooth; let Q be a codimension-0 submanifold of ∂M. Call f *proper along* Q, *relative to* $\mathrm{col}_{\partial N\subset N}$ with $\mathrm{Col}(\partial N \subset N) \subset \mathrm{Nb}(\partial N \subset N)$ and $\mathrm{col}_{Q\subset M}$, provided that: (1) $f(Q) \subset \partial N$ and $f(M \setminus Q) \subset \mathrm{Int}\, N$; (2) $f^{-1}(\mathrm{Nb}(\partial N \subset N)) \subset M$ is a submanifold, and $f|f^{-1}(\mathrm{Nb}(\partial N \subset N))$ is an embedding; (3) $f \circ \mathrm{col}_{Q\subset M}\colon\ \mathrm{Col}(Q \subset M) \hookrightarrow N$ is an embedding into $\mathrm{Nb}(\partial N \subset N)$ and $\mathrm{pr}_2 \circ (\mathrm{col}_{\partial N\subset N})^{-1} \circ f \circ \mathrm{col}_{Q\subset M} = \mathrm{pr}_2\colon\ \mathrm{Col}(Q \subset M) \to [0,1]$. Call f *proper along* Q in case there exist collarings $\mathrm{col}_{Q\subset M}$ and $\mathrm{col}_{\partial N\subset N}$ such that f is proper along Q relative to $\mathrm{col}_{Q\subset M}$ and $\mathrm{col}_{\partial N\subset N}$ ("along Q" is dropped when $Q = \partial M$). A properly embedded submanifold is simply *proper*.

Some special cases of low-dimensional immersions and embeddings of particular interest, and associated ancillary constructions, need extra terminology.

6. Definitions. Let M be a compact m–manifold, N an n–manifold. Let $f\colon\ M \looparrowright N$ be an immersion with $\mathrm{mult}(f) = \mathrm{doub}(f)$.

6.1. Let $m = 1$, and suppose M is an arc or an edge. Call f *half-proper* provided that it is proper along a single component of ∂M. A *half-proper* arc or edge is the image of a half-proper embedding. An n-star embedded in N is *proper* provided each of its edges is half-proper.

6.2. Let $m = 1$, $n = 2$. Call f *normal* provided that f is proper (whence $\mathrm{doub}(f) \subset \mathrm{Int}\, M$), $\mathrm{doub}(f)$ is finite, and if $f(x_1) = f(x_2)$ with $x_1 \neq x_2$, then the lines $Df(x_1)(T_{x_1}(M))$ and $Df(x_2)(T_{x_2}(M))$ in $T_{f(x_1)}(N)$ are transverse. A *crossing* of a normal immersion $f\colon\ M \looparrowright N$ is a double value of f; the *crossing number* of f is $\mathrm{cross}(f) := \mathrm{card}(f(\mathrm{doub}(f)))$. A *branch* of f at a crossing y is the (germ at y of the) f-image of either component of $\mathrm{Nb}(f^{-1}(y) \subset M \setminus (\mathrm{doub}(f) \setminus f^{-1}(y)))$. The image of a normal immersion of S^1 (resp., an arc; a finite disjoint union of copies of S^1; a finite disjoint union of arcs) is a *normal closed curve* (resp., a *normal arc*; a *normal collection of closed curves*; a *normal collection of arcs*); the *crossing number* of the normal collection of closed curves or arcs $f(M)$ is $\mathrm{cross}(f(M)) := \mathrm{cross}(f)$. A *bowtie* in N is the image of $\bowtie \subset h^{(1)}$ by an *embedding* of $\bowtie$ in N, that is, a map $\delta\colon\ \bowtie \to N$ that extends to an embedding $h^{(1)} \hookrightarrow N$. Let $C := f(M)$ be a normal collection

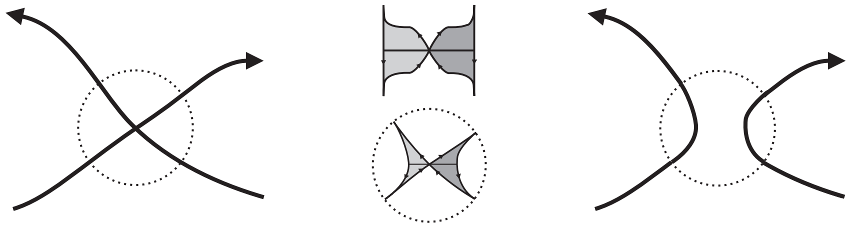

Figure 3. Local smoothing, using a bowtie.

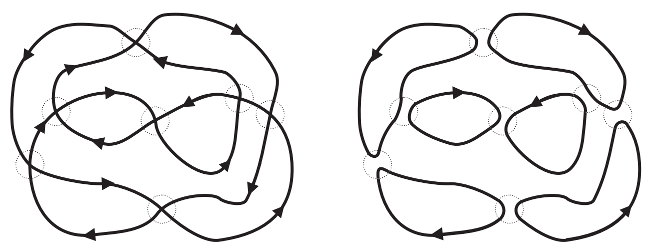

Figure 4. A normal closed curve in $\mathbb{R}^2$ and its smoothing.

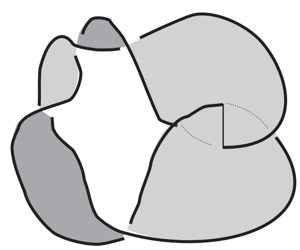

Figure 5. A clasp surface in $\mathbb{R}^3$.

of closed curves. For each $y \in f(\mathrm{doub}(f))$, let $\bowtie_y = \delta_y(\bowtie)$ be a bowtie with $\bowtie_y \subset \mathrm{Nb}(\{y\} \subset N \setminus f(\mathrm{doub}(f)) \setminus \{y\})$, such that the *crossed arcs* of $\bowtie_y$ (i.e., the δ_y-images of the crossed arcs of $\bowtie$) are the branches of f at y, correctly oriented. The *local smoothing* of C at y (see Figure 3) is the normal collection of closed curves $\mathrm{sm}(C; y) =: \mathrm{sm}(f; y)(\mathrm{sm}(M; y))$ created by replacing the crossed arcs of $\bowtie_y$ with $-\delta_y(A(\bowtie))$. Here, $\mathrm{sm}(M; y)$ is unique up to diffeomorphism, and $\mathrm{sm}(C; y)$ up to (arbitrarily small) isotopy; and $\mathrm{cross}(\mathrm{sm}(C; y)) = \mathrm{cross}(C) - 1$. The *smoothing* of C (see Figure 4) is the simple collection of closed curves $\mathrm{sm}(C) := \mathrm{sm}(\cdots \mathrm{sm}(\mathrm{sm}(C; y_1); y_2) \cdots ; y_{\mathrm{cross}(C)}) \subset N$, independent of the enumeration $\{y_1, \ldots, y_{\mathrm{cross}(C)}\}$ of $f(\mathrm{doub}(f))$.

6.3. Let $m = 2$, $n = 3$. Call f *clasp* provided that $\mathrm{doub}(f)$ is the union of finitely many pairwise disjoint edges $A'_1, A''_1, \ldots, A'_s, A''_s$ with $f(A'_i) = f(A''_i) \subset \mathrm{Int}\, N$, both A'_i and A''_i half-proper ($i = 1, \ldots, s$); call s the *clasp number* of the clasp immersion f, and denote it by $\mathrm{clasp}(f)$. The image $f(S)$ is a *clasp surface* (see Figure 5); the *clasp number* of $f(S)$ is $\mathrm{clasp}(f(S)) := \mathrm{clasp}(f)$.

6.4. Let $m = 2$, $n = 3$. Call f *ribbon* provided that $\mathrm{doub}(f)$ is the union of finitely many pairwise disjoint edges $A'_1, A''_1, \ldots, A'_s, A''_s$ with A'_i proper, A''_i interior, and $f(A'_i) = f(A''_i) \subset \mathrm{Int}\, N$. The image of a ribbon immersion of a

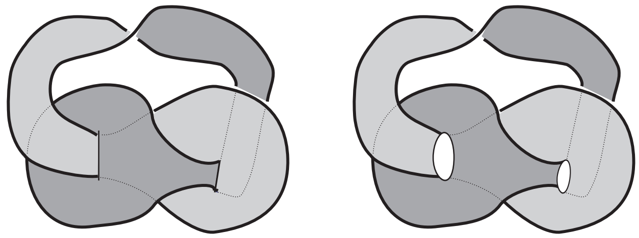

Figure 6. A ribbon surface in $\mathbb{R}^3$ and its smoothing.

surface is a *ribbon surface* (see Figure 6).For connected S, the *genus* of the ribbon surface $f(S)$ is $g(f(S)) := g(S)$. The *smoothing* of a ribbon surface $R = f(S)$ is an embedded surface $\mathrm{sm}(R) \subset N$, unique up to isotopy, constructed as follows. Let $S_0 := S \wr (\bigcup_{i=1}^s A'_i \cup \bigcup_{i=1}^s \operatorname{Int} A''_i)$. Let $\equiv$ be the equivalence relation on S_0 having as its non-trivial equivalence classes $\{x, y\}$ and $\{x^+, y^+\}$ $(x \in \operatorname{Int} A'_i,\ f(x) = f(y))$ and $\{x, x^+, y\}$ $(x \in \partial A'_i,\ f(x) = f(y))$, $i = 1, \ldots, s$. There is a natural way to smooth the quotient manifold $S_1 := S_0/{\equiv}$, and a natural map $f_1\colon\ S_1 \to N$ with $f_1(S_1) = f(S)$, which after an arbitrarily small perturbation yields an embedding $S_1 \hookrightarrow N$ with image $\mathrm{sm}(f(S))$. [5]

6.5. Let $m = 2$, $n = 4$. Call f *nodal* provided that f is proper, $\mathrm{doub}(f)$ is finite, and if $f(x_1) = f(x_2)$ and $x_1 \neq x_2$, then the planes $Df(x_1)(T_{x_1}(M))$ and $Df(x_2)(T_{x_2}(M))$ in $T_{f(x_1)}(N)$ are transverse. A *node* of a nodal immersion f is a double value of f; the *node number* of f is $\mathrm{node}(f) := \mathrm{card}(f(\mathrm{doub}(f)))$. A *branch* f at a node y is the (germ at y of the) f-image of either component of $\mathrm{Nb}(f^{-1}(y) \subset M \setminus \mathrm{doub}(f) \setminus f^{-1}(y))$. The *sign* $\varepsilon(y)$ of the node y is the sign ($+$ or $-$) of the given orientation of $T_y(N)$ with respect to its orientation as the direct sum of the oriented 2–planes $Df(x_1)(T_{x_1}(M))$ and $Df(x_2)(T_{x_2}(M))$ (in either order), where $f(x_1) = f(x_2) = y$, $x_1 \neq x_2$. The image of a nodal immersion of a surface is a *nodal surface*. The *node number* of a nodal surface $f(M)$ is $\mathrm{node}(f(M)) := \mathrm{node}(f)$, and its *smoothing* is an embedded surface $\mathrm{sm}(f(M)) \subset N$, unique up to isotopy, constructed by replacing $\mathrm{Nb}(f(\mathrm{doub}(f)) \subset f(M))$ with $\mathrm{card}(f(\mathrm{doub}(f)))$ annuli embedded in $\mathrm{Nb}(f(\mathrm{doub}(f)) \subset N)$ in a standard way. (In appropriate local coordinates on N, the neighborhood on $f(M)$ of a positive node is $\{(z, w) \in D^4 \subset \mathbb{C}^2 : zw = 0\}$, and is replaced by $\{(z, w) \in D^4 : zw = \epsilon(1 - \|(z, w)\|^4)\}$, where $\epsilon\colon\ [0, 1] \to \mathbb{R}_{\geq 0}$ is smooth, 0 near 0, positive near 1, and sufficiently small.)

[5] Fox (1962) introduced the word "ribbon" into knot theory (specifically, in the context of ribbon-immersed 2–disks). His usage was soon generalized (Tristram, 1969) and widely adopted. In a distinct chain of development, the biologist Crick (1976), followed by physicists (Grundberg, Hansson, Karlhede, and Lindström, 1989) and other scientists applying mathematics, gave the word quite a different meaning (perhaps closer to its everyday use), essentially to refer to twisted 2-dimensional bands. More recently this conflicting usage has been adopted by some knot-theorists (see, e.g., Reshetikhin and Turaev, 1990), particularly of a categorical bent. *Caveat lector.*

6.6. Let $m = 2$, $n = 4$. Call f *slice* provided that f is proper. A *slice surface* is the image of a slice embedding of a surface (i.e., a proper surface).[6]

6.7. Let $m = 2$, $n = 4$, and suppose that N is compact and $\rho\colon N \to \mathbb{R}$ is a topless Morse function. Call f *ρ-ribbon* provided that f is slice and $\rho \circ f$ is a topless Morse function on M. In case $N = D^4$ and $\operatorname{card}(\operatorname{crit}(\rho)) = 1$ (so that, up to diffeomorphisms of N and $\mathbb{R}$, $\rho = \|\cdot\|^2$), a ρ-ribbon embedding is called simply a *ribbon embedding*, and its image is called a *ribbon surface* in D^4.

There is a close relation between ribbon surfaces in D^4 and in S^3.

7. Proposition. *Let M be a surface. If $f\colon M \looparrowright S^3 = \partial D^4$ is a ribbon immersion, then there is a non-ambient isotopy $\{f_t\colon M \looparrowright D^4\}_{t\in[0,1]}$ such that $f_0 = f$, $f_t|\partial M = f_0|\partial M$ for $t \in [0,1]$, and $f_t\colon M \hookrightarrow D^4$ is a ribbon embedding for $t \in\,]0,1]$; conversely, if $g\colon M \hookrightarrow \partial D^4$ is ribbon, then $g = f_1$ for some such non-ambient isotopy $\{f_t\colon M \looparrowright D^4\}_{t\in[0,1]}$ with $f_0\colon M \looparrowright S^3$ ribbon.* (*Although the first of these non-ambient isotopies is unique up to ambient isotopy, the second enjoys no such uniqueness.*) □

See Tristram (1969), Hass (1983), or Rudolph (1985b) for more detailed statements and proofs.

A (smooth) *covering map* is an orientation-preserving immersion $f\colon M \looparrowright N$ that is also a topological covering map (it is not required that the domain of a topological covering map be connected). The usual theory of covering maps is assumed—in particular, the well-behaved, albeit many–many, correspondence for connected N between permutation representations $\rho\colon \pi_1(N;*) \to \mathfrak{S}_s$ and covering maps f with target N and *base fiber* $f^{-1}(*) = \mathsf{s}$. Given $s \in \mathbb{N}_{>0}$ and $\mathsf{g} \in \mathfrak{S}_s$, let $\rho_{\mathsf{g}}\colon \pi_1(S^1;1) \to \mathfrak{S}_s$ be the permutation representation $\circlearrowleft^k \mapsto \mathsf{g}^k$, where $\circlearrowleft$ denotes the *positive generator* of $\pi_1(S^1;1) \cong \pi_1(D^2 \setminus \{0\};1)$, that is, the homotopy class of id_{S^1}. Call $\Xi_{\rho_{\mathsf{g}}} := (D^2 \setminus \{0\}) \times \mathsf{s}/\langle\mathsf{g}\rangle$ the *standard covering space of D^2 of type* g and $\xi_{\rho_{\mathsf{g}}}\colon \Xi_{\rho_{\mathsf{g}}} \to D^2 : (z, t\langle\mathsf{g}\rangle) \mapsto z^{\operatorname{card}(t\langle\mathsf{g}\rangle)}$ the *standard covering map of D^2 of type* g.

The theory of *branched covering maps* originated in complex analysis and algebraic geometry. Following earlier work of Heegard (1898) and Tietze (1908), the theory was adapted to combinatorial manifolds by Alexander (1920) and Reidemeister (1926), then to more general spaces by Fox (1957). Durfee and Kauffman (1975) made the construction more precise in the smooth category. The *ad hoc* approach of **8** and **9**, below, suffices to handle the cases that are most important for the knot theory of complex plane curves.

8. Definitions. Let s, g, etc., be as above. The *standard branched covering*

[6] The redundant term "slice surface" has been retained for the sake of tradition. See Rudolph (1993, §1) for a history of the use of the word "slice" in knot theory.

space (resp., *map*) *of* D^2 *of type* g is $\widetilde{\widetilde{\Xi}}_{\rho_\mathsf{g}} := D^2 \times \mathbf{s}/\langle \mathsf{g}\rangle$ (resp., $\widetilde{\widetilde{\xi}}_{\rho_\mathsf{g}} \colon \widetilde{\widetilde{\Xi}}_{\rho_\mathsf{g}} \to D^2 : (z, t\langle \mathsf{g}\rangle) \mapsto z^{\operatorname{card}(t\langle \mathsf{g}\rangle)}$). Let N be a connected n–manifold equipped with a stratification $N/{\equiv}$ such that: (1) every stratum of $N/{\equiv}$ is smoothly immersed in N; (2) no stratum of $N/{\equiv}$ has dimension $n-1$ (whence there is a unique stratum N_0 of dimension n) or $n-3$; and (3) $N \setminus N_0$ is the closure in N of the union B of all $(n-2)$-dimensional strata. A smooth map $f\colon M \to N$ is a *branched covering map of* N *branched over* B, and M is a *branched covering space of* N *branched over* B, provided that: (4) $f(\operatorname{crit}(f)) \subset B$; (5) $f|(f^{-1}(N \setminus B)$ is a covering map of degree s; and (6) for every $x \in B$, there exist $\mathsf{g}(x) \in \mathfrak{S}_s$ (the *type of* f *at* x), an embedding $\varphi\colon D^2 \hookrightarrow N$ onto a meridional disk $⋔(B; x)$, and an embedding $\Phi\colon \widetilde{\widetilde{\Xi}}_{\rho_{\mathsf{g}(x)}} \hookrightarrow M$ with $(f|(f^{-1}(⋔(B; x)))) \circ \Phi = \varphi \circ \widetilde{\widetilde{\xi}}_{\rho_{\mathsf{g}(x)}}$. The conjugacy class of $\mathsf{g}(x)$ in $\mathfrak{S}_s$ is constant on each component of B, and $B = \{x \in f(\operatorname{crit}(f)) : \mathsf{g}(x) \neq \mathrm{id}_\mathbf{s}\}$. The branched covering f is called *simple* in case $\mathsf{g}(x)$ is a transposition for each $x \in B$.

9. Construction. Let N be a connected 2–manifold, $B \subset \operatorname{Int} N$ a non-empty finite subset. Fix an enumeration $B =: \{x_1, \ldots, x_q\}$. Let $*_0 \in \operatorname{Ext}(B \subset \operatorname{Int} N)$. The regular neighborhood $\operatorname{Nb}(B \subset N)$ is a union of pairwise disjoint meridional 2–disks $D_i^2 := ⋔(B; x_i)$, $i \in \mathbf{q}$. Let $*_i \in \partial D_i$. Fix a proper q-star $\psi \subset \operatorname{Ext}(B \subset N)$ with $\mathsf{V}(\psi) = \{*_0, *_1, \ldots, *_q\}$ and $\mathsf{E}(\psi) =: \{\mathbf{e}_1, \ldots, \mathbf{e}_q\}$, such that $\partial \mathbf{e}_k = \{*_0, *_k\}$ and the cyclic order of $\mathsf{E}(\psi)$ at $*_0$ (with respect to the orientation of N) is the cyclic order of their indices $1, \ldots, q$. Let $\Psi := \psi \cup \bigcup_{i=1}^q (D_i \setminus x_i)$. Let $\mathbf{g}_i \in \pi_1(\Psi; *_0)$ be the element represented by a loop that traverses the edge $\mathbf{e}_i$ of ψ from $*_0$ to $*_i$, represents $\circlearrowleft$ in $\pi_1(\partial D_i^2; *_i) \cong \pi_1(S^1; 1)$, and returns to $*_0$ along $\mathbf{e}_i$. Evidently $\pi_1(\Psi; *_0)$ is the free group $\operatorname{gp}\left(\mathbf{g}_i, i \in \mathbf{q}\colon \varnothing\right) \cong \pi_1(\partial D_1^2) * \cdots * \pi_1(\partial D_q^2)$. Call $\mathsf{g} =: (\mathsf{g}_1, \ldots, \mathsf{g}_q) \in \mathfrak{S}_s^q$ *compatible with* ψ (or Ψ) in case there exists a permutation representation $\rho_\mathsf{g}\colon \pi_1(\operatorname{Ext}(B \subset N); *_0) \to \mathfrak{S}_s$ such that $\rho_\mathsf{g}(\mathbf{g}_i') = \mathsf{g}_i$ ($i \in \mathbf{q}$), where $\mathbf{g} \mapsto \mathbf{g}'$ is the inclusion-induced homomorphism $\pi_1(\Psi; *_0) \to \pi_1(N \setminus B; *_0) \cong \pi_1(\operatorname{Ext}(B \subset N); *_0)$. (If N is closed then compatibility is a genuine restriction; if N is not closed, then every g is compatible with ψ, but ρ_g may not be unique if N is not contractible.) Finally, given Ψ, a compatible q-tuple g, and ρ_g, construct $\widetilde{\widetilde{\Xi}}_{\rho_\mathsf{g}}$ as the identification space (with an appropriate, easily defined smooth structure) of the disjoint union of copies of $\widetilde{\widetilde{\Xi}}_{\rho_{\mathsf{g}_i}}$ ($i \in \mathbf{q}$) and Ξ_{ρ_g} along their tautologously diffeomorphic boundaries $\widetilde{\widetilde{\xi}}_{\rho_{\mathsf{g}_i}}^{-1}(\partial D_i^2) = \partial \widetilde{\widetilde{\Xi}}_{\rho_{\mathsf{g}_1}}$ ($i \in \mathbf{q}$) and $\xi_{\rho_\mathsf{g}}^{-1}(\partial D_i^2) \subset \partial \Xi_{\rho_\mathsf{g}}$; let $\widetilde{\widetilde{\xi}}_{\rho_\mathsf{g}} | \widetilde{\widetilde{\Xi}}_{\rho_{\mathsf{g}_i}} = \widetilde{\widetilde{\xi}}_{\rho_{\mathsf{g}_i}}$ and $\widetilde{\widetilde{\xi}}_{\rho_\mathsf{g}} | \Xi_{\rho_\mathsf{g}} = \xi_{\rho_\mathsf{g}}$.

1.4 Knots, links, and Seifert surfaces

A *link* is a simple collection of closed curves embedded in S^3; except where otherwise stated, isotopic links are treated as identical. A 1-component link is a *knot*. For $n \in \mathbb{N}$, $O^{(n)}$ denotes the *n-component unlink*, that is, the boundary

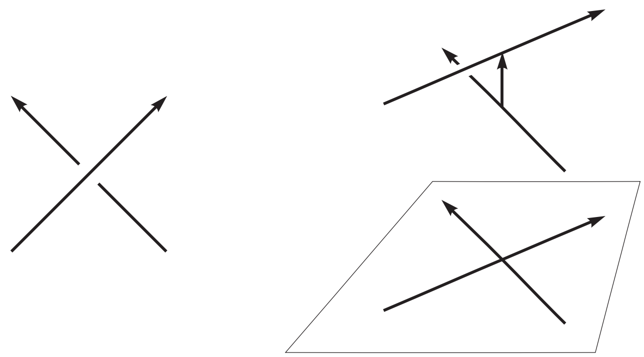

Figure 7. A positive crossing in a standard link diagram.

of n pairwise disjoint copies of D^2 embedded in S^3; the *unknot* is $O := O^{(1)}$.

10. Definitions. A *link diagram* is a pair $\mathbf{D}(L) =: (\mathbf{P}(\mathbf{D}(L)), \mathbf{I}(\mathbf{D}(L)))$, where (1) $L \subset \mathbb{R}^3 \subset S^3$ is a link (called *the link of* the diagram); (2) $\mathbf{P}(\mathbf{D}(L))$, the $\mathbf{D}(L)$-*picture* of L, is the image of L by an affine projection $\mathbf{p}_{\mathbf{D}(L)} \colon \mathbb{R}^3 \to \mathbb{C}$; and (3) $\mathbf{I}(\mathbf{D}(L))$, the $\mathbf{D}(L)$-*information* about L, is information sufficient to reconstruct from $\mathbf{P}(\mathbf{D}(L))$ the embedding of L into $\mathbb{R}^3$, up to isotopy respecting the fibers of $\mathbf{p}_{\mathbf{D}(L)}$. The *mirror image* of $\mathbf{D}(L)$ is the link diagram $\mathbf{D}(\operatorname{Mir} L)$, where $\mathbf{P}(\mathbf{D}(\operatorname{Mir} L)) := \overline{\mathbf{P}(\mathbf{D}(L))}$, and $\mathbf{p}_{\mathbf{D}(L)}$ and $\mathbf{I}(\mathbf{D}(L))$ are modified accordingly to produce $\mathbf{p}_{\mathbf{D}(\operatorname{Mir} L)}$ and $\mathbf{I}(\mathbf{D}(\operatorname{Mir} L))$; of course the link of $\mathbf{D}(\operatorname{Mir} L)$ is the mirror image $\operatorname{Mir} L$ of L as already defined.

The nature of the information $\mathbf{I}(\mathbf{D}(L))$ can be of various sorts, depending on context. For instance, $\mathbf{D}(L)$ is a *standard link diagram* for L provided that (1) $\mathbf{P}(\mathbf{D}(L))$ is a normal collection of closed curves, and (2) $\mathbf{I}(\mathbf{D}(L))$ consists of (a) the global information that $\mathbf{p}_{\mathbf{D}(L)}|L \colon L \to \mathbb{C}$ is a normal immersion with image $\mathbf{P}(\mathbf{D}(L))$, supplemented by (b) local information at each crossing specifying which branch is "under" and which is "over"—equivalently, which of the two points of $L \cap \mathbf{p}^{-1}_{\mathbf{D}(L)}(z)$ is the *undercrossing* $z_\frown$ (initial endpoint) and which is the *overcrossing* $z^\smile$ (terminal endpoint) of the interval between them on $\mathbf{p}^{-1}_{\mathbf{D}(L)}(z)$, when the standard orientations of $\mathbb{R}^3$ and $\mathbb{C}$ are used to orient the fibers of $\mathbf{p}_{\mathbf{D}(L)}$. (It is usual to depict crossings in the style of the left half of Figure 7.) Every link has many different standard link diagrams (some satisfying further conditions), as well as non-standard link diagrams of various types, some of which will be introduced later as needed.

11. Definitions. Let $\mathbf{D}(L)$ be a standard link diagram.

11.1. A *Seifert cycle*[7] of $\mathbf{D}(L)$ is any $\mathsf{o} \in \mathsf{O}_{\mathbf{D}(L)} := \pi_0(\operatorname{sm}(\mathbf{P}(\mathbf{D}(L))))$. The

[7] The more commonly used term "Seifert circle" seems to have been popularized, if not coined, by Fox (1962; see also a 1963 review in which Fox glosses Murasugi's "standard loops" as "Seifert circles"). Certainly Seifert's term "Kreis" does mean "circle", but it can also be translated as "cycle", and in the expositon of Seifert's

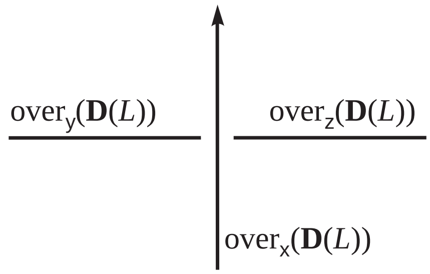

Figure 8. Labelled over-arcs (the unlabelled crossing need not be x).

inside of a Seifert cycle o is the 2–disk $\iota(\mathsf{o}) \subset \mathbb{C}$ oriented so that $\partial\iota(\mathsf{o}) = \mathsf{o}$. The *sign* $\varepsilon(\mathsf{o})$ of o is the sign, positive $(+)$ or negative $(-)$, of the orientation of $\iota(\mathsf{o})$ with respect to the standard orientation of $\mathbb{C}$. Let $\mathsf{O}^{+}_{\mathbf{D}(L)}$ (resp., $\mathsf{O}^{-}_{\mathbf{D}(L)}$) be the set of positive (resp., negative) Seifert cycles of $\mathbf{D}(L)$. Given $\mathsf{o}, \mathsf{o}' \in \mathsf{O}_{\mathbf{D}(L)}$, say that o *encloses* o', and write $\mathsf{o} \Supset \mathsf{o}'$, in case $\operatorname{Int}\iota(\mathsf{o}) \supset \mathsf{o}'$. Call $\mathbf{D}(L)$ *nested* (resp., *scattered*) in case $\mathsf{O}_{\mathbf{D}(L)}$ is an $\Supset$-chain (resp., $\Supset$-antichain).

11.2. A *crossing* of $\mathbf{D}(L)$ is any $\mathsf{x} \in \mathsf{X}_{\mathbf{D}(L)} := \mathbf{p}_{\mathbf{D}(L)}(\mathrm{doub}(\mathbf{p}_{\mathbf{D}(L)}))$. Let $\mathsf{x} \in \mathsf{X}_{\mathbf{D}(L)}$. An *over-arc of* $\mathbf{D}(L)$ *on* L *at* x is any arc $\mathrm{over}_{\mathsf{x}}(\mathbf{D}(L)) \subset L$ such that (a) $\mathsf{x}^{\smile} \in \mathrm{over}_{\mathsf{x}}(\mathbf{D}(L))$, (b) $\mathrm{over}_{\mathsf{x}}(\mathbf{D}(L)) \cap \{\mathsf{y}_{\frown} : \mathsf{y} \in \mathsf{X}_{\mathbf{D}(L)}\} = \varnothing$, and (c) among arcs $\alpha \subset L$ satisfying (a) and (b), $\mathrm{over}_{\mathsf{x}}(\mathbf{D}(L))$ maximizes $\mathrm{card}(\alpha \cap \{\mathsf{y}^{\smile} : \mathsf{y} \in \mathsf{X}_{\mathbf{D}(L)}\})$. An *over-arc of* $\mathbf{D}(L)$ *in* $\mathbf{P}(\mathbf{D}(L))$ *at* x is any arc $\mathbf{p}_{\mathbf{D}(L)}(\mathrm{over}_{\mathsf{x}}(\mathbf{D}(L))) \subset \mathbf{P}(\mathbf{D}(L))$. Let $\mathbf{u}^{\smile}$ (resp., $\mathbf{u}_{\frown}$) be a positively oriented basis vector for $T_{x^{\smile}}(L)$ (resp., $T_{x_{\frown}}(L)$). The *sign* $\varepsilon(\mathsf{x})$ of x is the sign, positive $(+)$ or negative $(-)$, of the frame $(\mathbf{u}_{\frown}, \mathsf{x}^{\smile} - \mathsf{x}_{\frown}, \mathbf{u}^{\smile})$ with respect to the standard orientation of $\mathbb{R}^3$. (The crossing in Figure 7 is positive.) Let $\mathsf{X}^{+}_{\mathbf{D}(L)}$ (resp., $\mathsf{X}^{-}_{\mathbf{D}(L)}$) be the set of positive (resp., negative) crossings of $\mathbf{D}(L)$.

11.3. Call $\mathsf{o} \in \mathsf{O}_{\mathbf{D}(L)}$ *adjacent* to $C \subset \mathsf{X}_{\mathbf{D}(L)}$ in case, for some $\mathsf{x} \in C$, o contains an attaching arc of the bowtie $\bowtie_{\mathsf{x}}$ used to construct $\mathrm{sm}(\mathbf{P}(\mathbf{D}(L)))$. Let $\mathsf{O}^{\geq}_{\mathbf{D}(L)}$ (resp., $\mathsf{O}^{<}_{\mathbf{D}(L)}$; $\mathsf{O}^{\varnothing}_{\mathbf{D}(L)}$) denote the set of Seifert cycles adjacent to $\mathsf{X}^{+}_{\mathbf{D}(L)}$ (resp., not adjacent to $\mathsf{X}^{+}_{\mathbf{D}(L)}$; not adjacent $\mathsf{X}_{\mathbf{D}(L)}$).

12. Theorem. *Let $\mathbf{D}(L)$ be a standard link diagram. If $C \subset \mathsf{X}_L$ is any set of minimal cardinality such that $\mathsf{X}_L \subset \cup_{\mathsf{x}\in C}\mathbf{p}_{\mathbf{D}(L)}(\mathrm{over}_{\mathsf{x}}(\mathbf{D}(L)))$, then the knot-group of L has a Wirtinger presentation*

$$(\mathbf{D}) \qquad \pi_1(S^3 \setminus L) = \mathrm{gp}\left(\mathbf{g}_{\mathsf{x}}\,(\mathsf{x} \in C), \mathbf{g}_{\mathsf{o}}\,(\mathsf{o} \in \mathsf{O}^{\varnothing}_{\mathbf{D}(L)}) : \mathbf{r}_{\mathsf{x}}\,(\mathsf{x} \in \mathsf{X}_L)\right)$$

where $\mathbf{r}_{\mathsf{x}} := {}^{\mathbf{g}_{\mathsf{x}}}\mathbf{g}_{\mathsf{y}}\mathbf{g}_{\mathsf{z}}^{-1}$ in case $\mathbf{P}(\mathbf{D}(L))$ looks locally like Figure 8 near x.

As Epple (1995) points out, Wirtinger (1905) discovered $(\mathbf{D})$ in the course of a study of the topology of holomorphic curves. A proof of **12** is given by Crowell and Fox (1977).

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construction the latter translation has two apparent advantages over the former: it does not connote geometric rigidity, and does connote intrinsic orientation.

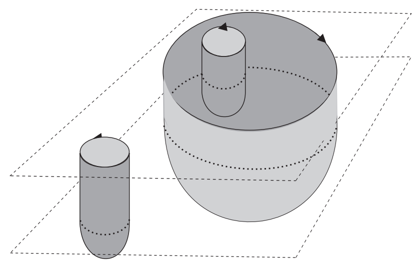

Figure 9. Seifert cycles o_i and proper 2-disks $\eta_{\mathsf{o}_i}(\iota(\mathsf{o})_i)$, with $\mathsf{o}_1 \Supset \mathsf{o}_2$, $\mathsf{o}_1 \not\Supset \mathsf{o}_3$.

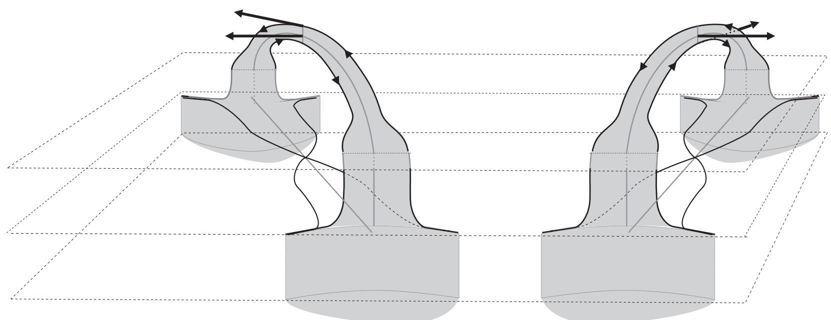

Figure 10. Attaching $\eta_{\mathsf{x}_i}(h^{(1)})$ to $\bigcup_{\mathsf{o}\in\mathsf{O}_{\mathbf{D}(L)}} \mathsf{o}$ (left, positive; right, negative).

A *Seifert surface* is a surface $S \subset S^3$. The boundary of a Seifert surface S is a link L, and S is called a Seifert surface *for* L. Similarly, a ribbon (resp., clasp) surface S with $L = \partial S$ is called a *ribbon* (resp., *clasp*) *surface for* L. It is a well known fact (apparently first stated by Frankl and Pontrjagin, 1930) that, if L is a link, then there exist Seifert surfaces (and, *a fortiori*, ribbon surfaces and clasp surfaces) for L. For the purposes of this survey the construction sketched by Seifert (1934, 1935) is especially convenient.

13. Construction. Let $\mathbf{D}(L)$ be a standard link diagram for a link $L \subset \mathbb{R}^3 \subset S^3$, equipped with a fixed smoothing $\mathrm{sm}(\mathbf{P}(\mathbf{D}(L)))$ of $\mathbf{P}(\mathbf{D}(L))$ given by a fixed family $\{\delta_{\mathsf{x}}\colon \bowtie \to \mathbb{C} : \mathsf{x} \in \mathsf{X}_{\mathbf{D}(L)}\}$ of embeddings of $\bowtie$. To implement Seifert's construction, choose embeddings $\eta_{\mathsf{o}}\colon \iota(\mathsf{o}) \hookrightarrow \mathbb{C} \times \mathbb{R}_{\leq 0}$ for $\mathsf{o} \in \mathsf{O}_{\mathbf{D}(L)}$, and $\eta_{\mathsf{x}}\colon h^{(1)} \hookrightarrow \mathbb{C} \times \mathbb{R}_{\geq 0}$ for $\mathsf{x} \in \mathsf{X}_{\mathbf{D}(L)}$, subject to the following conditions. As suggested in Figure 9, for each Seifert cycle o,

(1) η_{o} is proper relative to the standard collarings of o in $\iota(\mathsf{o})$ and $\mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{R}_{\leq 0}$,
(2) $\mathrm{pr}_1(\eta_{\mathsf{o}}(\iota(\mathsf{o}) \setminus \mathrm{Int}\,\mathrm{Col}(\mathsf{o} \subset \iota(\mathsf{o}))) = \iota(\mathsf{o}) \subset \mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{R}_{\leq 0}$, and
(3) if $\mathsf{o}' \neq \mathsf{o}$, then $\eta_{\mathsf{o}}(\iota(\mathsf{o}))$ and $\eta_{\mathsf{o}'}(\iota(\mathsf{o}'))$ are disjoint.

As suggested in Figure 10, for each crossing x,

(4) $\eta_{\mathsf{x}}\colon h^{(1)} \hookrightarrow \mathbb{C} \times \mathbb{R}_{\geq 0}$ is proper relative to the standard collarings of $\mathrm{Int}\,A(h^{(1)})$ in $h^{(1)}$ and $\mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{R}_{\geq 0}$,
(5) $\mathrm{pr}_1(\eta_{\mathsf{x}}(h^{(1)} \setminus \mathrm{Int}\,\mathrm{Col}(A(h^{(1)}) \subset h^{(1)})) = \delta_{\mathsf{x}}(\bowtie)$,
(6) $\eta_{\mathsf{x}}(h^{(1)}) \cap \mathbf{p}^{-1}_{\mathbf{D}(L)}(\mathsf{x}) = \kappa(h^{(1)})$,

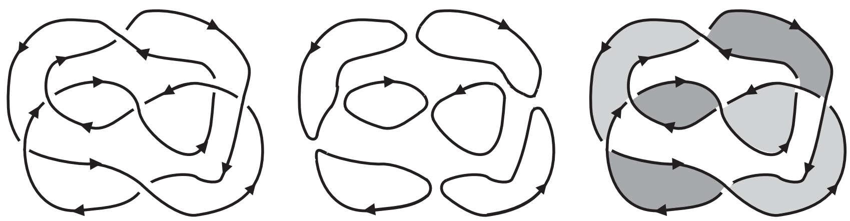

Figure 11. Seifert's construction applied to a scattered diagram.

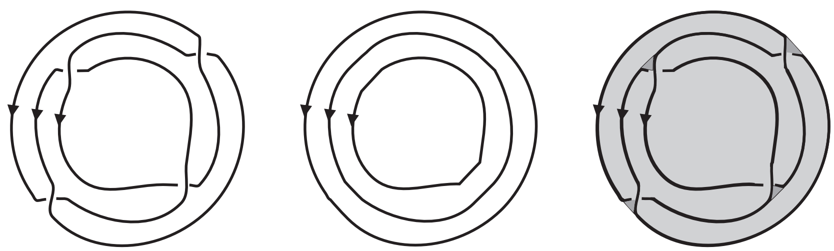

Figure 12. Seifert's construction applied to a nested diagram.

(7) $\eta_{\mathsf{x}}|A(h^{(1)})\colon\ \eta_{\mathsf{x}}(A(h^{(1)})) \hookrightarrow \bigcup_{\mathsf{o}\in\mathsf{O}_{\mathbf{D}(L)}} \mathsf{o}$ is orientation reversing, and
(8) the sign of the crossing of $\eta_{\mathsf{x}}(\partial h^{(1)} \setminus A(h^{(1)}))$ is equal to $\varepsilon(\mathsf{x})$.

It follows that

$$(\mathbf{E}) \qquad \Sigma := \bigcup_{\mathsf{o}\in\mathsf{O}_{\mathbf{D}(L)}} \eta_{\mathsf{o}}(\iota(\mathsf{o})) \cup \bigcup_{\mathsf{x}\in\mathsf{X}_{\mathbf{D}(L)}} \eta_{\mathsf{x}}(h^{(1)}) \subset \mathbb{C}\times\mathbb{R}$$

is a $(0,1)$-handle decomposition $(\mathbf{C})$ of a surface.

14. Proposition. (1) *There exists a diffeomorphism* $\delta\colon \mathbb{C}\times\mathbb{R} \to \mathbb{R}^3$ *such that* $\mathbf{p}_{\mathbf{D}(L)} \circ \delta = \mathrm{pr}_1\colon \mathbb{C}\times\mathbb{R} \to \mathbb{C}$ *and* $L = \delta(\partial\Sigma)$. (2) *The Seifert surface* $\delta(\Sigma) \subset \mathbb{R}^3 \subset S^3$ *for* L *is independent of* δ*, up to isotopy fixing* L *pointwise.* $\square$

Any Seifert surface for L of the form $\delta(\Sigma)$ in **14**(2) is denoted $S(\mathbf{D}(L))$, and called a *diagrammatic Seifert surface* for L. (A Seifert surface need not be isotopic to any diagrammatic Seifert surface.) Figure 11 and Figure 12 depict two diagrammatic Seifert surfaces.

Many operations on links are (most conveniently, and sometimes necessarily) defined using Seifert surfaces. Here are two examples: a *connected sum* of links L_1, L_2 bounding Seifert surfaces S_1, S_2 is $L_1 \# L_2 := \partial(S_1 \natural S_2) \subset S^3 \# S^3 \cong S^3$; the *split sum* of L_1, L_2 is $L_1 \sqcup L_2 := \partial(S_1 \# S_2)$. It is well-known that if $L_1 = K_1$ and $L_2 = K_2$ are knots, then $K_1 \# K_2$ is well-defined up to isotopy, and independent of S_1 and S_2; in any case, $L_1 \sqcup L_2$ is well-defined. In particular, for $n \in \mathbb{N}$ the n-component unlink is the split sum of n unknots. For any link L, let $L^{(n)}$ denote the (well-defined) link $L \# O^{(n)}$.

Similarly, many knot and link invariants are defined using Seifert surfaces.

15. Proposition. *If L and L' are disjoint links, then the algebraic number of intersections of L' with a Seifert surface S for L is independent of S (provided only that L' intersects S transversely).* $\square$

This integer invariant of the pair (L, L'), denoted $\operatorname{link}(L, L')$ and called the *linking number* of L and L', satisfies $\operatorname{link}(L, L') = \operatorname{link}(L', L) = -\operatorname{link}(-L, L')$ $= -\operatorname{link}(\operatorname{Mir} L, \operatorname{Mir} L')$.

16. Definitions. Let K be a knot, L a link. Define invariants

$$\begin{aligned}
g(K) &:= \min\{g(S) : \ S \text{ is a Seifert surface for } K\},\\
g_r(K) &:= \min\{g(S) : \ S \text{ is a ribbon surface for } K\},\\
g_s(K) &:= \min\{g(S) : \ S \text{ is a slice surface for } K\},\\
X(L) &:= \max\{\chi(S) : \ S \text{ is a Seifert surface for } L\},\\
X_r(L) &:= \max\{\chi(S) : \ S \text{ is a ribbon surface for } L\},\\
X_s(L) &:= \max\{\chi(S) : \ S \text{ is a slice surface for } L\},\\
\operatorname{clasp}(L) &:= \min\{\operatorname{clasp}(S) : \ S = f(D^2 \times \pi_0(L)) \text{ is a clasp surface for } L\ \},\\
\operatorname{node}(L) &:= \min\{\operatorname{node}(S) : \ S = f(D^2 \times \pi_0(L)) \text{ is a nodal surface for } L\}.
\end{aligned}$$

Call $g(K)$ (resp., $g_r(K)$; $g_s(K)$) the *genus* (resp., *ribbon genus*; *slice genus*) of K, and say K is a *slice* (resp., *ribbon*) knot in case $g_s(K) = 0$ (resp., $g_s(K) = 0$). Another name for $g_s(K)$ is the "Murasugi genus" of K.

17. Definition. Let $\mathbf{D}(L)$ be a standard link diagram, $n := \operatorname{card}(\pi_0(L))$. It is easy to prove and well known (see Hoste or Kauffman, this Handbook) that there is a standard link diagram $\mathbf{D}(O^{(n)})$ for an unlink $O^{(n)}$ such that $\mathbf{P}(\mathbf{D}(O^{(n)})) = \mathbf{P}(\mathbf{D}(L))$ and $\mathbf{I}(\mathbf{D}(O^{(n)}))$ differs from $\mathbf{I}(\mathbf{D}(L))$ precisely to the extent that some number $u \geq 0^{(n)}$ of crossings of $\mathbf{P}(\mathbf{D}(O^{(n)})) = \mathbf{P}(\mathbf{D}(L))$ have opposite signs in $\mathbf{I}(\mathbf{D}(O^{(n)}))$ and $\mathbf{I}(\mathbf{D}(L))$. The *unknotting number* of $\mathbf{D}(L)$ is the least such u. The *unknotting number* of L is the least unknotting number of all standard link diagrams for L. (The unknotting number is also called the *Überschneidungszahl* Wendt (1937); Milnor (1968) and the *Gordian number* Boileau and Weber (1983); Bennequin (1993); A'Campo (1998).) More generally, the *Gordian distance* $d_G(L, L')$ between two links L, L' with $\operatorname{card}(\pi_0(L)) = \operatorname{card}(\pi_0(L'))$ is the minimum number of sign changes at crossings needed to transform some standard link diagram $\mathbf{D}(L)$ to some standard link diagram $\mathbf{D}(L')$ (Murakami, 1985); thus $\ddot{u}(L) = d_G(L, O^{(n)})$.

Various more or less obvious inequalities relate $\ddot{u}$ and the several invariants named in **16** (see Shibuya, 1974; Rudolph, 1983b).

Let L be a link. A *framing* of L is a function $f\colon \pi_0(L) \to \mathbb{Z}$; the pair (L, f) is a *framed link*. A framing of a knot is identified with the integer which is its value. For framings f, g of L, write $f \preccurlyeq g$, and say f is *less twisted* than g provided that $f(K) \leq g(K)$ for every $K \in \pi_0(L)$.

18. Proposition. *The normal bundle of L is trivial. Given a Seifert surface S for L, there exists a trivialization $n\colon L \times \mathbb{C} \to \nu(L)$ such that, under the identification of* $\mathrm{Nb}(L \subset S^3)$ *with* $n(L \times D^2)$ *(as in* **4.3***),* $S \cap \mathrm{Nb}(L \subset S^3) = \mathrm{Nb}(L \subset S)$ *is identified with* $n(L \times [0, 1])$*. The homotopy class of n is well-defined, independent of S.* $\square$

Let (K, k) be a framed knot. A *k-twisted annulus of type K* is any annulus $A(K, k) \subset S^3$ such that $K \subset \partial A(K, f)$ and $\mathrm{link}(K, \partial A(K, k) \setminus K)$ is $-k$; note that, since $\partial A(K, k) \setminus K$ is clearly isotopic to $-K$, all four of $A(K, k)$, $A(-K, k)$, $-A(K, k)$, and $-A(-K, k)$ are isotopic. For a framed link (L, f), $A(L, f)$ is defined componentwise. Given a 2–submanifold $S \subset S^3$ and a link $L \subset S$, the *S-framing of L* is the framing $f_{L\subset S}$ such that $\mathrm{Col}(L \subset S) = A(-L, f_{L\subset S})$. A framed link (L, f) is *embedded* on a Seifert surface S in case $L \subset S$ and $f = f_{L\subset S}$.

Let S be a Seifert surface with collaring $\mathrm{col}_{S\subset S^3}$. The *Seifert pairing* (on S) of an ordered pair of links (L_0, L_1) with $L_0, L_1 \subset S$ is $(L_0, L_1)_S := \mathrm{link}(L_0, L_1^+)$; if $K \subset S$ is a knot, then $(K, K)_S = f_{K\subset S}$. Given an ordered μ-tuple $(L_1, \ldots, L_\mu)$ of links on S the homology classes of which form a basis for $H_1(S; \mathbb{Z})$, the *Seifert matrix* of S with respect to that basis is the $\mu \times \mu$ matrix $[(L_i, L_j)_S]$, and the *Seifert form* is the (typically non-symmetric) bilinear form on $H_1(S; \mathbb{Z})$ represented by $[(L_i, L_j)_S]$.

1.6 Fibered links, fiber surfaces, and open books

Let L be a link. Let $n\colon L \times \mathbb{C} \to \nu(L)$ be a trivialization, as in **18**, in the homotopy class corresponding to any Seifert surface S for L. Call L *fibered* in case there exists a map $\varphi\colon S^3 \setminus L \to S^1$ (called a *fiber map* for L) which is adapted to n, has $d(K) = 1$ for all $K \in \pi_0(L)$, and is a fibration (in particular, a Morse map). If L is a fibered link with fiber map φ, then for each $e^{i\theta} \in S^1$, $L \cap \varphi^{-1}(e^{i\theta})$ is a Seifert surface for L. A *fiber surface* is any Seifert surface F_L isotopic to $L \cap \varphi^{-1}(e^{i\theta})$ for any fibered link L with fiber map φ. The *Milnor number* of L is $\mu(L) := \dim_{\mathbb{R}} H_1(F; \mathbb{R})$.

Let S be a Seifert surface. The *top* of S is $\mathrm{top}(S) := \mathrm{Col}(S \subset S^3)$. A 2–disk $D \subset \mathrm{top}(S)$ is a *top-compression disk* in case $\partial D = D \cap S$ and ∂D bounds no disk on S. Call S *compressible* in case there exists a top-compression disk for at least one of S and $-S$, and *incompressible* in case it is not compressible. Call S *least-genus* provided that $\chi(S) = X(L)$. The following facts are well known (see Stallings, 1978; Gabai, 1983a,b, 1986).

19. Proposition. (1) *S is a fiber surface if and only if S is connected and a push-off map induces an isomorphism $\pi_1(\operatorname{Int} S; *) \to \pi_1(S^3 \setminus S; *^+)$.* (2) *A least-genus surface S is incompressible.* (3) *A fiber surface is least-genus, and up to isotopy it is the unique incompressible surface with its boundary.* (4) *$A(K,n)$ is least-genus if and only if $(K,n) \neq (O,0)$.* (5) *$A(K,n)$ is a fiber surface if and only if $(K,n) = (O,\mp 1)$.* $\square$

The fiber surface $A(O,-1)$ (resp., $A(O,1)$) is called a *positive* (resp., *negative*) *Hopf annulus* (sometimes "Hopf band"); the choice of the adjectives "positive" and "negative" reflects the linking number of the components of $\partial A(O,\mp 1)$.

Fibered links are slightly flaccid. They may be rigidified as follows. An *open book* is a map $\mathbf{f}\colon S^3 \to \mathbb{C}$ such that 0 is a regular value and $\arg(\mathbf{f}) := \mathbf{f}/|\mathbf{f}|\colon S^3 \setminus \mathbf{f}^{-1}(0) \to S^1$ is a fibration. The *binding* $\mathbf{f}^{-1}(0)$ of $\mathbf{f}$ clearly is a fibered link, and for each $e^{i\theta} \in S^1$, the *θth page* $F_\theta := \mathbf{f}^{-1}(\{re^{i\theta} : r \geq 0\})$ of $\mathbf{f}$ is a fiber surface. Every fibered link is the binding of various open books; any two fibered books with the same binding are equivalent in a straightforward sense (cf. Kauffman and Neumann, 1977).

Milnor (1968) discovered a rich source (now called *Milnor fibrations*; see **114**) of open books as part of his investigation of the topology of singular points of complex hypersurfaces. The simplest special cases are fundamental to the knot theory of complex plane curves and easy to write down. Let $m,n \in \mathbb{N}$, $(m,n) \neq (0,0)$.

20. Theorem. $\mathbf{o}\{m,n\}\colon S^3 \to \mathbb{C}^2 : (z,w) \mapsto z^m + w^n$ *is an open book.* $\square$

The binding $\mathbf{o}\{m,n\}^{-1}(0)$ is a *torus link of type* $\{m,n\}$, sometimes (as in Rudolph, 1982a, 1988, cf. Litherland, 1979) denoted $O\{m,n\}$. Call $\mathbf{o} := \mathbf{o}\{1,0\}$ (resp., $\mathbf{o}' := \mathbf{o}\{0,1\}$) the *vertical* (resp., *horizontal*) *unbook* and its binding $\mathbb{O} := O\{1,0\}$ (resp., $\mathbb{O}' := O\{0,1\}$) the *vertical* (resp., *horizontal*) *unknot* (Rudolph, 1988).

1.7 Polynomial invariants of knots and links

The intent of this section is to establish notations and conventions, and to state without proof two useful theorems. For a thorough treatment of polynomial link invariants, see Kauffman (this Handbook).

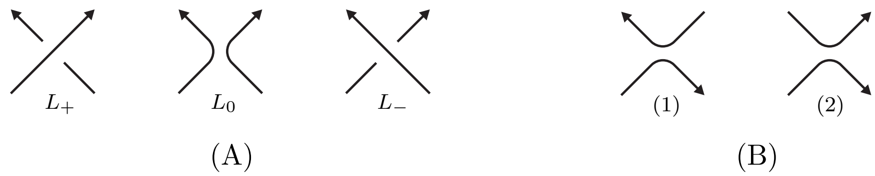

Figure 13. (A) L_+, L_0, and L_-. (B) L_∞ (homogeneous and heterogeneous cases).

For any ring $\mathcal{R}$, for any Laurent polynomial $H(s) \in \mathcal{R}[s^{\pm 1}]$, write $\operatorname{ord}_s H(s) := \sup\{n \in \mathbb{Z} : \ s^{-n}H(s) \in \mathcal{R}[s] \subset \mathcal{R}[s^{\pm 1}]\}$, $\deg_s H(s) := -\operatorname{ord}_s H(s^{-1})$.

21. Definition. Let K be a knot. Let S be a Seifert surface for K. Let $A = [(L_i, L_j)_S]$ be a Seifert matrix of S, with transpose A^{T}. The *unnormalized Alexander polynomial* of K is $\det(A^{\mathrm{T}} - tA) \in \mathbb{Z}[t]$. It is easily shown that $\mu := \deg_t \det(A^{\mathrm{T}} - tA)$ is even. The *Alexander polynomial* of K is $\Delta_K(t) := t^{-\mu/2} \det(A^{\mathrm{T}} - tA) \in \mathbb{Z}[t, t^{-1}]$.

22. Proposition. *Δ_K depends only on K, not on the choice of S or A.* $\square$

Let $L \subset \mathbb{R}^3 \subset S^3$ be a link. Let $\mathbf{D}(L)$ be a standard link diagram for L. Let $\mathsf{x} \in \mathsf{X}_{\mathbf{D}(L)}$. Call x *homogeneous* (resp., *heterogeneous*) in case $\operatorname{card}(\pi_0(\operatorname{sm}(L; \mathsf{x})))$ equals $\operatorname{card}(\pi_0(L)) + 1$ (resp., $\operatorname{card}(\pi_0(L)) - 1$). In case $\mathsf{x} \in \mathsf{X}^+_{\mathbf{D}(L)}$, let $L_+ := L$, and define standard link diagrams $\mathbf{D}(L_-)$ and $\mathbf{D}(L_0)$ (and thereby links L_- and L_0) as follows: (1) $\mathbf{P}(\mathbf{D}(L_-)) = \mathbf{P}(\mathbf{D}(L_+))$, and $\mathbf{I}(\mathbf{D}(L_-))$ differs from $\mathbf{I}(\mathbf{D}(L_+))$ precisely to the extent that $\mathsf{x} \in \mathsf{X}^-_{\mathbf{D}(L_-)}$; (2) $\mathbf{P}(\mathbf{D}(L_0))$ is the local smoothing $\operatorname{sm}(\mathbf{P}(\mathbf{D}(L_\pm)); \mathsf{x})$ of $\mathbf{P}(\mathbf{D}(L_\pm))$ at x, and $\mathbf{I}(\mathbf{D}(L_0))$ differs from $\mathbf{I}(\mathbf{D}(L_\pm))$ precisely to the extent that $\mathsf{x} \notin \mathsf{X}_{\mathbf{D}(L_0)}$. In case $\mathsf{x} \in \mathsf{X}^-_{\mathbf{D}(L)}$, modify these definitions accordingly; the two sets of definitions are consistent.

Up to isotopy, the local situation at x is as in Figure 13(A). In case x is homogeneous (resp., heterogeneous), let $\mathbf{D}(L_\infty)$ be the standard link diagram differing from $\mathbf{D}(L_\pm)$ and $\mathbf{D}(L_0)$ only as required by case (1) (resp., case (2)) of Figure 13(B), let p (resp., q) be the linking number of the right-hand visible component of L_0 with the rest of L_0 (resp., of the lower visible component of L_+ with the rest of L_+), and define $r := 4p + 1$ (resp., $r := 4q - 1$).

23. Definition. The *oriented polynomial* $P_L(v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ and *semi-oriented polynomial* $F_L(a, x) \in \mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$ of L are defined recursively as follows, with the initial conditions $P_O(v, z) = 1 = F_O(a, x)$.

$$(\mathbf{F}) \qquad P_{L_+}(v, z) = vzP_{L_0}(v, z) + v^2 P_{L_-}(v, z)$$
$$(\mathbf{G}) \qquad F_{L_+}(a, x) = a^{-1}xF_{L_0}(a, x) - a^{-2}F_{L_-}(a, x) + a^{-r}xF_{L_\infty}(a, x)$$

The nomenclature is that of Lickorish (1986); the choice of variables v, z in (**F**) follows Morton (1986). The oriented (resp., semi-oriented) polynomial is often known, eponymously, as the *FLYPMOTH* (Freyd et al., 1985; Przytycki and Traczyk, 1988) (resp., *Kauffman* (Kauffman, 1987)) polynomial.

24. Definitions. Several other polynomials, though mere adaptations of the oriented or semi-oriented polynomials, nonetheless have their uses.

24.1. Let (L, f) be a framed link. The *framed polynomial* $\{L, f\}(v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ is

$$\textbf{(H)} \qquad (-1)^{\operatorname{card}(\pi_0(L))}(1 + (v^{-1} - v)z^{-1} \sum_{\varnothing \neq C \subset \pi_0(L)} (-1)^{\operatorname{card}(C)} P_{\partial A(\cup C, f|\cup C)})$$

(see Rudolph, 1990). For any (L, f), $\{L, f\} = v^{-2\sum_{K \in \pi_0(L)} f(K)}\{L, 0\}$.

24.2. Let $R_L(v) := (z^{\operatorname{card}(\pi_0(L))-1}P_L(v, z))\big|_{z=0}$. R_L can be calculated from $R_O(v) = 1$ and $R_{L_+}(v) = hR_{L_0}(v) + v^2R_{L_-}(v)$, where h is 0 (resp., 1) in case x is heterogeneous (resp., homogeneous).

24.3. Let $F^*_L(a, x) := (F_L \pmod 2) \in (\mathbb{Z}/2\mathbb{Z})[a^{\pm 1}, x^{\pm 1}]$. For $k = 0, 1$, let $G^k_L(a) := (x^{1-c(L)}F^*_L(a, x))\big|_{x=k} \in (\mathbb{Z}/2\mathbb{Z})[a^{\pm 1}]$, so $G^0_L(a) = R_L(a^{-1}) \pmod 2$ and can be calculated using **24.2**, while G^1_L can be calculated from $G^1_O(a) = 1$ and $G^1_{L_+}(a) = a^{-2}G^1_{L_-}(a) + a^{-1}G^1_{L_0}(a) + a^{-r}G^1_{L_\infty}(a)$.

The result underlying most applications of polynomial invariants to the knot theory of complex plane curves, due to Morton (1986) and Franks and Williams (1987), is rephrased here to fit the expository order of this survey; the usual statement, in terms of braids, is given in **58**.

25. Theorem. *If* $\mathbf{D}(L)$ *is a standard link diagram such that* (1) $\mathbf{D}(L)$ *is nested and* (2) $\mathsf{O}_{\mathbf{D}(L)} = \mathsf{O}^+_{\mathbf{D}(L)}$, *then*

$$\operatorname{ord}_v P_L \geq \operatorname{card}(\mathsf{X}^+_{\mathbf{D}(L)}) - \operatorname{card}(\mathsf{X}_{\mathbf{D}(L)^-}) - \operatorname{card}(\mathsf{O}_{\mathbf{D}(L)}) + 1. \qquad \square$$

The framed polynomial provides a bridge between the oriented and semi-oriented polynomials, as the following result (Rudolph, 1990) makes plain.

26. Theorem. $(1+(v^{-2}+v^2)z^{-2})F_L(v^{-2}, z^2) \equiv v^{4\tau(L)}\{L, 0\}(v, z) \pmod 2$. $\square$

1.8 Polynomial and analytic maps; algebraic and analytic sets

This section recalls needed definitions from real and complex algebraic and analytic geometry, and establishes notations. General background, and proofs of stated results, can be found in Whitney (1957, 1972); Milnor (1968); Narasimhan

(1960); Gunning and Rossi (1965), and, for **27**(1), Abraham and Robbin (1967, Appendix B).

Let $\mathbb{F}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, with its metric topology. The algebra of polynomials (resp., somewhere-convergent power series) in n variables with ground field $\mathbb{F}$ is denoted $\mathbb{F}[\varphi_1, \ldots, \varphi_n]$ (resp., $\mathbb{F}\{\varphi_1, \ldots, \varphi_n\}$), where φ stands for x (resp., z) in case $\mathbb{F}$ is $\mathbb{R}$ (resp., $\mathbb{C}$). As usual, $f \in \mathbb{F}[\varphi_1, \ldots, \varphi_n]$ is conflated with the *polynomial function* $f\colon\ \mathbb{F}^n \to \mathbb{F}$ that it defines, and $f \in \mathbb{F}\{\varphi_1, \ldots, \varphi_n\}$ with both the $\mathbb{F}$-analytic function that it defines in a neighborhood of $0 \in \mathbb{F}^n$ and the germ of that function at $0 \in \mathbb{F}^n$. (In particular φ_s is conflated with the coordinate projection $\mathrm{pr}_s\colon\ \mathbb{F}^n \to \mathbb{F}$ for every $s \in \mathbf{n}$, no notational distinction being made between the functions φ_s on F^m and F^n so long as $s \in \mathbf{m} \cap \mathbf{n}$.) Given a non-empty open set $\Omega \subset \mathbb{F}^n$, a function $f\colon\ \Omega \to \mathbb{F}$ is called $\mathbb{F}$-*analytic* (simply *analytic* when $\mathbb{F}$ is clear from context; also *holomorphic*for $\mathbb{F} = \mathbb{C}$, and *entire*when $\Omega = \mathbb{C}^n$) in case, for every $(\varphi_1^{(0)}, \ldots, \varphi_n^{(0)} \in \Omega$, the germ of $f(\varphi_1 - \varphi_1^{(0)}, \ldots, \varphi_n - \varphi_n^{(0)})$ at $(0, \ldots, 0)$ belongs to $\mathbb{F}\{\varphi_1, \ldots, \varphi_n\}$. The set $\mathcal{O}(\Omega)$ of all $\mathbb{F}$-analytic functions on Ω is an algebra containing (a natural isomorphic image of) $\mathbb{F}[\varphi_1, \ldots, \varphi_n]$.

A *polynomial* (resp., $\mathbb{F}$-*analytic*) *map* $F = (f_1, \ldots, f_m)$ from $\mathbb{F}^n$ (resp., Ω) to $\mathbb{F}^m$ is one with components f_s that are polynomial (resp., $\mathbb{F}$-analytic) functions. An *algebraic set* (resp., *global analytic set*) is a subset $\mathcal{V}_F := F^{-1}(0, \ldots, 0)$ of $\mathbb{F}^m$ (resp., Ω), where F is a polynomial (resp., analytic) map. An *analytic set* is a subset X of Ω for which every point of Ω has an open neighborhood U such that $X \cap U$ is a global analytic set $\mathcal{V}_F$ for some $F \in \mathcal{O}(U)$. Every global analytic set (in particular, every algebraic set) is an analytic set; the converse fails for many Ω when $n > 1$.

Let $F\colon\ \Omega \to \mathbb{F}^m$ be an analytic map (allowing the possibility that $\Omega = \mathbb{F}^n$ and F is polynomial). It may happen that

$$\mathrm{Reg}(F) := \{(y_1, \ldots, y_m) \in \mathcal{V}_F \ :\ \mathrm{rank}_{\mathbb{F}}\, DF(y_1, \ldots, y_m) = n - m\}$$

is not dense in $\mathcal{V}_F$. However, theorems of commutative algebra show that there is another analytic map $F_0\colon\ \Omega \to \mathbb{F}^m$ such that $\mathcal{V}_{F_0}$ and $\mathcal{V}_F$ are equal *as sets* (that is, ignoring multiplicities), and $\mathrm{Reg}(F_0)$ is dense in $\mathcal{V}_{F_0} = \mathcal{V}_F$. Call $(y_1, \ldots, y_m) \in \mathcal{V}_F$ a *regular* (*resp., singular point*) of the global analytic set $\mathcal{V}_F$ in case $\mathrm{rank}_{\mathbb{F}}\, DF_0(y_1, \ldots, y_m)$ equals (resp., is less than) $n - m$; these definitions are independent of the particular choice of F_0. Regularity in $\mathcal{V}_F$ is clearly a local property, and is therefore well-defined in any analytic set X. The set $\mathrm{Reg}(X)$ of regular points of X is called the *regular locus* of X; it is an $\mathbb{F}$-analytic manifold. The *singular locus* $X \setminus \mathrm{Reg}(X) =: \mathrm{Sing}(X)$ of X is an algebraic, global analytic, or analytic set according as X is. Let $\mathrm{Sing}^0(X)$ and $\mathrm{Reg}^1(X)$ both mean $\mathrm{Reg}(X)$; for $s \in \mathbb{N}_{>0}$, let $\mathrm{Sing}^{s+1}(X) := \mathrm{Sing}(\mathrm{Sing}^s(X))$, $\mathrm{Reg}^{s+1}(X) := \mathrm{Reg}(\mathrm{Sing}^s(X))$. An $\mathbb{F}$-analytic set X is partitioned by the (finitely many) non-empty sets in the sequence $\{\mathrm{Reg}^s(X)\}_{s\in\mathbb{N}_{>0}}$. In case X is algebraic, the refinement of this partition obtained by separating each $\mathrm{Reg}^s(X)$ into its connected components is a finite stratification (Whitney, 1957; see Milnor, 1968, Theorems 2.3 and 2.4); call it the *naïve stratification* of X.

A *basic semi-algebraic set* in $\mathbb{R}^n$ is the intersection of an $\mathbb{R}$-algebraic set $\mathcal{V}_F$ and finitely many sets of the form $G^{-1}(\mathbb{R}_{>0})$, with $G \in \mathbb{R}[x_1, \ldots, x_n]$; a *semi-algebraic set* is the union of finitely many basic semi-algebraic sets. An algebraic set is semi-algebraic.

27. Proposition. (1) *The image of a semi-algebraic set by a polynomial map is semi-algebraic.* (2) *A semi-algebraic set has a finite naïve stratification.* $\square$

27(1) is due to Tarski and to Seidenberg. **27**(2) is a result of Whitney (1957).

Let $U \subset \mathbb{C}^2$ be an open set. A *holomorphic curve* in U is a $\mathbb{C}$-analytic set Γ such that the complex manifold $\mathrm{Reg}(\Gamma)$ is non-empty, everywhere of real dimension 2, and dense in Γ.

28. Proposition. *Let $\Gamma \subset U$ be a holomorphic curve in an open set in $\mathbb{C}^2$. There exists a complex manifold G of real dimension 2, and a holomorphic map $R\colon G \to U$, such that:* (1) $R(G) = \Gamma$; (2) $\mathrm{crit}(R) \subset R^{-1}(\mathrm{Sing}(\Gamma))$, *and $R|R^{-1}(\mathrm{Sing}(\Gamma))$ has finite fibers;* (3) *$R|R^{-1}(\mathrm{Reg}(\Gamma))$ is a holomorphic diffeomorphism.* $\square$

The map R is essentially unique, and is called *the resolution of* Γ. A *branch* of Γ at $P \in \Gamma$ is the image by R of a component of $\mathrm{Int}\,\mathrm{Nb}(\{Q\} \subset G \setminus \mathrm{crit}(R))$ for some $Q \in R^{-1}(P)$(or the germ of the image of such a component). If $P \in \mathrm{Reg}(\Gamma)$ then there is only one branch of Γ at P, but there can also be singular points of Γ at which there is only one branch of Γ.

29. Examples. Classical algebraic geometers gave names to quite a few special cases of branches (and resolutions). Two examples are of particular importance in the knot theory of complex plane curves.

29.1. Define $f \in \mathcal{O}(\mathrm{Int}\, D^4)$ by $f(z,w) = z^2 + w^2$. The holomorphic curve $\mathcal{V}_f$ has two branches at $(0,0)$. Its resolution is $R\colon\ \mathrm{Int}\, D^2 \times \{+,-\} \to \mathcal{V}_f\ :\ (\zeta, \pm) \mapsto 2^{-1/2}(\zeta, \pm i\zeta)$. A point P of a holomorphic curve $\Gamma \subset \Omega$ such that there exist an open neighborhood U of P in Ω and a diffeomorphism (which may in fact be required to be holomorphic) $h\colon\ (U, U\cap\Gamma, P) \to (\mathrm{Int}\, D^4, \mathcal{V}_f, (0,0))$ is called a *node* of Γ.

29.2. Define $f \in \mathcal{O}(\mathrm{Int}\, D^4)$ by $f(z,w) = z^2 + w^3$. The holomorphic curve $\mathcal{V}_f$ has one branch at $(0,0)$. Its resolution is $R\colon\ \mathrm{Int}\, D^2 \to \mathcal{V}_f\ :\ (\zeta, \pm) \mapsto 2^{-1/2}(\zeta^3, -\zeta^2)$. A point P of a holomorphic curve $\Gamma \subset \Omega$ such that there exist an open neighborhood U of P in Ω and a diffeomorphism (which may in fact be required to be holomorphic) $h\colon (U, U\cap\Gamma, P) \to (\operatorname{Int} D^4, \mathcal{V}_f, (0,0))$ is called a *cusp* of Γ.

A holomorphic curve Γ such that every point of $\operatorname{Sing}(\Gamma)$ is a node (resp., either a node or a cusp) is called a *node* (resp., *cusp*) *curve*.

1.9 Configuration spaces and spaces of monic polynomials

Let X be a topological space. For $n \in \mathbb{N}$, the sets $\mathcal{MP}^{[n]}(X)$, $\binom{X}{n}$, and $\Delta_n(X)$ are endowed with topologies by the application of the bijection (**B**) to the quotient topology induced on $X^n/\mathfrak{S}_n$ from the product topology on X^n; with these topologies, they are called the nth *multipower space*, the nth *configuration space*, and the nth *discriminant space* of X, respectively.

If M is a manifold, then clearly each equivalence class of the partition by type $\mathcal{MP}^{[n]}(M)/\tau$ of $\mathcal{MP}^{[n]}(M)$ is a manifold. However, even for connected M it often happens that not every fiber of τ is connected. The stratification $\mathcal{MP}^{[n]}(M)/\tau^c$ of $\mathcal{MP}^{[n]}(M)$ by the connected components of the fibers of τ will be called the *standard stratification* of $\mathcal{MP}^{[n]}(M)$; the standard stratification of $\mathcal{MP}^{[n]}(M)$ induces a *standard stratification* on each of $\mathcal{MP}^{[n]}(M)/\tau^c$ and $\Delta_n(M)$, since they are evidently unions of strata of $\mathcal{MP}^{[n]}(M)/\tau^c$.

If $\mathbb{F}$ is a field, then of course $\mathbb{F}^n$ is an algebraic set over $\mathbb{F}$, and the standard action of $\mathfrak{S}_n$ on $\mathbb{F}^n$ is algebraic. In general this is not enough to ensure that the set $\mathbb{F}^n/\mathfrak{S}_n$ can be endowed with as much structure as might be desirable to full-fledged algebraic geometers. However, for $\mathbb{F} = \mathbb{C}$ (or any algebraically closed field), on general principles $\mathbb{C}^n/\mathfrak{S}_n$ does have a natural structure as an algebraic set (more or less naturally embedded in an affine space $\mathbb{C}^N$; see, e.g., Cartan, 1957) with respect to which the unordering map $U_{\mathbb{C},n}\colon \mathbb{C}^n \to \mathbb{C}^n/\mathfrak{S}_n$ is a polynomial map. By contrast, if $n > 1$, then $\mathbb{R}^n/\mathfrak{S}_n$ not an algebraic set (in any natural way), although it is semi-algebraic.

Denote by $MP_n := \{p(w) \in \mathbb{C}[w] : p(w) = w^n + c_1 w^{n-1} + \cdots + c_{n-1}w + c_n\}$ the n-dimensional complex affine space of *monic polynomials* of degree $n \in \mathbb{N}$. Define the *roots map* $r\colon MP_n \to \mathcal{MP}^{[n]}(\mathbb{C})$ by $r(p) := (p^{-1}(0), \mathbf{m}_p|p^{-1}(0))$. where $\mathbf{m}_p(z)$ is the usual *multiplicity of* $p(w) \in MP_n$ *at* $z \in \mathbb{C}$. Let V be the polynomial map $\mathbb{C}^n \to MP_n : (z_1, \ldots, z_n) \mapsto (w - z_1)\cdots(w - z_n)$. The diagram

$$
\text{(J)}\qquad
\begin{tikzcd}
\mathbb{R}^n \arrow[d, "U_{\mathbb{R},n}"] & \mathbb{C}^n \arrow[l, "\mathrm{Re}"'] \arrow[r, "V"] \arrow[d, "U_{\mathbb{C},n}"] & MP_n \arrow[d, "r"] \\
\mathbb{R}^n/\mathfrak{S}_n \arrow[d, "\tau^c"] & \mathbb{C}^n/\mathfrak{S}_n \arrow[l, "\mathrm{Re}"'] \arrow[r, "\cong", "\textbf{(B)}"'] \arrow[d, "\tau"] & \mathcal{MP}^{[n]}(\mathbb{C}) \\
(\mathbb{R}^n/\mathfrak{S}_n)/\tau^c & (\mathbb{C}^n/\mathfrak{S}_n)/\tau &
\end{tikzcd}
$$

is commutative, and defines $R\colon\ MP_n \to \mathbb{C}^n/\mathfrak{S}_n$the following results are standard; some go back, in essence, to Viète (1593) [8] and Descartes (1637).

30. Proposition. *The naïve stratification of $\mathbb{C}^n/\mathfrak{S}_n$ as an algebraic set coincides with its stratification by type $(\mathbb{C}^n/\mathfrak{S}_n)/\tau$. In particular:*

(1) *there are no strata of odd (real) codimension;*
(2) *the only codimension-0 stratum is $\mathrm{Reg}(\mathbb{C}^n/\mathfrak{S}_n) = \binom{\mathbb{C}}{n} = \tau^{-1}(\{1,\dots,1\})$, so $\mathrm{Sing}(\mathbb{C}^n/\mathfrak{S}_n) = \Delta_n(\mathbb{C})$ is the union of the strata of codimension > 0;*
(3) *the only codimension-2 stratum is $\mathrm{Reg}^2(\mathbb{C}^n/\mathfrak{S}_n) = \tau^{-1}(\{1,\dots,1,2\})$.* □

Call the stratification in **30** the *complex stratification of* $\mathbb{C}^n/\mathfrak{S}_n$, and denote it by $(\mathbb{C}^n/\mathfrak{S}_n)/{\equiv_{\mathbb{C}}}$.

31. Proposition. *R is a homeomorphism and $R|r^{-1}(\Delta_n(\mathbb{C}))$ is a diffeomorphism; in fact, R is a normalization and minimal resolution of the algebraic set $\mathbb{C}^n/\mathfrak{S}_n$.* □

The resolution R and the complex stratification of $\mathbb{C}^n/\mathfrak{S}_n$ together impose a *complex stratification* $MP_n/{\equiv_{\mathbb{C}}}$ on MP_n.

32. Proposition. $(\mathbb{R}^n/\mathfrak{S}_n)/\tau^c$ *is a cellulation; in particular, $\binom{\mathbb{R}}{n}$, its unique codimension-0 stratum, is an n-cell.* □

33. Proposition. (1) *If S is a stratum of $(\mathbb{C}^n/\mathfrak{S}_n)/{\equiv_{\mathbb{C}}}$, then $S/(\tau^c \circ \mathrm{Re})$ is a stratification of S, and in fact a cellulation of S; thus the partition $(\mathbb{C}^n/\mathfrak{S}_n)/{\equiv_{\mathbb{R}}}$, such that each $\equiv_{\mathbb{R}}$-class is a $\tau^c \circ \mathrm{Re}$-class of some stratum S of $(\mathbb{C}^n/\mathfrak{S}_n)/{\equiv_{\mathbb{C}}}$, is a cellulation.* (2) *Each cell of $(\mathbb{C}^n/\mathfrak{S}_n)/{\equiv_{\mathbb{R}}}$ is a real semi-algebraic set.*

Proof. (1) is apparently originally due to Fox and Neuwirth (1962) (see also Fuchs, 1970; Vaĭnšteĭn, 1978; Napolitano, 1998). (2) follows from **27**(1). □

[8] V stands for *Viète map*, a coinage due (apparently) to Arnol'd, now widely used.

Call the cellulation in **33** the *real cellulation of* $\mathbb{C}^n/\mathfrak{S}_n$, and denote it by $(\mathbb{C}^n/\mathfrak{S}_n)/{\equiv_{\mathbb{R}}}$. The resolution R and the real cellulation of $\mathbb{C}^n/\mathfrak{S}_n$ together impose a *real cellulation* $MP_n/{\equiv_{\mathbb{R}}}$ on MP_n. The real cellulations of $\mathbb{C}^n/\mathfrak{S}_n$ and MP_n in turn define real cellulations of $\mathcal{MP}^{[n]}(\mathbb{C})$, $\Delta_n(\mathbb{C})$, $R^{-1}(\mathcal{MP}^{[n]}(\mathbb{C}))$, and $R^{-1}(\Delta_n(\mathbb{C}))$.

34. Examples. For small n, very explicit descriptions of the complex stratifications are easily given.

34.1. $MP_1/{\equiv_{\mathbb{C}}}$ consists of a single stratum, necessarily of codimension-0.

34.2. $MP_2/{\equiv_{\mathbb{C}}}$ consists of two incident strata: $R^{-1}(\binom{\mathbb{C}}{2})$, of codimension 0, is diffeomorphic to $\mathbb{C}\times(\mathbb{C}\setminus\{0\})$; $R^{-1}(\Delta_2(\mathbb{C}))$, of codimension 2, is diffeomorphic to $\mathbb{C}$. Explicitly, $\mathbb{C}^2/\mathfrak{S}_2 \to \mathbb{C}^2 : \{w_1, w_2\} \mapsto (w_1 + w_2, (w_1 - w_2)^2)$ is a homeomorphism that maps $\binom{\mathbb{C}}{2}$ (resp., $\Delta_2(\mathbb{C})$) diffeomorphically onto $\mathbb{C}\setminus\{0\}$ (resp., $\mathbb{C}\times 0$).

34.3. $MP_3/{\equiv_{\mathbb{C}}}$ consists of three mutually incident strata: $\binom{\mathbb{C}}{3}$, of codimension 0, is diffeomorphic to $\mathbb{C}\times(\mathbb{C}^2\setminus\mathcal{V}_f)$, where $f(z_1, z_2) = 4z_1^3 + 9z_2^2$ and so $\mathcal{V}_f$ is a cuspidal cubic curve, homeomorphic to $\mathbb{C}$ and having a single singular point; $\operatorname{Reg}(\Delta_3(\mathbb{C}))$, of codimension 2, is diffeomorphic to $\mathbb{C}\times\operatorname{Reg}(\mathcal{V}_f)$ and therefore to $\mathbb{C}\times\mathbb{C}\setminus\{0\}$; and $\operatorname{Sing}(\Delta_3(\mathbb{C}))$, of codimension 4, is diffeomorphic to $\mathbb{C}\times\operatorname{Sing}(\mathcal{V}_f)$ and therefore to $\mathbb{C}$. It is easy to write down an explicit polynomial homeomorphism $\mathbb{C}^3/\mathfrak{S}_3 \to \mathbb{C}^3$ giving an isomorphic stratification.

The real cellulations of MP_n, and thus of $\binom{\mathbb{C}}{n}$ and $\Delta_n(\mathbb{C})$, can be described very explicitly, in all dimensions (see Fox and Neuwirth, 1962; Napolitano, 2000). For the purposes of this survey, it is sufficient to describe the cells of dimension $2n$, $2n-1$, and $2n-2$ only, along with their incidence relations; this can be done in a uniform manner for all n.

35. Example. In $\binom{\mathbb{C}}{n}/{\equiv_{\mathbb{R}}}$ there is exactly one cell of dimension $2n$, exactly $n-1$ cells of dimension $2n-1$, and exactly $(n-1)(n-2)/2$ cells of dimension $2n-2$.

(1) The cell C_0 of dimension $2n$ consists of all $\{z_1, \dots, z_n\}$ with $\operatorname{Re} z_1 < \cdots < \operatorname{Re} z_n$.
(2) For $k = 1, \dots, n-1$, there is a cell C_k of dimension $2n-1$ consisting of all $\{z_1, \dots, z_n\}$ with $\operatorname{Re} z_1 < \cdots < \operatorname{Re} z_k = \operatorname{Re} z_{k+1} < \cdots < \operatorname{Re} z_n$ and $\operatorname{Im} z_k \neq \operatorname{Im} z_{k+1}$; C_k is transversely oriented by the complex orientation of MP_n, and C_0 is incident on C_k from both sides—more precisely, there is a simple closed curve in $\binom{\mathbb{C}}{n}$ that intersects C_k in a single point, transversely, and is otherwise contained in C_0.
(3) For $1 \le k \le n-2$, there is a cell $C_{k,k+1}$ of dimension $2n-2$ consisting of all $\{z_1, \dots, z_n\}$ with $\operatorname{Re} z_1 < \cdots < \operatorname{Re} z_k = \operatorname{Re} z_{k+1} = \operatorname{Re} z_{k+2} < \cdots < \operatorname{Re} z_n$

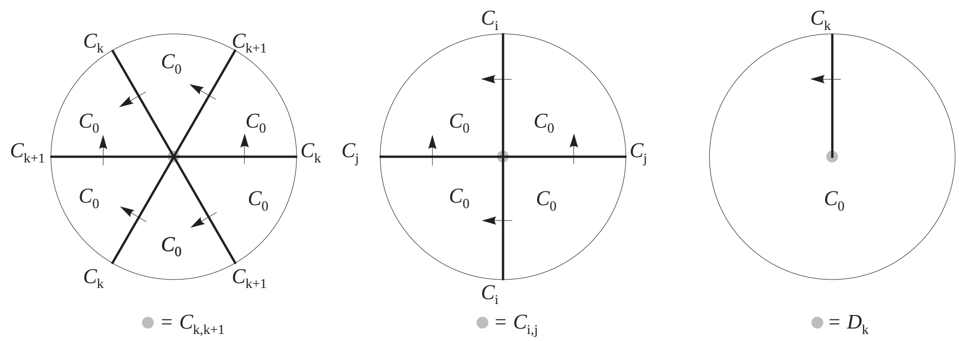

Figure 14. The transverse structure of $MP_n/{\equiv_{\mathbb{R}}}$ along its codimension-2 cells.

and card$(\{)\operatorname{Im} z_k, \operatorname{Im} z_{k+1}, \operatorname{Im} z_{k+2}\} = 3$. The two cells C_k and C_{k+1} of dimension $2n-1$ are each triply incident on $C_{k,k+1}$—more precisely, the stratification induced on a small 2–disk in MP_n intersecting $C_{k,k+1}$ in a single point, transversely, is as pictured to the left of Figure 14.

(4) For $1 \le i < j-1 \le n-2$, there is a cell $C_{i,j}$ of dimension $2n-2$ consisting of all $\{z_1, \ldots, z_n\}$ with $\operatorname{Re} z_1 < \cdots < \operatorname{Re} z_i = \operatorname{Re} z_{i+1} < \cdots < \operatorname{Re} z_j = \operatorname{Re} z_{j+1} < \cdots < \operatorname{Re} z_n$, $\operatorname{Im} z_i \ne \operatorname{Im} z_{i+1}$, and $\operatorname{Im} z_j \ne \operatorname{Im} z_{j+1}$. The two cells C_i and C_j of dimension $2n-1$ are each doubly incident on $C_{i,j}$—more precisely, the stratification induced on a small 2–disk in MP_n intersecting $C_{i,j}$ in a single point, transversely, is as pictured in the middle of Figure 14.

In $\Delta_n(\mathbb{C})/{\equiv_{\mathbb{R}}}$, there are no cells of dimension $2n$ or $2n-1$. For $1 \le k \le n-2$, there is one cell D_k of dimension $2n-2$ consisting of all $\{z_1, \ldots, z_n\}$ with $\operatorname{Re} z_1 < \cdots < \operatorname{Re} z_k = \operatorname{Re} z_{k+1} < \cdots < \operatorname{Re} z_n$ and $\operatorname{Im} z_k = \operatorname{Im} z_{k+1}$. The stratification induced on a small 2–disk in MP_n intersecting D_k transversely at a single point is as pictured at the right of Figure 14.

1.10 Contact 3-manifolds, Stein domains, and Stein surfaces

This section simply resumes basic definitions and needed results. For details on contact structures and contact 3–manifolds, see Etnyre (this Handbook). For the topology of Stein domains and Stein surfaces, see Gompf (1998). For the complex function theory of Stein domains, Stein manifolds, and Stein spaces (in general dimensions), see Gunning and Rossi (1965).

36. Definitions. Let M be a 3–manifold without boundary. A *contact structure* on M is a completely non-integrable field ξ of tangent 2-planes on M; for instance, on any round 3–sphere $S^3 \subset \mathbb{C}^2$, the field ξ of tangent 2-planes that are actually complex lines is a contact structure, called the *standard* contact structure on that sphere. A 3–manifold with a contact structure is called a *contact manifold*. A (closed) 1–submanifold L in a contact manifold M with contact structure ξ is *Legendrian* in case $T_x(L) \subset \xi_x \subset T_x(M)$ for all $x \in L$. Of course L is Legendrian if and only if $-L$ is.[9] In any contact manifold M, a Legendrian 1–submanifold L is naturally endowed with a normal line field $\xi_L^\perp$ (unique up to isotopy), which determines an annular surface $A \subset M$ (also unique up to isotopy) containing L as a retract and such that $\nu(\mathrm{i}_{L\hookrightarrow A}) = \xi_L^\perp$. In particular, a Legendrian link L in S^3 with its standard contact structure has a *natural framing* $f_L^{\mathcal{L}}$ for which $A(L, f_L^{\mathcal{L}})$ is such an annular surface. The *Thurston–Bennequin number* of a Legendrian knot $K \subset S^3$ is $\mathrm{tb}(K) := f_K^{\mathcal{L}}(K)$. For an arbitrary knot $K \subset S^3$, denote by $\mathrm{TB}(K)$ the *maximal Thurston–Bennequin number* $\max\{\mathrm{tb}(K') \ : \ K' \text{ is a Legendrian knot isotopic to } K\}$ of K; $\mathrm{TB}(K)$ is an integer (i.e., neither $-\infty$ nor ∞; see Bennequin, 1983).

37. Definitions. An *open Stein manifold* is a complex manifold that is holomorphically diffeomorphic to a topologically closed complex submanifold of some complex affine space $\mathbb{C}^N$ (equivalently, to a non-singular global analytic set $\mathcal{V}_f$ with $f \in \mathcal{O}(\mathbb{C}^N)$; see Gunning and Rossi, 1965). For instance, $\mathbb{C}^N$ itself is an open Stein manifold, as is (non-obviously) any open subset of $\mathbb{C}$. An *open Stein surface* is an open Stein manifold of real dimension 4. Let M be an open Stein surface. An *exhausting strictly plurisubharmonic function* on a non-empty open set U of M is a smooth function $\rho\colon U \to \mathbb{R}$ that is bounded below, *proper* (in the sense that $\rho^{-1}([a,b])$ is compact for all $a, b \in \mathbb{R}$), and such that for each $c \in \mathbb{R}$, the field of tangent complex lines on the 3–manifold $\rho^{-1}(c) \setminus \mathrm{crit}(\rho)$ is a contact structure, called the *natural* contact structure on that 3–manifold. (Thus, the standard structure on S^3 is the natural structure for the standard embedding $S^3 \hookrightarrow \mathbb{C}^2$.) A *Stein domain* in M is a compact codimension-0 submanifold $X \subset M$ such that X is a sublevel set $\rho^{-1}(\mathbb{R}_{\le c})$ ($c \in \mathbb{R} \setminus \rho(\mathrm{crit}(\rho))$) of an exhausting strictly plurisubharmonic Morse function $\rho\colon U \to \mathbb{R}$ on an open set $U \subset M$. A *Stein surface with boundary* is a compact 4–manifold X with a complex structure on $\mathrm{Int}\, Y$ such that X is diffeomorphic to a Stein domain in some open Stein surface by a diffeomorphism that is holomorphic on $\mathrm{Int}\, X$. A *Stein disk* is a Stein domain D in $\mathbb{C}^2$ diffeomorphic to D^4 (for instance, D^4 itself, where $\rho(z,w)$ can be taken to be $\|z,w\|^2$). Any non-singular level 3–manifold of an exhausting strictly plurisubharmonic function on a Stein surface is called a *strictly pseudoconvex* 3–manifold. A closed contact manifold is *Stein-fillable* in case it is diffeomorphic to a strictly pseudoconvex 3–manifold N by a diffeomorphism carrying its contact structure to the natural contact structure on N.

Let $X \subset \mathbb{C}^2$ be a Stein domain. It is convenient to establish notation for

[9] In particular, in the context of links in S^3 (equipped with its standard contact structure) it is common practice to refer to either L or $|L|$ as a *Legendrian link* (or knot, as the case may be), and—contrary to the conventions established earlier, which require that a link (or knot) be oriented—this practice will be followed here.

several subsets of $C(X) := \{f\colon X \to \mathbb{C} : f \text{ is continuous}\}$; although $C(X)$, equipped with its sup norm, is a well known Banach algebra, here it (and its subsets) will not be endowed with any topology.

$$
(\mathbf{K}) \qquad
\begin{aligned}
\mathcal{A}(X) &:= \{f \in C(X) : f|\operatorname{Int} X \text{ is holomorphic}\}\\
\mathcal{O}(X) &:= \{f \in C(X) : f = F|X \text{ for some open neighborhood } U \text{ of}\\
&\qquad\qquad X \text{ in } \mathbb{C}^2 \text{ and some holomorphic } F\colon U \to \mathbb{C}\}\\
\mathcal{U}(X) &:= \{f \in \mathcal{O}(X) : 0 \notin f(X)\}\\
\mathcal{W}(X) &:= \{f \in \mathcal{O}(X) : 0 \notin f(\operatorname{Int} X)\}.
\end{aligned}
$$

An element of $\mathcal{A}(X)$ is called a *germ* of a holomorphic function on X. Both $\mathcal{A}(X)$ and $\mathcal{O}(X)$ are algebras. By a standard argument, $\mathcal{U}(X)$ is the group of units of $\mathcal{O}(X)$. Clearly $\mathcal{W}(X)$ is a multiplicative subsemigroup of $\mathcal{O}(X)$ properly containing $\mathcal{U}(X)$.

Major reasons for complex analysts' interest in Stein manifolds include general theorems of which the following are special cases.

38. Theorem. *If $X \subset \mathbb{C}^2$ is a Stein disk and $\Gamma \subset U$ is a holomorphic curve in an open neighborhood U of X, then $\Gamma \cap X = f^{-1}(0)$ for some $f \in \mathcal{O}(X)$.* $\square$

39. Theorem. *Any holomorphic function on a Stein disk X can be arbitrarily closely uniformly approximated, along with any finite number of its derivatives, by the restriction to X of a polynomial function.* $\square$

The following results are especially useful for topological applications.

40. Theorem. *If $X \subset \mathbb{C}^2$ is a Stein disk with exhausting plurisubharmonic Morse function ρ, and $f \in \mathcal{O}(X)$ is such that $\operatorname{Sing}(\mathcal{V}_f) = \varnothing$, then $\mathcal{V}_f$ is ρ-ribbon.* $\square$

41. Theorem. *Every covering space of an open Stein manifold is an open Stein manifold. A finite-sheeted branched covering space of a Stein disk branched along a non-singular holomorphic curve is a Stein surface with boundary.* $\square$

2 Braids and braided surfaces

Much of the following material is treated (usually more generally and often from a different perspective) by Birman (1975) and Birman and Brendle (this Handbook), references to which should be assumed throughout.

2.1 Braid groups

42. Definition. For any $n \in \mathbb{N}_{>0}$ and $X \in \binom{\mathbb{C}}{n}$, let $B_X := \pi_1(\binom{\mathbb{C}}{n}; X)$, and call B_X *an n-string braid group.*[10] The *standard* n-string braid group is $B_n := B_{\mathbf{n}}$. By convention, the (unique) 0-string braid group is $B_0 := \{o\}$. Write o_X for the identity of B_X, and let $o^{(n)} := o_{\mathbf{n}}$.

Of course, since $\binom{\mathbb{C}}{n}$ is connected, every n-string braid group is isomorphic to B_n, but it is very convenient to allow more general basepoints. With the conventions in **42**, B_0 is isomorphic but not identical to $B_1 = \{o^{(1)}\}$; this is consistent with the obvious fact that the groups B_n, $n \in \mathbb{N}_{>0}$, being fundamental groups of pairwise distinct spaces, are pairwise disjoint.

43. Theorem.

$$(\mathbf{L}) \qquad B_n = \mathrm{gp}\left(\sigma_s, s \in \{1, \ldots, n-1\} \colon \begin{matrix} [\sigma_s, \sigma_t] \ (|s-t| > 1), \\ \lll \sigma_s, \sigma_t \ggg \ (|s-t| = 1) \end{matrix}\right) \qquad \square$$

It is usual to call (**L**) the *standard presentation* of B_n, the generators σ_s of (**L**) the *standard generators* of B_n, and the relators of (**L**) the *standard relators* of B_n. (It is also usual, and perhaps regrettable, to conflate $\sigma_s \in B_n$ with $\sigma_s \in B_{n'}$ for all $n, n' > s$. A more precise notation was proposed by Rudolph, 1985b, see **52**.) The detailed proof of **43** given by Fox and Neuwirth (1962) is, more or less exactly, an application of the usual algorithm (as in Magnus, Karrass, and Solitar, 1976), which produces a presentation of the fundamental group of a 2-dimensional cell complex with one 0-cell (the basepoint) having a generator for each 1-cell and a relator for each 2-cell, to the 2-skeleton of the cellulation of $\binom{\mathbb{C}}{n}$ that is dual to $\binom{\mathbb{C}}{n}/\equiv_{\mathbb{R}}$.

---

[10] As Magnus (1974, 1976) points out, in effect $\pi_1(\binom{\mathbb{C}}{n})$ was investigated, and recognized as a "braid group", by Hurwitz as early as 1891. Apparently this insight had been long forgotten when Fox and Neuwirth (1962, p. 119) described as "previously unnoted" their "remark that B_n may be considered as the fundamental group of the space ... of configurations of n undifferentiated points in the plane." It is interesting to speculate as to possible reasons for this instance of what Epple calls "elimination of contexts" (see especially Epple, 1995, p. 386, n. 32).

2.2 Geometric braids and closed braids

Let $I \subset \mathbb{R}$ be a closed interval. Let $p\colon I \to \binom{\mathbb{C}}{n}$ be a closed path. The multigraph $\mathrm{gr}(p) \subset I \times \mathbb{C}$ of p is called a n-string *geometric braid* for the (algebraic) braid in $B_{p(\partial I)}$ represented by p.

Let $q\colon \mathbb{O}' \to \binom{\mathbb{C}}{n}$ be a loop with domain the horizontal unknot $\mathbb{O}' \subset S^3$. The multigraph $\mathrm{gr}(q) \subset S^1 \times \mathbb{C}$ of q is a simple collection of closed curves in the open solid torus $\mathbb{O}' \times \mathbb{C}$. Using the vertical unbook $\mathbf{o}$, it is easy to create a (nearly standard) identification of $\mathbb{O}' \times \mathbb{C}$ with $S^3 \setminus \mathbb{O}$, and thus a (nearly standard) embedding of $\mathrm{gr}(q)$ into $S^3 \setminus \mathbb{O}$. This embedding has the property that $\arg(\mathbf{o})|\,\mathrm{gr}(q)\colon\ \mathrm{gr}(q) \to S^1$ is a covering map of degree n, and every simple collection of closed curves $L \subset \mathbb{O}' \times \mathbb{C}$ such that $\arg(\mathbf{o})|L\colon\ L \to S^1$ is a covering map of degree n arises in this way from some q. Call any such L an n-string *closed* $\mathbf{o}$*-braid* (Rudolph, 1988). In general, if $\mathbf{a}$ is an *unbook* (that is, an open book on S^3 with unknotted binding A), then $\mathbf{a}$ is equivalent to $\mathbf{o}$, and the class of *closed* $\mathbf{a}$*-braids* (sometimes called, slightly abusively, simply "closed braids with axis A") is defined by any such equivalence. Given a basepoint $* \in \mathbb{O}'$, q naturally represents an element of $\beta_q \in B_{q(*)}$. The notation $\widehat{\beta}(\beta_q)$ is often used for the closed $\mathbf{o}$-braid $\mathrm{gr}(q)$.

2.3 Bands and espaliers

For $n = 2$, (**L**) says that B_2 is infinite cyclic. More specifically and directly, **34**(2) shows that, for any edge $\mathbf{e} \subset \mathbb{C}$, the 2-string braid group $B_{\partial \mathbf{e}}$ is infinite cyclic with a preferred generator, say $\sigma_{\mathbf{e}}$; in fact $\sigma_{\mathbf{e}}$ depends only on $\partial \mathbf{e}$. In particular, any two 2-string braid groups are canonically isomorphic. For $n > 2$, typically no isomorphism between distinct n-string braid groups has much claim to be called canonical; however, the following is an immediate consequence of **32**.

44. Proposition. *If $X \in \binom{\mathbb{R}}{n}$, then $B_X \cong B_n$, and any path in $\binom{\mathbb{R}}{n}$ from X to $\mathbf{n}$ induces this canonical isomorphism.* □

For $n \geq 2$, if $X \cap \mathbf{e} = \partial \mathbf{e}$, then there is a *natural injection* $\mathfrak{i}_{\mathbf{e};X}\colon\ B_{\partial \mathbf{e}} \to B_X$, and $\mathfrak{i}_{\partial \mathbf{e};X}$ depends only on the isotopy class of $\mathbf{e}$ (rel. $\partial \mathbf{e}$) in $\mathbb{C} \setminus (X \setminus \mathbf{e})$.

45. Definitions. A *positive X-band* is any $\sigma_{\mathbf{e};X} := \mathfrak{i}_{\mathbf{e};X}(\sigma_{\mathbf{e}}) \in B_X$. (When X is understood, or irrelevant, $\sigma_{\mathbf{e};X}$ may be abusively abbreviated to $\sigma_{\mathbf{e}}$.) A *negative X-band* is the inverse of a positive X-band. An *X-band* is a positive or negative X-band. Let $|\sigma^{\pm 1}_{\mathbf{e};X}| := \sigma_{\mathbf{e};X}$ denote the *absolute value* of the band $\sigma^{\pm 1}_{\mathbf{e};X}$, $\varepsilon(\sigma^{\pm 1}_{\mathbf{e};X}) := \pm$ the *sign* (positive or negative) of $\sigma^{\pm 1}_{\mathbf{e};X}$, and $\mathsf{e}(\sigma^{\pm 1}_{\mathbf{e};X})$ the *edge-class* of $\sigma^{\pm 1}_{\mathbf{e};X}$, that is, the isotopy class (rel. $\partial\mathbf{e}$) of $\mathbf{e}$ in $\mathbb{C} \setminus (X \setminus \mathbf{e})$. An X-*bandword* of *length* k is a k-tuple $\mathbf{b} =: (b(1), \ldots, b(k))$ such that each $b(i)$ is an X-band.[11] An X-bandword $\mathbf{b}$ is *quasipositive* in case each $b(i)$ is a positive X-band. The *braid of* $\mathbf{b}$ is $\beta(\mathbf{b}) := b(1) \cdots b(k) \in B_X$. Every braid in B_X is the braid of some X-bandword. A braid in B_X is *quasipositive* in case it is the braid of a quasipositive X-bandword.

Orevkov (2004) gives an algorithm to determine whether or not a given braid in B_3 is quasipositive. Bentalha (2004) generalizes this to all B_n.

46. Proposition. *Any two positive X-bands are conjugate in B_X.* $\square$

An n-string braid group is the knotgroup of $\Delta_n(\mathbb{C})$ in $\binom{\mathbb{C}}{n}$ (see Rudolph, 1983b); a positive band is a meridian.

47. Proposition. *Let $X \in \binom{\mathbb{C}}{n}$, $n \geq 2$. Let $\mathbf{e}, \mathbf{f} \subset \mathbb{C}$ be two edges with $X \cap \mathbf{e} = \partial\mathbf{e}$, $X \cap \mathbf{f} = \partial\mathbf{f}$. If $\mathbf{e} \cap \mathbf{f} = \varnothing$ (resp., $\mathbf{e} \cap \mathbf{f} = \{x\} \subset \partial\mathbf{e} \cap \partial\mathbf{f}$), then $[\sigma_{\mathbf{e};X}, \sigma_{\mathbf{f};X}] = o_X$ (resp., $\lll \sigma_{\mathbf{e};X}, \sigma_{\mathbf{f};X} \ggg = o_X$).*

Proof. This is readily proved directly. Alternatively, note that: (1) the general case is isotopic to a special case in which $X = \mathbf{n}$, $\mathbf{e} = [1, 2]$, and $\mathbf{f} = [s, s+1]$ with $s \in \{2, \ldots, n\}$; (2) in such a special case, $B_X = B_n$, $\sigma_{\mathbf{e};X} = \sigma_1$ and $\sigma_{\mathbf{f};X} = \sigma_s$ are two standard generators of (**L**), and the claimed commutator and yangbaxter relators are two standard relators of (**L**). $\square$

48. Proposition. *Let $\mathcal{T} \subset \mathbb{C}$ be a planar tree. The positive $\mathsf{V}(\mathcal{T})$-bands $\sigma_{\mathbf{e};\mathsf{V}(\mathcal{T})}$, $\mathbf{e} \in \mathsf{E}(\mathcal{T})$, generate $B_{\mathsf{V}(\mathcal{T})}$; no proper subset of them does so. If $\mathbf{e} \cap \mathbf{f} = \varnothing$ (resp., $\mathbf{e} \cap \mathbf{f} = \{z\}, z \in \mathsf{V}(\mathcal{T})$), then $[\sigma_{\mathbf{e};\mathsf{V}(\mathcal{T})}, \sigma_{\mathbf{f};\mathsf{V}(\mathcal{T})}] = o_{\mathsf{V}(\mathcal{T})}$ (resp., $\lll \sigma_{\mathbf{e};\mathsf{V}(\mathcal{T})}, \sigma_{\mathbf{f};\mathsf{V}(\mathcal{T})} \ggg = o_{\mathsf{V}(\mathcal{T})}$).* $\square$

Call the $\mathsf{V}(\mathcal{T})$-bands $\sigma_{\mathbf{e};\mathsf{V}(\mathcal{T})}$, $\mathbf{e} \in \mathsf{E}(\mathcal{T})$, the $\mathcal{T}$-*generators* of $B_{\mathsf{V}(\mathcal{T})}$. Note that **48** asserts that the braid group $B_{\mathsf{V}(\mathcal{T})}$ is a quotient of

$$(\mathbf{M}) \qquad \operatorname{gp}\left(\sigma_{\mathbf{e}}, \mathbf{e} \in \mathsf{E}(\mathcal{T}) \colon \begin{array}{c} [\sigma_{\mathbf{e}}, \sigma_{\mathbf{f}}] \ (\operatorname{card}(\mathbf{e} \cap \mathbf{f}) = 0), \\ \lll \sigma_{\mathbf{e}}, \sigma_{\mathbf{f}} \ggg \ (\operatorname{card}(\mathbf{e} \cap \mathbf{f}) = 1) \end{array}\right)$$

but not that these groups are identical. In fact they are easily seen to be so if and only if $\mathcal{T}$ has no intrinsic vertices.

An *espalier* is a planar tree $\mathcal{T}$ such that each $\mathbf{e} \in \mathsf{E}(\mathcal{T})$ is a proper edge in $\mathbb{C}_-$ and $\operatorname{Re}|(\mathbf{e} \setminus \operatorname{Col}(\partial\mathbf{e} \subset \mathbf{e}))$ is injective. Rudolph (2001a) gives proofs of the following facts about espaliers.

[11] In particular, $\mathbf{n}$-bandwords are just what (since 1983a) I have long called "band representations"—a coinage which I would like to suppress, due to its misleading suggestion of a relation to group representation theory.

49. Proposition. (1) *Every planar tree is isotopic, in $\mathbb{C}$, to an espalier.* (2) *The embedding of an espalier $\mathcal{T}$ in $\mathbb{C}_-$ is determined, up to isotopy of $(\mathcal{T}, \mathsf{V}(\mathcal{T}))$ in $(\mathbb{C}_-, \mathbb{R})$, by the combinatorial structure of its cellulation together with the order induced on $\mathsf{V}(\mathcal{T})$ by its embedding in $\mathbb{R} = \partial\mathbb{C}_-$. In particular, given $X = \{x_1 < \cdots < x_n\} \subset \mathbb{R}$ and $n-1$ pairs $\{x_{i(p)} < x_{j(p)}\} \subset X$, the following are equivalent.* (a) *There is an espalier $\mathcal{T}$ with $\mathsf{V}(\mathcal{T}) = \{x_1, \ldots, x_n\}$, $\mathsf{E}(\mathcal{T}) = \{\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}\}$, and $\partial\mathbf{e}_p = \{x_{i(p)}, x_{j(p)}\}$.* (b) *For $1 \le p < q \le n-1$ the pairs $\{x_{i(p)} < x_{j(p)}\}$ and $\{x_{i(q)} < x_{j(q)}\}$ are not linked (i.e., they are either in touch or unlinked).* $\square$

50. Definition. Let $X \in \binom{\mathbb{R}}{n}$. An X-band σ is *embedded* provided that some $\mathbf{e} \in \mathsf{e}(\sigma)$ is a proper edge in $\mathbb{C}_-$; when it is given that $\sigma = \sigma^{\pm 1}_{\mathbf{e};X}$ is embedded, it is assumed without further comment that $\mathbf{e} \in \mathsf{e}(\sigma)$ is such an edge. Clearly there are exactly $n(n-1)/2$ embedded positive X-bands, one—which may be denoted $\sigma_{x_i,x_j;X}$—for each $\{x_i < x_j\} \subset X$. The embedded positive X-bands generate B_X. (A presentation (**A**) of B_n with the positive **n**-bands as generators is given by Birman et al., 1998, and many interesting conclusions drawn therefrom.) An X-bandword $\mathbf{b}$ is *embedded* in case each $b(i)$ is embedded. An X-bandword $\mathbf{b}$ is called *positive* in case it is both quasipositive and embedded.

Let $X_0 \subset X \in \binom{\mathbb{C}}{n}$. Unless $n_0 := \operatorname{card}(X_0)$ equals n, typically no non-trivial homomorphism $B_{X_0} \to B_X$ has much claim to be called canonical. For $X \subset \mathbb{R}$, however, the situation is much better (cf. **44**). In fact, given a positive embedded X_0-band $\sigma_{x_i,x_j;X_0} \in B_{X_0}$, let $\mathfrak{i}_{X_0;X}(\sigma_{x_i,x_j;X_0}) := \sigma_{x_i,x_j;X} \in B_X$.

51. Proposition. *There is a unique homomorphism $\mathfrak{i}_{X_0;X}\colon B_{X_0} \to B_X$ extending $\mathfrak{i}_{X_0;X}$ as defined on the embedded X_0-bands, and it is injective.* $\square$

Call $\mathfrak{i}_{X_0;X}$ the *canonical injection* of B_{X_0} into B_X. (The collision of notation with $\mathfrak{i}_{\mathbf{e};X}$ is unproblematic; if $X \subset \mathbb{R}$ and $\mathbf{e}$ is embedded with $\partial\mathbf{e} = \{x_i, x_j\}$, then $\mathfrak{i}_{x_i,x_j;X} = \mathfrak{i}_{\partial\mathbf{e};X}$.) **51** excuses the conflation, under the single name σ_{x_i,x_j}, of all the positive embedded bands $\sigma_{x_i,x_j;X}$ for $\xi_i < \xi_j \subset X \subset \mathbb{R}$, X finite.

52. Example. For $n \ge m$, the canonical injection $\mathfrak{i}_{\mathbf{m};\mathbf{n}}$ is implicit in the identification of B_m as a subgroup of B_n, discussed following **43**. Rudolph (1985b) proposed the notation $\beta^{(n-m)}$ for $\mathfrak{i}_{\mathbf{m};\mathbf{n}}(\beta)$, and the convention (extending the notations o and $o^{(n)}$ introduced in **42**) that for each $m \in \mathbb{N}$, σ_{m-1} denote only an element of B_m, the other standard generators of B_m being $\sigma_1^{(m-2)}, \ldots, \sigma_{m-2}^{(1)}$. This notation and convention have been widely unadopted.

53. Definition. Let $\mathcal{T}$ be an espalier. A $\mathcal{T}$-*bandword* is a $\mathsf{V}(\mathcal{T})$-bandword $\mathbf{b}$ such that every $|b(i)|$ is a $\mathcal{T}$-generator (so, in particular, a $\mathcal{T}$-bandword is an embedded $\mathsf{V}(\mathcal{T})$-bandword). A positive $\mathcal{T}$-bandword $\mathbf{b}$ is called *strictly* $\mathcal{T}$-*positive* in case every $\mathcal{T}$-generator appears among the bands $b(s)$.

For $X = \{x_1 < \cdots < x_n\} \in \mathcal{MP}^{[n]}(\mathbb{R})$, let $\mathcal{I}_X$ (resp., $\mathcal{Y}_X$) denote any espalier $\mathcal{T}$ with $\mathsf{V}(\mathcal{T}) = X$ and $\{\partial\mathbf{e} : \mathbf{e} \in \mathsf{E}(\mathcal{T})\} = \{\{x_p, x_{p+1}\} : 1 \le p < n\}$ (resp., $\{\{x_1, x_p\} : 1 < p \le n\}$). Among the combinatorial types of trees $\mathcal{T}$ with $\mathsf{V}(\mathcal{T}) = X$, $\mathcal{I}_X$ and $\mathcal{Y}_X$ represent two extreme types, which may be called *linear* (minimal number of endpoints) and *star-like* (maximal number of endpoints), respectively; further, among the linear (resp., star-like) espaliers, $\mathcal{I}_X$ (resp., $\mathcal{Y}_X$) is again extreme, in a sense that is obvious and easily formalized.

54. Example. The $\mathcal{I}_n$-generators of B_n are the standard generators of (**L**), an $\mathcal{I}_\mathbf{n}$-bandword is just a "braid word" in the usual sense (Birman, 1975, p. 70 ff.), and the braid of an $\mathcal{I}_n$-positive (or, for some authors, strictly $\mathcal{I}_n$-positive) $\mathcal{I}_n$-bandword is a "positive braid" in the usual sense (Birman, 1975; Rudolph, 1982b; Franks and Williams, 1987, etc., etc.).

55. Question. As noted after **43**, the standard generators and standard relators of (**L**) correspond naturally to the codimension-1 and codimension-2 cells of $\binom{\mathbb{C}}{n}/\!\equiv_{\mathbb{R}}$. As noted in **54**, the standard generators also correspond naturally to the edges of $\mathcal{I}_n$. Clearly, given any linear planar tree $\mathcal{T}$ with $\operatorname{card}(\mathsf{V}(\mathcal{T})) = n$, from an isotopy of $\mathcal{T}$ to $\mathcal{I}_n$ may be contrived a cellulation $\binom{\mathbb{C}}{n}/\!\equiv_{\mathcal{T}}$ with a unique codimension-0 cell, such that (1) the codimension-1 cells of $\binom{\mathbb{C}}{n}/\!\equiv_{\mathcal{T}}$ correspond to the $\mathcal{T}$-generators of $B_{\mathsf{V}(\mathcal{T})}$, and (2) the codimension-2 cells of $\binom{\mathbb{C}}{n}/\!\equiv_{\mathcal{T}}$ correspond to the relators of **L**. Now suppose that $\mathcal{T}$ is a planar tree for which there exists a (natural or contrived) cellulation $\binom{\mathbb{C}}{n}/\!\equiv_{\mathcal{T}}$, with a unique codimension-0 cell, that has property (1) but, rather than property (2), satisfies both (3) some of the codimension-2 cells of $\binom{\mathbb{C}}{n}/\!\equiv_{\mathcal{T}}$ correspond to the relators of (**M**), and (4) the remaining codimension-2 cells of $\binom{\mathbb{C}}{n}/\!\equiv_{\mathcal{T}}$ correspond to some family of extra relators exactly sufficient to convert (**M**) into a presentation of $B_{\mathsf{V}(\mathcal{T})}$. Does this imply that, in fact, $\mathcal{T}$ is linear? Specifically, does there exist a cellulation $\binom{\mathbb{C}}{4}/\!\equiv_{\mathcal{Y}_\mathbf{4}}$ with properties (1), (3), and (4)?

The methods of Birman, Ko, and Lee (1998) might help answer **55** (although the presentation they give is not obviously associated to a cellulation).

2.4 Embedded bandwords and braided Seifert surfaces

Early versions of the construction of braided Seifert surfaces described in this section appeared in Rudolph (1983b,c).

56. Construction. Let $X \in \binom{\mathbb{R}}{n}$, $T \in \binom{\mathbb{R}}{k}$, say $X =: \{x_1 < \cdots < x_n\}$, $T =: \{t_1 < \cdots < t_k\}$. Let $0 < \epsilon < \min\{|t - t'|/2 : t, t' \in T, t \ne t'\}$; in case

$k > 0$, let $I = [\min T - \epsilon, \max T + \epsilon]$. Let $\mathbf{b}$ be an embedded X-bandword of length k. To implement this construction of braided Seifert surfaces, choose a proper arc $\mathbf{e}_s \subset \mathbb{C}_-$ in the edge-class $\mathsf{e}(b(s))$ for each $s \in \mathbf{k}$, and embeddings $\eta_x\colon\ h^{(0)} \hookrightarrow \mathbb{C}_+ \times \mathbb{R}$ $(x \in X)$ and $\eta_t\colon\ h^{(1)} \hookrightarrow \mathbb{C}_- \times]t-\epsilon, t+\epsilon[$ $(t \in T)$, subject to the following conditions. For each $x \in X$,

(1) η_x is proper along a boundary arc of $h^{(0)}$,
(2) $\eta_x(h^{(0)}) \subset \{x + iy :\ y \in \mathbb{R}_{\geq 0}\} \times \mathbb{R}$,
(3) and $I \subset \partial\eta_x(h^{(0)})$.

For each $t \in T$, say $t = t_p$, $p \in \mathbf{k}$,

(4) η_{t_p} is proper along the attaching arcs of $h^{(1)}$,
(5) $\mathrm{pr}_1 \circ \eta_{t_p}|\kappa(h^{(1)})\colon\ \kappa(h^{(1)}) \to \mathbb{C}_-$ is a diffeomorphism onto $\mathbf{e}_p$,
(6) $(\mathrm{Re} \circ \mathrm{pr}_1, \mathrm{pr}_2) \circ \eta_t\colon\ h^{(1)} \to \mathbb{R} \times \mathbb{R}$ is a bowtie, and
(7) the sign of the crossing of $(\mathrm{Re} \circ \mathrm{pr}_1, \mathrm{pr}_2) \circ \eta_t|\partial h^{(1)}$ is equal to $\varepsilon(b(t))$.

It follows that

$$(\mathbf{N}) \qquad \Sigma(\mathbf{b}) := \bigcup_{x\in X} \eta_x(h^{(0)}) \cup \bigcup_{t\in T} \eta_t(h^{(1)}) \subset \mathbb{C} \times \mathbb{R}$$

is a $(0,1)$-handle decomposition $(\mathbf{C})$ of a surface. Any Seifert surface $S[\mathbf{b}] := \delta(\Sigma(\mathbf{b})) \subset \mathbb{R}^3 \subset S^3$, where $\delta\colon\ \mathbb{C} \times \mathbb{R} \to \mathbb{R}^3 :\ (z,t) \mapsto (\mathrm{Re}\, z, \mathrm{Im}\, z, t)$ and $\Sigma(\mathbf{b})$ is constructed as in **56**, is called a *braided Seifert surface of* the embedded X-bandword $\mathbf{b}$. It is easy to see that $\gamma(\mathbf{b}) := \Sigma(\mathbf{b}) \cap (\mathbb{C}_- \times I)$ is a geometric braid for $\beta(\mathbf{b})$, and that the link $\partial S[\mathbf{b}]$ is isotopic in $S^3 \supset \mathbb{R}^3$ to $\widehat{\beta}(\mathbf{b})$.

The *braid diagram of* $\mathbf{b}$ is the pair $\mathbf{BD}(\mathbf{b}) =: (\mathbf{P}(\mathbf{BD}(\mathbf{b})), \mathbf{I}(\mathbf{BD}(\mathbf{b})))$, where $\mathbf{P}(\mathbf{BD}(\mathbf{b})) := (\mathrm{Re}, \mathrm{pr}_2)(\gamma(\mathbf{b})) \subset \mathbb{R} \times I$ and $\mathbf{I}(\mathbf{BD}(\mathbf{b}))$ is the information about the signs of the crossings of $\mathbf{P}(\mathbf{BD}(\mathbf{b}))$ (indicated graphically in the style of **11**(2); see Figure 15). A *braid diagram for* $\beta \in B_X$ is a braid diagram of any bandword $\mathbf{b}$ with $\beta(\mathbf{b}) = \beta$. A *standard braid diagram for* $\beta \in B_n$ is a braid diagram of any $\mathcal{I}_{\mathbf{n}}$-bandword $\mathbf{b}$ with $\beta(\mathbf{b}) = \beta$.

Of course $\mathbf{b}$ determines $S[\mathbf{b}]$ and $\widehat{\beta}(\mathbf{b})$ up to isotopy; and clearly, up to isotopy (even isotopy through braided Seifert surfaces), none of X, T, or the collection of specific edge-class representatives $\mathbf{e}_s$ is necessary *per se* to the construction of $S[\mathbf{b}]$—all that is needed is a modicum of combinatorial information extracted from $\mathbf{b}$. That information can be encoded as the k-tuple of triples $((i_{\mathbf{b}}(1), j_{\mathbf{b}}(1), \varepsilon_{\mathbf{b}}(1)), \ldots, (i_{\mathbf{b}}(k), j_{\mathbf{b}}(k), \varepsilon_{\mathbf{b}}(k)))$ with $\{x_{i_{\mathbf{b}}(s)} < x_{j_{\mathbf{b}}(s)}\} := \partial\mathbf{e}_s \subset X$ and $\varepsilon_{\mathbf{b}}(s) := \varepsilon(b(s))$. Another convenient encoding is graphical, using *charged fence diagrams* (see Rudolph, 1992a, 1998). Figure 16 pictures a (very simple) charged fence diagram.[12]

[12] In Rudolph (1992a), I described fence diagrams as "my synthesis of some diagrams that H. Morton used to describe certain Hopf-plumbed fiber surfaces in 1982 . . . and 'square bridge projections' as described by H. Lyon" (Lyon, 1980). By 2001a, I had recalled that I first saw fences (and braid groups!) "c. 1959, in one of Martin Gard-

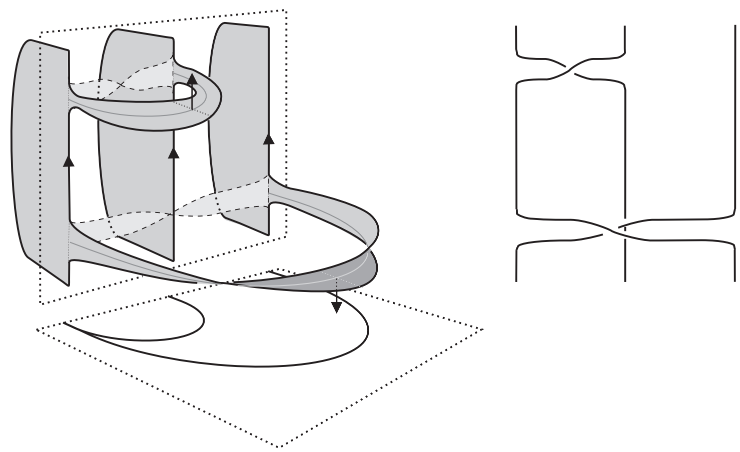

Figure 15. Left: a surface $\Sigma(\mathbf{b})$; $\varepsilon(b(1)) = -$, $\varepsilon(b(1)) = +$. Right: a braid diagram for $\beta(\mathbf{b})$.

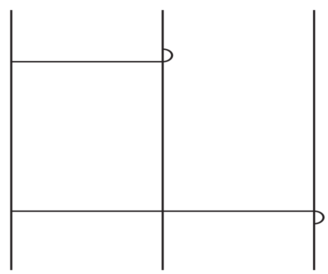

Figure 16. A charged fence diagram for the surface in Figure 15.

There is a more or less canonical way to turn an n-string braid diagram $\mathbf{BD}(\mathbf{b})$ into a standard link diagram $\mathbf{D}(\widehat{\beta}(\mathbf{b}))$: identify $\mathbb{C}$ with $\mathbb{R}^2 \supset \mathbf{P}(\mathbf{BD}(\mathbf{b}))$ by $(\mathrm{Re}, \mathrm{Im})$; in $\mathbb{C}$, attach n arcs to $\mathbf{P}(\mathbf{BD}(\mathbf{b})) \subset \mathbb{C}$, preserving orientation, so as to create a normal collection $\mathbf{P}(\mathbf{D}(\widehat{\beta}(\mathbf{b})))$ of closed curves with no new crossings; and let $\mathbf{I}(\mathbf{D}(\widehat{\beta}(\mathbf{b})))$ be $\mathbf{I}(\mathbf{BD}(\mathbf{b}))$.

57. Proposition. (1) *Let* $\mathbf{b}$ *be an* $\mathcal{I}_{\mathbf{n}}$*-bandword. Let* $L = \widehat{\beta}(\mathbf{b})$*. The arcs* $\mathbf{P}(\mathbf{D}(L)) \setminus \mathrm{Int}\,\mathbf{P}(\mathbf{BD}(\mathbf{b}))$ *can be chosen so that* $\mathsf{O}_{\mathbf{D}(L)} = \mathsf{O}^+_{\mathbf{D}(L)}$*; if they are, then* $\mathbf{D}(L)$ *is nested and the diagrammatic Seifert surface* $S(\mathbf{D}(L))$ *is isotopic to* $S[\mathbf{b}]$*.* (2) *Conversely, if* $\mathbf{D}(L)$ *is nested and* $\mathsf{O}_{\mathbf{D}(L)} = \mathsf{O}^+_{\mathbf{D}(L)}$*, then* (*up to isotopy of* $\mathbf{P}(\mathbf{D}(L))$) *there exists an* $\mathcal{I}_{\mathbf{n}}$*-bandword* $\mathbf{b}$*,* $n := \mathrm{card}(\mathsf{O}_{\mathbf{D}(L)})$*, such that* $L = \widehat{\beta}(\mathbf{b})$ *and* $S(\mathbf{D}(L))$ *is isotopic to* $S[\mathbf{b}]$*.* (3) *If* $\mathbf{b}$ *is an embedded*

ner's 'Mathematical Games' columns" (reprinted in Gardner, 1966, as Chapter 2, "Group Theory and Braids"). Since 2001 I have become aware of "amida-diagrams" (introduced by Yamamoto, 1978, precisely to construct certain special Seifert surfaces) and "wiring diagrams" (cf. Cordovil and Fachada, 1995, and other literature from the theory of line arrangements), both closely related to fences, as are (via their essential identity with square bridge projections) the "barber-pole projections" attributed in Rudolph (1992a) to Thurston (unpublished), Erlandsson (1981), and Kuhn (1984). Is this another instance of Epple's "elimination of contexts", or a mere multiplication of contexts?

$\mathbf{n}$*-bandword but not an* $\mathcal{I}_{\mathbf{n}}$*-bandword, and* $L = \widehat{\beta}(\mathbf{b})$*, then* $\mathbf{D}(L)$ *is nested but* $\chi(S(\mathbf{D}(L))) < \chi(S[\mathbf{b}])$*, so* $S(\mathbf{D}(L))$ *and* $S[\mathbf{b}]$ *are not diffeomorphic, let alone isotopic.* □

57 shows that **25** is equivalent to the following.

58. Theorem. *For all* n*, for all* $\beta \in B_n$*,*

$$\text{(P)} \qquad \operatorname{ord}_v P_{\widehat{\beta}} \geq e(\beta) - n + 1,$$

where $e\colon B_n \to \mathbb{Z}$ *is abelianization (exponent sum).* □

Proofs of the following theorem apppear in Rudolph (1983b,c); it also follows from results of Bennequin (1983) on "Markov surfaces" (see Rudolph, 1985b).

59. Theorem. *If* $S \subset \mathbb{R}^3 \subset S^3$ *is a Seifert surface, then there exist* X *and an embedded* X*-bandword* $\mathbf{b}$ *such that* S *is isotopic to* $S[\mathbf{b}]$*.* □

60. Question. The *braid index* $\operatorname{brin}(L)$ of a link L is the minimum $n \in \mathbb{N}_{>0}$ such that L is isotopic to $\widehat{\beta}(\mathbf{b})$ for some n-bandword $\mathbf{b}$ (which can, since its length is not an issue, be taken to be embedded); see Birman and Brendle (this Handbook). Let the *braided Seifert surface index* of a Seifert surface S be the minimum $n \in \mathbb{N}_{>0}$ such that S is isotopic to $S[\mathbf{b}]$ for some embedded $\mathbf{n}$-bandword $\mathbf{b}$; by **59**, this invariant of S is an integer. How can it be calculated or estimated? This question is the analogue for Seifert surfaces of "Open Problem 1" of Birman and Brendle (this Handbook), which asks how to calculate $\operatorname{brin}(L)$. Clearly the braided Seifert surface index of S is at least as great as $\operatorname{brin}(\partial S)$. Examples of Hirasawa and Stoimenow (2003) show that this inequality can be strict.

61. Definition. A Seifert surface S is *quasipositive* in case S is isotopic to a braided Seifert surface $S[\mathbf{b}]$ for some positive X-bandword $\mathbf{b}$. A link is *strongly quasipositive* in case it has a quasipositive Seifert surface.

2.5 Plumbing and braided Seifert surfaces

Let $X \in \binom{\mathbb{R}}{n}$. For $x \in X$, write $X_{\le x} := X \cap \mathbb{R}_{\le x}$, $X_{\ge x} := X \cap \mathbb{R}_{\ge x}$.

62. Definitions. Let $x \in X$. Let $\mathbf{b}$ be an embedded X-bandword of length k. Let $T_{\mathbf{b};\le x} := \{s \in \mathbf{k} : b(s) \in \mathbb{i}_{X_{\le x},X}\}$ and $T_{\mathbf{b};\ge x} := \{s \in \mathbf{k} : b(s) \in \mathbb{i}_{X_{\ge x},X}\}$; let $k_{\le} := \operatorname{card}(T_{\mathbf{b};\le x})$, $k_{\ge} := \operatorname{card}(T_{\mathbf{b};\ge x})$, so in every case $k_{\le} + k_{\ge} \le k$. In case $k_{\le} + k_{\ge} = k$, say that $\mathbf{b}$ is *deplumbed by* $x \in X$. Writing $T_{\mathbf{b};\le x} =: \{u_1 < \cdots < u_{k_\le}\}$, $T_{\mathbf{b};\ge x} =: \{v_1 < \cdots < v_{k_\ge}\}$, let $\mathbf{b}_{\le x} := (b_{\le x}(1), \ldots, b_{\le x}(k_\le))$ (resp., $\mathbf{b}_{\ge x} := (b_{\ge x}(1), \ldots, b_{\ge x}(k_\ge)))$ be the embedded $X_{\le x}$-bandword (resp.,

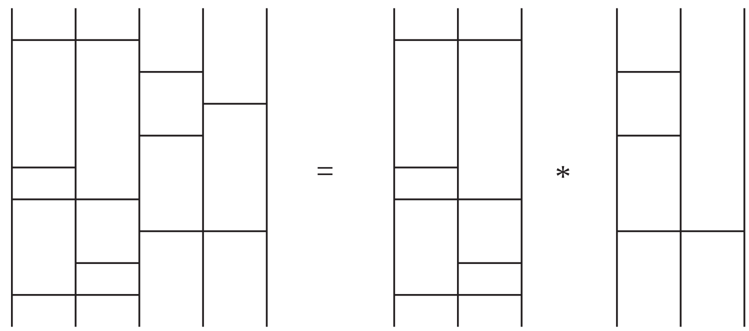

Figure 17. An example of deplumbing.

$X_{\geq x}$-bandword) for which $\mathbf{i}_{X_{\leq x},X})b_{\leq x}(s) = b(u_s)$ (resp., $\mathbf{i}_{X_{\geq x},X})b_{\geq x}(t) = b(v_t)$). By construction **56**, if $\mathbf{b}$ is deplumbed by $x \in X$, then $\Sigma(\mathbf{b}) = \Sigma(\mathbf{b}_{\leq x}) \cup \Sigma(\mathbf{b}_{\geq x})$ and $\Sigma(\mathbf{b}_{\leq x}) \cap \Sigma(\mathbf{b}_{\geq x}) = \Sigma(\mathbf{b} \cap \mathbb{C} \times \{x\}$ is a disk (in fact, a 0-handle of the constructed $(0,1)$-handle decompositions of $\Sigma(\mathbf{b}_{\leq x}) \cup \Sigma(\mathbf{b}_{\geq x})$); the situation is illustrated using fence diagrams in Figure 17. The inverse of this operation, which combines two braided Seifert surfaces into one, *braided Stallings plumbing*.

Although it appears to be a rather special case of *Stallings plumbing*, or *Murasugi sum*, a geometric operation on Seifert surfaces that has been extensively studied (by Murasugi, 1963; Conway, 1970; Siebenmann, 1975; Stallings, 1978; Gabai, 1983a,b, 1986, and many others; notations, definitions, and some history are given by Rudolph, 1998), braided Stallings plumbing is in fact perfectly general, according to Rudolph (1998).

63. Theorem. *Up to isotopy, every Murasugi sum $S_0 * S_1$ of Seifert surfaces S_0, S_1 is a braided Stallings plumbing of braided Seifert surfaces.* □

2.6 Labyrinths, braided surfaces in bidisks, and braided ribbons

Let X be a topological space. Given $f\colon X \to MP_n$, let $f_\triangle\colon X \times \mathbb{C} \to \mathbb{C} : (x, w) \mapsto f(x)(w)$, so that $\mathrm{gr}(R \circ f) = f_\triangle^{-1}(0)$. Generally it can be a subtle matter to determine whether a map $F\colon X \times \mathbb{C} \to \mathbb{C}$ (with fibers of cardinality bounded by $n \in \mathbb{N}$) is $f_\triangle$ for some (continuous) $f\colon X \to MP_n$, but in case F is known to be such, write $f = F_\triangledown$.

64. Definition. A map $f\colon X \to MP_n$ is *amazing* provided that the partition of M into connected components of inverse images $f^{-1}(C)$ of the various cells C of $MP_n/\equiv_{\mathbb{R}}$ has some finite refinement that is a stratification of X. A *labyrinth* for f is any such stratification.

65. Examples. Amazing maps have proved useful in several applications.

65.1. If $F(z,w) := f_0(z)w^n + f_1(z)w^{n-1} + \cdots + f_n(z) \in \mathbb{C}[z,w]$ is a polynomial in Weierstrass form, then $F_\triangledown \colon \mathbb{C} \to MP_\mathbb{C}$ is (in effect) F considered as an element of $\mathbb{C}[z][w] \cong \mathbb{C}[z,w]$, and is amazing. Classically, in this situation w is said to be the "n-valued algebraic function of z without poles" determined by the algebraic curve $\mathcal{V}_F$ (Bliss, 1933; Hansen, 1988), and a more or less canonical set of "branch cuts" for w can be extracted systematically from the labyrinth of $F_\triangledown$. Rudolph (1983a) determined the topology of the labyrinth of $F_\triangledown$ in the case $F_\epsilon(z,w) = (w-1)(w-2)\cdots(w-n+1)(w-z)+\epsilon$, first for $\epsilon = 0$, then for small $\epsilon \neq 0$, and used it to give the first proof of **77**.

65.2. The term labyrinth was introduced by Dung and Hà (1995), in their study of line arrangements in $\mathbb{C}^2$. In a linear coordinate system chosen so that no line in the arrangement is vertical, such an arrangement is $\mathcal{V}_F$, where $F(z,w) \in \mathbb{C}[z,w]$ is a product of (unrepeated) factors $w + az + b$ (as in (1), where F_0 corresponds to an arrangement of $n-1$ horizontal lines crossed by a single diagonal line). Again, $F_\triangledown$ exists and is amazing. (See §6.4 for further references on arrangements.)

65.3. Orevkov (1988) made excellent use of the observation that, if $V \subset \mathbb{C}^2$ is both a $\mathbb{C}$-algebraic curve and a nodal surface, then, after an arbitrarily small linear change of coordinates in $\mathbb{C}^2$, $V = \mathcal{V}_F$ where $F \in \mathbb{C}[z,w]$ is in Weierstrass form and the labyrinth of the amazing map $F_\triangledown$ has a very special form. (See §6.2 for further discussion of this application.)

65.4. Orevkov (1996) developed the theory of labyrinths in careful detail, and applied it to the Jacobian Conjecture. (See §6.3).

Amazing maps, and the facts about the low-codimension cells of $MP_n/\equiv_\mathbb{R}$ laid out in **35**, together allow two closely related constructions—of braided surfaces in bidisks, and braided ribbons in D^4—to be described and carried out more precisely than in the original sources (Rudolph, 1983b, 1985b).

66. Proposition. *Let M be a surface. If $f \colon M \to MP_n$ is transverse to all the cells of $MP_n/\equiv_\mathbb{R}$, then f is amazing. More specifically, f has a labyrinth $M/\equiv$ with the following properties.*

(1) *The union of the vertices and edges of $M/\equiv$ is a graph $\Lambda(f)$ with no vertex of degree $d \notin \{1,4,6\}$.*
(2) *The association by f to each edge $\mathbf{e}$ of $\Lambda(f)$ of a codimension-1 cell of $MP_n/\equiv_\mathbb{R}$ endows $\mathbf{e}$ with* (a) *a natural transverse orientation* (*and therefore a natural orientation, so that $\mathbf{e}$ is naturally either a simple closed curve or an arc*) *and* (b) *a clew $\sigma(\mathbf{e})$ in the set $\{\sigma_1, \ldots, \sigma_{n-1}\}$ of standard generators of the presentation* (**L**) *of B_n.*
(3) *The association by f to each vertex x of $\Lambda(f)$ of a codimension-2 cell of $MP_n/\equiv_\mathbb{R}$ endows x with either* (a) *a standard relator of* (**L**), *in case* $\mathrm{val}_{\Lambda(f)}(x) \in \{4,6\}$, *or* (b) *a trivial relator $\sigma_s^{\pm1}\sigma_s^{\mp1}$ in case* $\mathrm{val}_{\Lambda(f)}(x) = 1$.

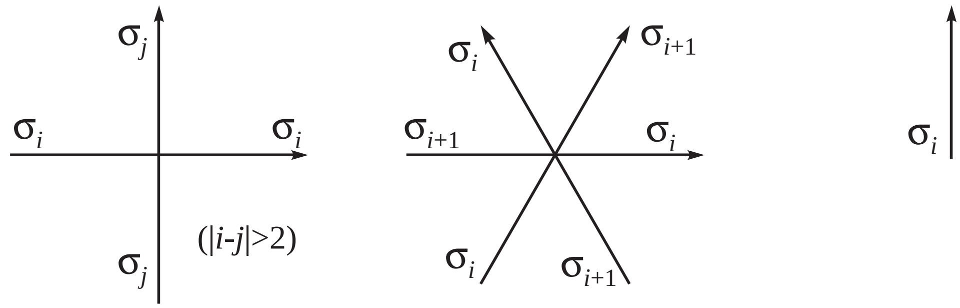

Figure 18. Clews in the labyrinth of a $MP_n/{\equiv_{\mathbb{R}}}$-transverse map on a surface.

(4) *The clews* (2b) *are consistent with the relators* (3) *as pictured in Figure* 18. (*In each local picture, the mirror image of the illustrated situation is also allowed, as is simultaneous reversal of all edge orientations.*) $\square$

67. Construction. A *braided surface* of *degree n* in $D^2 \times \mathbb{C}$ is $S_f := f_{\triangle}^{-1}(0)$, where $f\colon\ D^2 \to MP_n$ is transverse to all the cells of $MP_n/{\equiv_{\mathbb{R}}}$ as in **66**.

68. Proposition. *A braided surface S_f in $D^2 \times \mathbb{C}$ is a surface, and* $\mathrm{pr}_1|S\colon\ S \to D^2$ *is a simple branched covering map branched over* $\mathsf{V}(\Lambda(f))$. $\square$

In these terms, various facts about the constructions in (Rudolph, 1983b, 1985b) can be phrased as follows. Since any braided surface S_f is compact, it is actually contained in $D^2 \times rD^2$ for some r; clearly only braided surfaces in $D^2 \times D^2$ need be considered. The obvious identification of $S^1 \times D^2 \subset \partial(D^2 \times D^2)$ with $\mathrm{Nb}(\mathbb{O}' \subset S^3)$ identifies the boundary of a braided surface $S_f \in D^2 \times D^2$ with a 1–submanifold of $\mathrm{Nb}(\mathbb{O}' \subset S^3)$, which is a closed n-string $\mathbf{o}$-braid isotopic to some $\widehat{\beta}(\mathbf{b})$ of length $\mathrm{card}(\mathsf{V}(\Lambda(f)))$. The bandword $\mathbf{b}$ can be read off from the clews of $\Lambda(f)$. In particular, $\beta(\mathbf{b})$ is the ordered product of the clews on the (oriented) edges of $\Lambda(f)$ that intersect S^1. Every bandword determines a braided surface which is unique up to isotopy (through braided surfaces). Rudolph (1983a) characterizes quasipositive braided surfaces as follows.

69. Theorem. *A braided surface S_f, where f is holomorphic, determines a quasipositive bandword. Any braided surface that determines a quasipositive bandword is isotopic* (*through braided surfaces*) *to S_f with f a polynomial.* $\square$

There is a straightforward way to "round the corners" of $D^2 \times D^2$ by a *smoothing* $D^2 \times D^2 \to D^4$—that is, a homeomorphism which is a diffeomorphism off $S^1 \times S^1$—so that a braided surface in $D^2 \times D^2$ is carried to a ribbon surface in D^4. The converse is also true (Rudolph, 1983b, 1985b).

70. Theorem. *Up to isotopy, every ribbon surface in D^4 is the image of a braided surface in $D^2 \times D^2$ by a smoothing $D^2 \times D^2 \to D^4$.* $\square$

71. Theorem. *Let* $\mathbf{b}$ *be an embedded bandword. The ribbon surface S in D^4 with $\partial S = \partial S[\mathbf{b}] \subset S^3$, produced by pushing* $\mathrm{Int}\,S[\mathbf{b}]$ *into* $\mathrm{Int}\,D^4$*, is isotopic* *to the ribbon surface which is the image of the braided surface in $D^2 \times D^2$ determined by the bandword* $\mathbf{b}$. $\square$

72. Definitions. A *node* (resp., *cusp*) in an n-string braid group B_X is the square (resp., cube) of a X-band. A *nodeword* (resp., *cuspword*) in B_X is a k-tuple $\mathbf{b} =: (b(1), \ldots, b(k))$ such that each $b(i)$ is a band or a node (resp., a band, a node, or a cusp) in B_X.

The proof of **70** can be generalized (cf. Rudolph, 1983b) to show that every "nodal ribbon surface" in D^4 can be realized, up to smoothing and isotopy, by a "nodal braided surface" in $D^2 \times D^2$. Orevkov (1998) proved an immense generalization of **69**; the following special case of Orevkov's result is easy to state and more than adequate for this review.

73. Theorem. *If* $\mathbf{b}$ *is a quasipositive cuspword in* B_n*, then there is a holomorphic cusp curve in* $D^2 \times D^2$ *that corresponds to* $\mathbf{b}$ *via an appropriate holomorphic map* $f\colon D^2 \to MP_n$. $\square$

The map f in **73** is amazing; it is transverse to all the cells of $MP_n/{\equiv_{\mathbb{R}}}$ if and only if the cuspword $\mathbf{b}$ is a bandword.

3 Transverse $\mathbb{C}$-links

74. Definition. A link $L \subset S^3$ is a *transverse* $\mathbb{C}$*-link* provided that (S^3, L) is diffeomorphic to $(\Sigma, \mathcal{V}_f \cap \Sigma)$, where: (1) $D \subset \mathbb{C}^2$ is a Stein disk bounded by a strictly pseudoconvex 3–sphere Σ, and (2) $f \in \mathcal{O}(D)$ is a holomorphic function with $\operatorname{Sing}(\mathcal{V}_f) \cap \Sigma = \varnothing$, such that (3) the complex manifold $\operatorname{Reg}(\mathcal{V}_f)$ intersects Σ transversely and (4) $\mathcal{V}_f \cap \Sigma$ is non-empty (and therefore a smooth, naturally oriented closed 1–submanifold of Σ).

Various modifications may be made to the specifications in **74** without changing the class of links so specified.

75. Proposition. (1) *The requirement that Σ be strictly pseudoconvex can be somewhat relaxed.* (2) *The requirement that Σ be strictly pseudoconvex can be considerably strengthened: Σ can be required to be convex, or even to be a round sphere.* (3) *$\mathcal{V}_f$ can be required to be non-singular, to be algebraic, or to be both at once.*

Proof (*sketches*). (1) Here, no attempt will be made to state the most general theorem (see **76**(2)). A sufficient example for present purposes is the following. Let $\gamma \subset \mathbb{C}$ be an arbitrary simple closed curve and $G \subset \mathbb{C}$ the 2–disk that it bounds, so that for any $r > 0$ the *bicylinder* $D := G \times rD^2$ is diffeomorphic (indeed, biholomorphic) to the bidisk $D_{1,1}$. Although D is not a Stein domain, $\operatorname{Int} D$ is an open Stein manifold, and $(D, \partial D)$ can be arbitrarily closely approximated by Stein domains. The result follows from **39** and transversality. (2) As pointed out by Boileau and Orevkov (2001), theorems of Eliashberg (1990) show that for this purpose all Stein disks are equally good. (3) A nearby level set of f will have no singularities and give the same link type; by **39**, f is arbitrarily closely approximated by its sufficiently high-degree Taylor polynomials, and again we get the same link type. □

76. Questions. **74** and **75** immediately suggest a number of questions.

76.1. Is there an algorithm for determining whether or not a given link is a transverse $\mathbb{C}$-link? (It follows from **77** that the set of isotopy classes of transverse $\mathbb{C}$-links is recursive: there exists an algorithm that produces a link in every such isotopy class. Thus the question becomes: Is the set of isotopy classes of transverse $\mathbb{C}$-links recursively enumerable?)

76.2. How much can the requirement that Σ be strictly pseudoconvex be relaxed (as in **75**(1))? See Rudolph (1985a) for a cautionary result.

76.3. Let L be a transverse $\mathbb{C}$-link. By **75**(3), up to isotopy $L = S^3 \cap \mathcal{V}_f$ for some $f \in \mathbb{C}[z, w]$. What is the minimum degree of such a polynomial f?

In light of **75**(2), the same question can be asked with the roundness of S^3 weakened to convexity or strict pseudoconvexity of a 3–sphere Σ. Can either of these weakenings strictly decrease the minimum? Calculations of, or even good upper bounds for, this (or these) invariant(s) of L would have applications to finding embeddings of certain Stein domains (namely, cyclic branched covers of D^4 branched over holomorphic curves) into algebraic surfaces of unexpectedly low degree in $\mathbb{C}^3$ (Boileau and Rudolph, 1995).

3.1 Transverse $\mathbb{C}$-links are the same as quasipositive links

77. Theorem. *Every quasipositive link is a transverse $\mathbb{C}$-link.* □

Proofs of **77** (with some variation in the details) are presented in (Rudolph, 1983a, 1984, 1985b). A remarkable theorem of Boileau and Orevkov (2001) asserts the converse.

78. Theorem. *Every transverse $\mathbb{C}$-link is a quasipositive link.* □

79. Corollary. *L is a transverse $\mathbb{C}$-link if and only if L is quasipositive.* □

80. Corollary. *Every isotopy class of transverse $\mathbb{C}$-links is represented by a transverse intersection $\mathcal{V}_f \cap \{(z,w) \in \mathbb{C}^2 : \|(z,w)\| = 1\}$, where $f \in \mathbb{C}[z,w]$ and $\operatorname{Sing}(\mathcal{V}_f) = \varnothing$.* □

As Boileau and Orevkov note, their proof (the only one known to date) is completely non-constructive, relying strongly as it does on the theory of pseudoholomorphic curves (Gromov, 1985).

81. Question. Is there a constructive proof of **78**? Biding such a proof, can an upper bound on $\{n : L \text{ is isotopic to a quasipositive } n\text{-string braid}\}$ be deduced from other invariants of a transverse $\mathbb{C}$-link L?

3.2 Slice genus and unknotting number of transverse $\mathbb{C}$-links

82. Theorem. *If $f \in \mathcal{O}(D^4)$ is such that $\operatorname{Sing}(\mathcal{V}_f) = \varnothing$, and L is the transverse $\mathbb{C}$-link $\mathcal{V}_f \cap S^3$, then $X_s(L) = \chi(\mathcal{V}_f \cap D^4)$.* □

83. Corollary. *If $\mathbf{b}$ is a quasipositive n-string bandword of length k, then $X_s(\widehat{\beta}(\mathbf{b})) = n - k$.* □

It had been known at least as early as 1982 (Boileau and Weber, 1983; Rudolph, 1983d) that these results would follow from a local version of the Thom Conjecture (see **A.6**): either (in the case of **82**) that L be a torus link $O\{n,n\}$ for some $n \in \mathbb{N}_{>0}$, or (in the case of **83**) that $\mathbf{b}$ be the important $\mathcal{I}_{\mathbf{n}}$-bandword [13]

$$(\mathbf{Q}) \qquad (\overbrace{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \ldots, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}}^{n \text{ repetitions}}) =: \nabla_n$$

with closed braid $\widehat{\beta}(\nabla_n) = O\{n, n\}$. The first proof of the local Thom Conjecture was given by Kronheimer and Mrowka (1993), using gauge theory for embedded surfaces, a 4-dimensional technique. The full Thom Conjecture (and more) was proved using Seiberg–Witten theory—also a 4-dimensional technique—by Kronheimer and Mrowka (1994); Morgan, Szabó, and Taubes (1996). Recently Rasmussen (2004) announced a purely 3-dimensional combinatorial proof based on Khovanov homology.

84. Corollary. *If L is a transverse $\mathbb{C}$-link then $X_r(L) = X_s(L)$.* □

It was also known (Boileau and Weber, 1983; Rudolph, 1983d) that the truth of the local Thom Conjecture implies an affirmative answer to "Milnor's Question" (see **A.5**) on unknotting numbers of links of singularities (§4.1) or—more generally (see Rudolph, 1983b)—closed positive braids in the sense of **54**.

85. Corollary. *If L is the link of a singularity (e.g., a torus link $O\{m, n\}$ with $0 \le m < n$), then $\ddot{\mathrm{u}}(L) = \mathrm{node}(L) = (\mathrm{card}(\pi_0(L)) - X(L))/2$.* □

3.3 Strongly quasipositive links

82–**85** immediately imply the following.

86. Theorem. *If L is a strongly quasipositive link with quasipositive Seifert surface S, then $\chi(S) = X(L) = X_r(L) = X_s(L)$. If K is a strongly quasipositive knot with quasipositive Seifert surface S, then $g(S) = g(K) = g_r(K) = g_s(K) \le \mathrm{node}(K) \le \ddot{\mathrm{u}}(K)$.* □

To fully exploit **86** it is necessary to have a good supply of transverse $\mathbb{C}$-links or, equivalently, quasipositive (closed) braids. Clearly every strongly quasipositive link (**61**) is quasipositive. The converse is false; for instance, the closed

[13] There is a long and worthy tradition of designating (the braid of) this $\mathcal{I}_{\mathbf{n}}$-bandword as Δ^2 (Birman, 1975; Birman and Brendle, this Handbook, etc., etc.), a usage apparently introduced by Fadell (1962) in homage to P. A. M. Dirac (who wrote that he "first thought of the string problem about 1929" in a letter quoted by Gardner, 1966, addendum to Chapter 2; see also Newman, 1942). Nonetheless, in the context of the knot theory of complex plane curves, where discriminants (which have their own, algebro-geometric, traditional claim on the notation Δ) abound, I believe that the makeshift of using ∇ to stand in for Δ^2 is preferable to the further overloading of the symbol Δ.

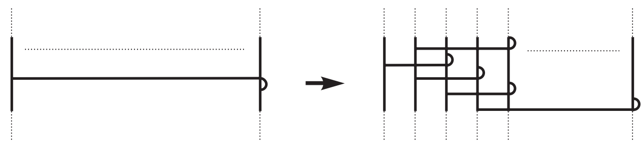

Figure 19. “Trefoil insertion” on a fence diagram preserves the Seifert form while eliminating a negative band.

braid of ${}^{\sigma_2^3}\sigma_1\sigma_1 \in B_3$ is a quasipositive knot, but not strongly quasipositive (by **83**). Nonetheless, the class of strongly quasipositive links is very varied.

3.3.1 S-equivalence and strong quasipositivity

87. Proposition. *No link invariant calculable from a Seifert matrix* (*for instance, the Alexander polynomial of a knot, the signatures of a knot or link, etc., etc.*) *can tell whether or not a link is quasipositive.*

Proof. Let $[(L_i, L_j)_S]$ be a Seifert matrix for a link L with Seifert surface S. By **59**, we may assume that $S = S[\mathbf{b}]$, for some embedded X-bandword $\mathbf{b}$ of length k. Let there be $m \le k$ negative bands in $\mathbf{b}$. If $m = 0$, we are done. Otherwise, let $b(s)$ be negative, say $b(s) = \sigma_{x_p,x_q}^{-1}$, with $\{x_p < x_q\} \subset X$. Let $X' := X \cup \{x_1', x_2', x_3', x_4'\}$, where $x_p < x_1' < x_2' < x_3' < x_4' < x_q$. Let $b'(t) = \mathfrak{i}_{X,X'}b(t)$ for $1 \le t < s$, $b'(s) = \sigma_{x_3',x_q}$, $b'(s+1) = \sigma_{x_2',x_4'}$, $b'(s+2) = \sigma_{x_1',x_3'}$, $b'(s+3) = \sigma_{x_p,x_2}$, $b'(s+4) = \sigma_{x_1',x_4'}$, and $b'(t) = \mathfrak{i}_{X,X'}b(t+4)$ for $s+4 < t \le k+4$ (see Figure 19 for a rendition of this operation in terms of fence diagrams). Manifestly $S[\mathbf{b}]$ is diffeomorphic to $S[\mathbf{b}']$ by a diffeomorphism that is the identity of a single 1-handle, and an easy calculation shows that they have identical Seifert matrices. There are only $m-1$ negative bands in $\mathbf{b}'$, so this result of Rudolph (1983c) is true by induction on m. □

88. Corollary. (1) *Every S-equivalence class of knots contains strongly quasipositive knots.* (2) *Every knot can be converted to a strongly quasipositive knot by a sequence of “doubled-delta moves”.*

Proof. Knots K_0 and K_1 are *S-equivalent* (in the first instance, Trotter, 1962, “h-equivalent”) just in case K_i has a Seifert surface S_i such that some homology bases of S_0 and S_1 produce identical Seifert matrices; so (1) is immediate from (the proof of) **87**. Naik and Stanford (2003) introduced the “doubled-delta move”, an operation on standard link diagrams illustrated in Figure 20, and proved that the equivalence relation on knots generated by applying such moves is precisely S-equivalence; (2) follows directly. □

89. Question. Conway (1970) introduced an operation on knots and links called *mutation*. The result L' of applying mutation to a link L is called a

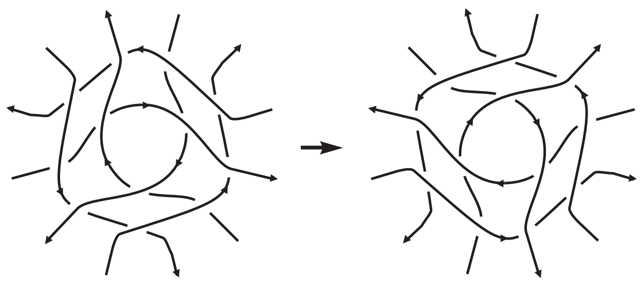

Figure 20. The "doubled-delta" move.

mutant of L. Although mutant links can be non-isotopic, they are indistinguishable by a wide variety of link invariants. Mutation was generalized by Anstee, Przytycki, and Rolfsen (1989). The result L' of applying the operation defined by Anstee, Przytycki, and Rolfsen to L is called a *rotant* of L. The doubled-delta move is a special case of Anstee, Przytycki, and Rolfsen's operation. Obviously **88**(2) can be rephrased as "the doubled-delta move need not preserve strong quasipositivity". In what circumstances is a rotant of a strongly quasipositive link strongly quasipositive? In particular, is a mutant of a strongly quasipositive link necessarily strongly quasipositive?

3.3.2 Characterization of strongly quasipositive links

There is a simple characterization of quasipositive Seifert surfaces, and thus—in some sense—of strongly quasipositive links.

90. Theorem. *A Seifert surface S is quasipositive if and only if, for some $n \in \mathbb{N}_{>0}$, S is ambient isotopic to a full subsurface of the fiber surface of the torus link $O\{n, n\}$.* □

91. Questions. Does there exist an algorithm to determine whether or not a given Seifert surface is quasipositive? Does there exist an algorithm to determine whether a given link is strongly quasipositive?

3.3.3 Quasipositive annuli

92. Lemma. *Let (L, g) be a framed link such that the annular Seifert surface $A(L, g)$ is quasipositive. If the framing f of L is less twisted than g, then $A(L, f)$ is quasipositive.*

Proof. This is the "Twist Insertion Lemma" of Rudolph (1983a) (with the sign conventions in force here, it should be called the "Twist Reduction Lemma"). A proof using fence diagrams is shown in Figure 21. □

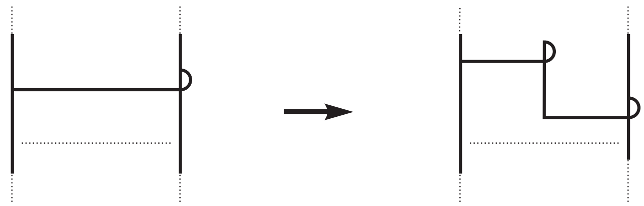

Figure 21. “Twist reduction” on a fence diagram.

93. Definition. The *modulus of quasipositivity* of a knot K is

$$(\mathbf{R}) \qquad q(K) := \sup\{t \in \mathbb{Z} : \ A(K,t) \text{ is a quasipositive Seifert surface}\}.$$

Rudolph (1990) applied **58** (via **26**) and **90** to deduce the following.

94. Proposition. *If $A(L,f)$ is quasipositive, then $\operatorname{ord}_v\{L,f\} \geq 0$.* □

95. Corollary. *If K is a knot, then $q(K) \leq \frac{1}{2}\operatorname{ord}_v\{K\} \leq -1-\deg_a F^*_K(a,x)$.* □

96. Corollary. *If K is a knot, then $q(K) \in \mathbb{Z}$.* □

Since $q(K) = \mathrm{TB}(K)$ (Rudolph, 1995), **96** also follows from Bennequin (1983).

97. Corollary. $q(O) = -1$. □

98. Proposition. *For $K \neq O$, $q(K) = \sup\{f \in \mathbb{Z} :$ the framed knot (K,f) embeds on the fiber surface of $O\{n,n\}$ for some $n \in \mathbb{N}_{>0}\}$.*

Proof. If a framed knot (K,f) embedded on a Seifert surface S and K is not full on S, then $(K,f) = (O,0)$; so **98** follows immediately from the characterization theorem **90**. □

99. Corollary. *A quasipositive Seifert surface is incompressible.* □

Of course **99** also follows from **85**. Yet another proof can be extracted from Bennequin (1983).

3.3.4 Quasipositive plumbing

A somewhat lengthy but straightforward combinatorial proof of the following theorem appears in Rudolph (1998).

100. Theorem. *A Murasugi sum is quasipositive if and only if its summands are quasipositive.* □

101. Corollary. *Iterated Murasugi sums of quasipositive annuli are quasipositive Seifert surfaces.* □

102. Examples. **101** covers a surprising amount of ground.

102.1. A *positive Hopf-plumbed fiber surface* is an iterated Murasugi sum of positive Hopf annuli $A(O,-1)$ starting from the trivial Murasugi sum D^2. A fiber surface S is *stably positive Hopf-plumbed* in case some positive Hopf-plumbed fiber surface F can also be constructed as an iterated Murasugi sum of positive Hopf annuli $A(O,-1)$ starting from S. A remarkable theorem of Giroux (2002) states that a fibered link L has a stably positive Hopf-plumbed fiber surface if and only if a certain contact structure on S^3 (constructed from the fibration in a way he describes) is the standard contact structure. In combination with **90** and **100**, Giroux's theorem implies that a fibered link is strongly quasipositive if and only if it is stably positive Hopf-plumbed.

102.2. A *basket* is a Seifert surface produced by repeatedly plumbing unknotted annuli $A(O,k_i)$ to a single fixed $D^2 \subset S^3$. (See Rudolph, 2001a, for details and examples.) A fundamental theorem of Gabai (1983b, 1985) implies that a basket is a fiber surface if and only if it is Hopf-plumbed (allowing both positive and negative Hopf annuli as plumbands). In particular, according to Rudolph (2001a), a fiber surface is a quasipositive basket if and only if it is isotopic to a braided Seifert surface $S[\mathbf{b}]$ where $\mathbf{b}$ is a strictly $\mathcal{T}$-positive $\mathcal{T}$-bandword for some espalier $\mathcal{T}$.

102.3. General *arborescent links* are constructed as boundaries of unoriented (possibly non-orientable) 2–manifolds formed by *unoriented plumbing*, in which not just unknotted annuli $A(O,k_i)$ but also unknotted Möbius bands are used as building blocks, while their mode of assembly is suitably restricted (it is coded by a planar tree with integer weights; see Conway, 1970; Siebenmann, 1975; Gabai, 1986). So-called *special arborescent links* (Sakuma, 1994) are those obtained by disallowing Möbius plumbands. As noted by Rudolph (2001a), the surfaces defining special arborescent links are baskets; in particular, when all weights of an arborescent presentation of an arborescent link L are strictly negative even integers, then L is strongly quasipositive.

102.4. The *positively-clasped, k-twisted Whitehead double* $\mathcal{D}(K;k;+)$ of a knot K is defined to be the boundary of the non-trivial plumbing (that is, not a boundary-connected sum) of $A(K,k)$ and $A(O,-1)$ (Figure 22 illustrates this operation with a fence diagram). By **101** and **90**, $\mathcal{D}(K;k;+)$ is strongly quasipositive if and only if $k \leq \mathrm{TB}(K)$ (see Rudolph, 1993).

102.5. If a set of Seifert surfaces contains $A(O,-1)$ and is closed under isotopy, Murasugi sum, and the operation of passing from a surface to a full subsurface, then it contains all quasipositive Seifert surfaces; and the set of all quasipositive Seifert surfaces is the smallest set with these properties.

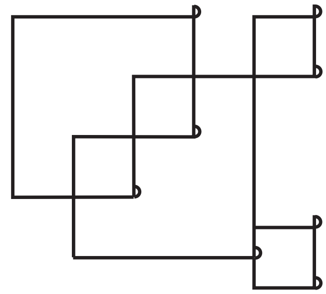

Figure 22. A fence diagram for the plumbed Seifert surface of $\mathcal{D}(O\{2,3\};0;+)$.

3.3.5 Positive links are strongly quasipositive

A standard link diagram $\mathbf{D}(L)$ is called *positive* in case every crossing $\mathsf{x} \in \mathsf{X}_{\mathbf{D}(L)}$ is positive; a link L is *positive* if L has a positive standard link diagram. Nakamura (1998, 2000) and Rudolph (1999) independently proved the following.

103. Theorem. *If the standard link diagram $\mathbf{D}(L)$ is positive, then the diagrammatic Seifert surface $S(\mathbf{D}(L))$ is quasipositive.* □

104. Corollary. *A positive link is strongly quasipositive.* □

105. Questions. Two questions posed by Rudolph (1999) remain open and are relevant here.

105.1. Can positive links be characterized as strongly quasipositive links that satisfy some extra geometric conditions?

105.2. Does $K \geq O\{2,3\}$ imply that K is strongly quasipositive? (Here $K_1 \geq K_2$ means that K_1 is concordant to K_2 inside a 4-manifold with positive intersection form; this partial order was defined and studied by Cochran and Gompf (1988), who showed that if K is a positive knot, then $K \geq O\{2,3\}$.)

3.3.6 Quasipositive pretzels

Braidzels are generalizations of the well-known (oriented) pretzel surfaces, with braiding data supplementing the twisting data that specifies an ordinary pretzel surface (a braidzel with trivial braiding); two typical pretzel surfaces and two typical braidzel surfaces are pictured in Figure 23. Braidzels were defined by Rudolph (2001b) and have been further studied by Nakamura (2004), who showed that every link has a Seifert surface which is a braidzel. Using braidzels, Rudolph (2001b) proved the following.

106. Proposition. *The oriented pretzel surface $\mathfrak{P}(t_1,\ldots,t_k)$ is quasipositive if and only if the even integer $t_i + t_j$ is less than 0 for $1 \leq i < j \leq k$.* □

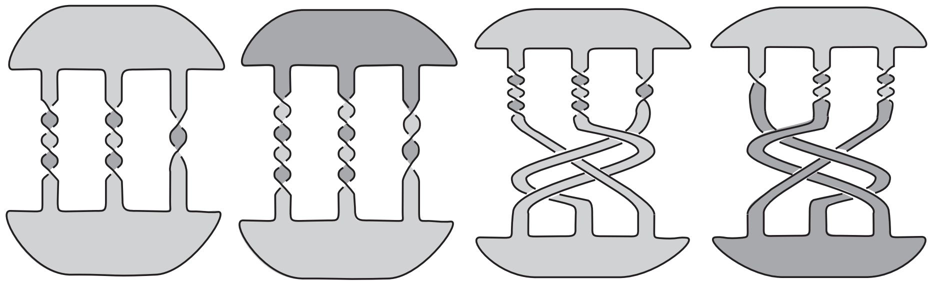

Figure 23. The quasipositive pretzel surfaces $\mathfrak{P}(4,4,-2)$ and $\mathfrak{P}(5,5,-3)$; the braidzel surfaces $\mathfrak{P}(\sigma_1,{\sigma_2}^{-1},\sigma_1,{\sigma_2}^{-1};4,4,-2)$ and $\mathfrak{P}({\sigma_2}^{-1},\sigma_1,{\sigma_2}^{-1},\sigma_1;1,-3,-3)$.

The question **89** was somewhat motivated by **106**, since up to repeated mutation the pretzel link $\partial\mathfrak{P}(t_1,\ldots,t_k)$ depends only on $\{t_1,\ldots,t_k\}$.

107. Question. What are necessary and sufficient conditions on the defining data of a braidzel that it be quasipositive?

3.3.7 Links of divides

Identify $\mathbb{C}^2$ with the tangent bundle of $\mathbb{C}$. A'Campo (1998) calls a normal collection P of edges in D^2 a *divide*, and constructs the *link of the divide* P as $L(P) := \{(z,w) \in S^3 : z \in P, w \in T_z(P)\}$. This construction has been extended—first by allowing non-proper edges ("free divides"; Gibson and Ishikawa, 2002a), more generally by allowing immersed unoriented circle components (Kawamura, 2002), and more generally yet (and most recently) by allowing immersions of graphs that are not 1–manifolds ("graph divides"; Kawamura, 2004).

108. Theorem. *If P is a divide, then:* (1) *$L(P)$ is fibered, and every link L of a singularity is of the form $L(P)$ for some divide P* (A'Campo, 1998)*;* (2) *$L(P)$ is strongly quasipositive* (Hirasawa, 2000)*;* (3) *$L(P)$ is stably positive Hopf-plumbed* (Hirasawa, 2002)*; and indeed* (4) *$L(P)$ is positive Hopf-plumbed* (Ishikawa, 2002)*. If P is a free divide, possibly with circle components, then* (5) *$L(P)$ is strongly quasipositive* (Kawamura, 2002)*. If P is a graph divide, then* (6) *$L(P)$ is quasipositive* (Kawamura, 2004)*.* □

See also Couture and Perron (2000); Hongler and Weber (2000); Ishikawa (2001); Gibson and Ishikawa (2002b); Gibson (2002); Goda, Hirasawa, and Yamada (2002); Chmutov (2003).

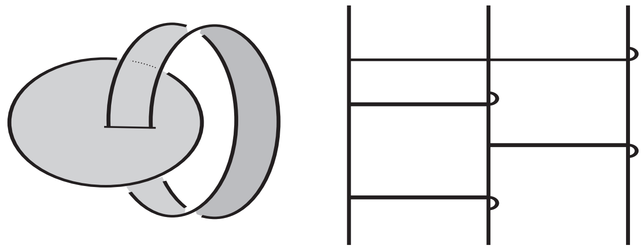

Figure 24. A quasipositive fibered link which is not strongly quasipositive.

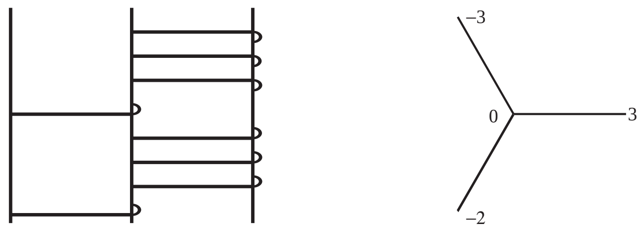

Figure 25. Two representations of a quasipositive Lover's Knot.

3.4 Non-strongly quasipositive links

Other than the tautologous construction by forming the closed braid of a quasipositive braid, little is yet known about systematic constructions of non-strongly quasipositive transverse $\mathbb{C}$-links.

109. Example. There exist fibered links which are quasipositive but not strongly quasipositive (compare with **1**). Examples include all the links $H_{n+m,m}$ obtained from $O\{n+2m, n+2m\}$ $(m, n \geq 1)$ by reversing the orientation of (any) m components. Figure 24 pictures a ribbon surface $R = f(S^1 \times [0,1] \coprod D^2)$ for $H_{1,1}$ and a fence diagram for the fiber surface $\mathrm{sm}(R)$ of $H_{1,1}$.

110. Example. There exist arborescent links which are quasipositive but are not special arborescent and not strongly quasipositive (compare with 3). An example is the quasipositive arborescent ribbon knot pictured, along with its defining weighted tree, in Figure 25.

111. Question. Every pretzel link is arborescent. (Caution: this does not mean that every pretzel link bounds an oriented pretzel surface, and in fact there is a natural sense in which those which do are a small minority. Nor is a typical oriented pretzel surface itself an arborescent Seifert surface.) What are necessary and sufficient conditions on an arborescent link that it be quasipositive, or strongly quasipositive?

Besides **106**, some partial but systematic progress towards answering this question has been made by Tanaka (1998) (for the of rational links).

4 Complex plane curves in the small and in the large

As noted in **80**, every transverse $\mathbb{C}$-link L is "algebraic" in the sense that, up to isotopy, L is the transverse intersection of $rS^3 = \{(z,w) \in \mathbb{C}^2 : \|(z,w)\| = r\}$ and a complex algebraic curve $\mathcal{V}_f$; clearly the same class of links arises whether $r > 0$ is fixed once for all, or allowed to vary throughout $\mathbb{R}_{>0}$, so long as f is allowed to vary throughout $\mathbb{C}[z,w]$. By contrast, interesting additional restrictions on the isotopy type of L arise in case, for each fixed $\mathcal{V}_f$, r is constrained to be either "very small" or "very large". More precisely, the situation is as follows (essentially this is proved by Milnor, 1968, in general dimensions). Let $F \in \mathbb{C}[z,w]$ be non-constant and without repeated factors. The *argument* of F is $F/|F| \colon \mathbb{C}^2 \setminus \mathcal{V}_F \to S^1$. The *Milnor map of* F is is $\varphi_F := \arg(()F)|(S^{2n-1} \setminus \mathbb{D}(F))$; for $r > 0$, the *Milnor map of* F *at radius* r, denoted by $\varphi_{F,r}$, is the Milnor map of $(z,w) \mapsto F(z/r, w/r)$. Let $m(F) := \inf\{\|(z,w)\|^2 : F(z,w) = 0\} = \sup\{r : rS^3 \cap \mathcal{V}_F = \varnothing\}$.

112. Proposition. *There is a finite set* $\mathcal{X}(F) \subset \mathbb{R}_{>m(F)}$ *of radii* r *with the following properties.* (1) *If* $r \in \mathbb{R}_{>m(F)} \setminus \mathcal{X}(F)$*, then* $\mathcal{V}_F$ *intersects* rS^3 *transversally, so that* $L(F,r) := (1/r)(\mathcal{V}_F \cap rS^3)$ *is a link in* S^3*.* (2) *If* r *and* r' *are in the same component of* $\mathbb{R}_{>m(F)} \setminus \mathcal{X}(F)$ *then* $L(F,r)$ *and* $L(F,r')$ *are isotopic.* (3) *If* $m(F) < r \notin \mathcal{X}(F)$*, then there is a trivialization* $n \colon L(F,r) \times \mathbb{C} \to \nu(L(F,r))$*, as in* **18***, that lies in the homotopy class corresponding to any Seifert surface for* $L(F,r)$ *and such that* $\varphi_{F,r}$ *is adapted to* $\varphi_{F,r}$ *with* $d(K) = 1$ *for all* $K \in \pi_0(L(F,r))$*.* □

113. Definitions. In case $F(0,0) = 0$, for any $r \in \,]0, m(F)[$ the transverse $\mathbb{C}$-link $L(F,r)$ is called the *link of the singularity* of F (or of $\mathcal{V}_F$) at $(0,0)$. In any case, for any $r > \max \mathcal{X}(F)$ the transverse $\mathbb{C}$-link $L(F,r)$ is called the *link at infinity* of F (or of $\mathcal{V}_F$).

4.1 Links of singularities as transverse $\mathbb{C}$*-links*

It can happen that $\varphi_{F,r}$ has degenerate critical points, and so is not a Morse map, or that $\varphi_{F,r}$ is a Morse map but not a fibration (the latter in fact being common: as noted in the first paragraph of this section, every transverse $\mathbb{C}$-link can be realized as $L(F,r)$ for some F and $r > 0$, while according to the previous section many, many transverse $\mathbb{C}$-links are not fibered links). However, Milnor (1968) proved the following (and its analogue in higher dimensions).

114. Theorem. *If* $m(F) < r < \min \mathcal{X}(F)$*, then* $L(F,r)$ *is a fibered link and* $\varphi_{F,r}$ *is a fibration.* □

In other words, if L is a link of a singularity, then L is fibered.

Let L be the link of a singularity. There is a huge literature devoted to studying the algebraic and geometric topology L, and especially of its fibration (which, at the geometric level, completely determines L and all its invariants—though not necessarily in a perspicuous fashion). Some starting points for the dedicated reader are Eisenbud and Neumann (1985); Durfee (1999); Neumann (2001); Lê Dũng Tráng (2003). Here it may simply be noted that many—though certainly not all—of the interesting topological features of the fibration of L can profitably be explored using one or another of the representations of L alluded to in earlier sections. On one hand, L is the link of a divide (A'Campo, 1998), from which one representation of L as a positive Hopf-plumbed link can be derived and exploited (A'Campo, 1998). On the other hand, it is easy to use labyrinths to see that L is a positive closed braid (and not overwhelmingly difficult actually to derive a specific positive braidword for L), whence by Rudolph (2001a) one may derive (and, at least potentially, exploit) an apparently different—though surely closely related—representation of L as a positive Hopf-plumbed link.

4.2 Links at infinity as transverse $\mathbb{C}$-links

Let L be a link at infinity. In general, L need not be fibered (for instance, every unlink $O^{(n)}$ is a link at infinity, but only $O = O^{(n)}$ is fibered), but as Neumann (1989) puts it, L is "nearly" fibered. The literature on links at infinity is smaller than that on links of singularities, but still too large and multifaceted to summarize here. The reader is referred to Boileau and Fourrier (1998) for a review up to 1998. Some more recent articles on various aspects of the subject are Bodin (1999); Némethi (1999); Neumann (1999); Bartolo and Cassou-Noguès (2000); Neumann and Norbury (2000); Păunescu and Zaharia (2000); Gwoździewicz and Płoski (2001, 2002); Rudolph (2002); Neumann and Norbury (2003); Cimasoni (2004).

5 Totally tangential $\mathbb{C}$-links

115. Definition. A link $L \subset S^3$ is a *totally tangential* $\mathbb{C}$*-link* provided that $L = \mathcal{V}_g \cap S^3$, where $g \in \mathcal{O}(\mathbb{C}^2)$ is such that $\mathcal{V}_g \cap \operatorname{Int} D^4 = \varnothing$ and $\mathcal{V}_g \cap S^3$ is a non-empty non-degenerate critical manifold of index 1 of $\rho|\operatorname{Reg}(\mathcal{V}_g)$, with $\rho(z,w) := \|(z,w)\|^2$.

This section follows Rudolph (1992b, 1995, 1997).

116. Lemma. *L is a totally tangential $\mathbb{C}$- if and only if* (1) *L is Legendrian with respect to the standard contact structure on S^3, and* (2) *L is an $\mathbb{R}$-analytic submanifold of S^3.* □

By 116, a totally tangential $\mathbb{C}$-link has a natural framing $f = f_L^{\mathcal{L}}$.

117. Proposition. *Let $\rho\colon U \to \mathbb{R}$ be an exhausting strictly plurisubharmonic function on an open set $U \subset C^2$ such that $D = \rho^{-1}(\mathbb{R}_{\leq 1})$ is a Stein disk, so $\Sigma := \partial D$ is a strictly pseudoconvex 3–sphere with a natural contact structure. If $g \in \mathcal{W}(D) \subset \mathcal{O}(U)$ is such that $\mathcal{V}_g \cap \Sigma$ is a non-empty non-degenerate critical manifold M of index 1 of $\rho|\operatorname{Reg}(\mathcal{V}_g)$, then M is a Legendrian 1–submanifold of the contact manifold Σ. Further, there exists a totally tangential $\mathbb{C}$-link $L \subset S^3$ such that (Σ, M) is diffeomorphic to (S^3, L) by a diffeomorphism carrying the natural normal line field of M in Σ to the natural normal line field of L in S^3.* □

118. Question. Does every totally tangential $\mathbb{C}$-link arise (up to isotopy) as in **115** with g required to be a polynomial? (Rudolph, 1995, shows that g may be be required to be entire.)

119. Theorem. *If $L \subset S^3$ is Legendrian, then L is isotopic through Legendrian links to a totally tangential $\mathbb{C}$-link L' with* $\operatorname{tb}(L') = \operatorname{tb}(L)$.

Proof. The proof for knots (Rudolph, 1995) extends directly to links. □

120. Corollary. *If $L \subset S^3$ is Legendrian, then there is a polynomial map $p\colon \mathbb{C}^2 \to \mathbb{C}$ such that* (1) $\operatorname{Sing}(\mathcal{V}_p) = \varnothing$, (2) *$\mathcal{V}_p$ intersects S^3 transversely,* (3) *$\mathcal{V}_p \cap D^4$ is an annular surface, and* (4) $\mathcal{V}_p \cap S^3 = \partial A(L, \operatorname{tb} L)$. □

Coupled with **82–85**, the connection **120** between totally tangential $\mathbb{C}$-links and transverse $\mathbb{C}$-links, leads to several interesting conclusions.

121. Theorem. *If* $\operatorname{TB}(K) \geq 0$*, then K is not slice.* □

A *front* (see Eliashberg, 1993; Etnyre, this Handbook) is a piecewise-smoothly immersed curve $\gamma \subset \mathbb{R}^2$ of the sort illustrated in Figreffront figure: γ has

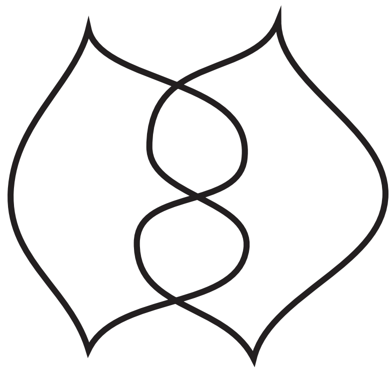

Figure 26. A front.

finitely many cuspidal corners and finitely many normal crossings; the corners are exactly the local extrema of $\mathrm{pr}_2|\gamma$; at a corner, the common tangent line of the two branches is parallel to $\{0\} \times \mathbb{R}$; and elsewhere the tangent lines are parallel neither to $\{0\} \times \mathbb{R}$ nor to $\mathbb{R} \times \{0\}$. A *front diagram* is a pair $\mathbf{FD}(L) =: (\mathbf{P}(\mathbf{FD}(L)), \mathbf{I}(\mathbf{FD}(L)))$, where $\mathbf{P}(\mathbf{FD}(L))$ is a front, and $\mathbf{I}(\mathbf{D}(L))$ is the information that L is a closed Legendrian 1–submanifold of $\mathbb{R}^3$ equipped with a certain standard contact structure (consistent with its embedding as the complement of a point in S^3 equipped with its standard contact structure).

122. Lemma. (1) *Given any front* $\mathbf{P}(\mathbf{FD}(L))$*, the information* $\mathbf{I}(\mathbf{D}(L))$ *is true, and* $\mathbf{FD}(L)$ *determines* $|L|$ *up to a vertical translation in* $\mathbb{R}^3$*.* (2) *A standard link diagram* $\mathbf{D}(L)$ *for* L *can be derived from* $\mathbf{P}(\mathbf{FD}(L))$ *by* (a) *letting* $\mathbf{P}(\mathbf{D}(L))$ *be* $\mathbf{P}(\mathbf{FD}(L))$ *with its cusps rounded off, and* (b) *letting* $\mathbf{I}(\mathbf{D}(L))$ *declare that at each crossing of* $\mathbf{P}(\mathbf{D}(L))$*, the over-arc is the branch of which the tangent line has larger positive slope.* □

123. Proposition. *For any knot* K*, if* $r \leq \mathrm{TB}(K)$ *then* $\mathrm{TB}(D(K, r, +)) \geq 1$*.* □

124. Theorem. *For any knot* $K \subset S^3$*,* $g_s(K) \geq (\mathrm{TB}\, K + 1)/2$*.* □

6 Relations to other research areas

This section is a bibliography of papers that relate the knot theory of complex plane curves to other research areas; in some of the subsections, one (or a few) paper(s) are highlighted. The choice of highlights, and indeed the selections themselves, reflect both my bias and my ignorance; the latter, at least, is corrigible, and I welcome pointers to sources I have neglected. The division into subsections is necessarily somewhat arbitrary.

6.1 Low-dimensional real algebraic geometry; Hilbert's 16th problem.

An interesting series of recent papers (Orevkov, 1999; Orevkov and Polotovskiĭ, 1999; Orevkov, 2000, 2001a,b; Orevkov and Shustin, 2002) has applied quasipositivity to Hilbert's 16th problem Hilbert's 16th problem.

6.2 The Zariski Conjecture; knotgroups of complex plane curves

125. Theorem. *Let* $f \in \mathbb{C}[z,w]$. *If* (1) $\mathcal{V}_f \subset \mathbb{C}^2$ *is a node curve, and* (2) *the link at infinity of* $\mathcal{V}_f$ *is a positive* (*and not merely quasipositive*) *closed braid, then the knotgroup* $\pi_1(\mathbb{C}^2 \setminus \mathcal{V}_f)$ *is abelian.* □

125 is the version of "Zariski's Conjecture" proved by Orevkov (1988). The original version of the conjecture (in which the link at infinity is required to be, not just any positive closed braid, but actually $\widehat{\beta}(\nabla_{\deg f})$) had been proved by Fulton (1980); Deligne (1981); Nori (1983); Harris (1986). The particular beauty of Orevkov's proof consists, first, in his use of a labyrinth for f (assumed to be in Weierstrass form) to construct a presentation of $\pi_1(\mathbb{C}^2 \setminus \mathcal{V}_f)$ with very many generators ($\deg_w(f)$ generators for each 0-dimensional stratum of the labyrinth) and very many relations, which are however all of very simple forms; and, second, in his ingenious application of the clews of the labyrinth, coupled with the positivity of the closed braid at infinity, to inductively reduce that presentation to a smaller presentation that visibly presents an abelian group.

The background and motivation for the conjecture, and related material on knotgroups of complex plane curves, may be found in Enriques (1923); Lefschetz (1924); Brauner (1928); Zariski (1929); van Kampen (1933); Burau (1934); Zariski (1936, 1971); Reeve (1954/1955); Cheniot (1973); Lê Dũng Tráng (1974); Oka (1975/76, 1974); Oka and Sakamoto (1978); Chang (1979); Randell (1980); Kaliman (1992); Moishezon and Teicher (1996); Kulikov (1997); Garber (2003).

The ideas of Orevkov (1998) are extended by Orevkov (1990a); Dethloff, Orevkov, and Zaidenberg (1998); Neumann and Norbury (2003).

6.3 Keller's Jacobian Problem; embeddings and injections of $\mathbb{C}$ in $\mathbb{C}^2$

For somewhat unclear reasons, Keller's Jacobian Problem inspired Segre (1956/1957) to state—though not to prove correctly—a theorem classifying embeddings of $\mathbb{C}$ in $\mathbb{C}^2$, which has now been proved and generalized to injections.

126. Theorem. *Let $p, q \in \mathbb{C}[t]$ be such that $F\colon\ \mathbb{C} \to \mathbb{C}^2 : t \mapsto (p(t), q(t))$ is a (set-theoretical) injection.* (1) *If F is a smooth embedding, then one of* $\deg(p)$, $\deg(q)$ *divides the other; as a consequence, there is a polynomial change of coordinates $Q\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that $Q(F(\mathbb{C}))$ is a straight line.* (2) *If F is not smooth, then either one of* $\deg(p)$, $\deg(q)$ *divides the other or* $\deg(p)$ *and* $\deg(q)$ *are relatively prime; as a consequence, there is a polynomial change of coordinates $Q\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that $Q(F(\mathbb{C}))$ is one of the curves $\{(z, w) \in \mathbb{C}^2 : z^m + w^n\}$ for some relatively prime positive integers m and n.* $\square$

126(1) was first proved by Abhyankar and Moh (1975) (algebraically) and by Suzuki (1974) (analytically); a topological (knot-theoretical) proof was given by Rudolph (1982a). **126**(2) was first proved by Zaĭdenberg and Lin (1983) (analytically); a topological (knot-theoretical) proof was given by Neumann and Rudolph (1987). Many other proofs have appeared since. Those proofs, related theorems, and some of the more knot-theoretical work on the original Jacobian Problem may be found in Abhyankar and Singh (1978); Oka and Sakamoto (1978); Orevkov (1990b, 1991); Chang (1991); Kang (1991); Chang and Wang (1993); Kaliman (1993); Abhyankar (1994); Artal-Bartolo (1995); Gurjar and Miyanishi (1996); Orevkov (1996); A'Campo and Oka (1996); van den Essen (1997); Żołądek (2003).

6.4 Chisini's statement; braid monodromy

It was known early on (Zariski, 1929, see) by algebraic geometers that, for proposed applications to the study of algebraic surfaces, the most important knotgroups of complex plane curves were those of cusp curves. Starting in 1933 and continuing through the 1950s, Oscar Chisini and his school published a series of papers around that subject (Dedò, 1950; Chisini, 1952; Tibiletti, 1952; Chisini, 1954; Tibiletti, 1955a,d,b,c; Chisini, 1955). The tools they brought to bear were what would later be called "braid monodromy" and "the arithmetic of braids" (Moishezon, 1981, 1983, 1985; Moishezon and Teicher, 1988, 1991; Moishezon, 1994; Moishezon and Teicher, 1994a,b, 1996).

127. Theorem. *Let* $\mathbf{b}$ *be a quasipositive cuspword in* B_n *such that* $\beta(\mathbf{b}) = \nabla_n$.
(1) *For any* n *and any such* $\mathbf{b}$, *there exists a holomorphic cusp curve in* $D^2 \times D^2$ *that corresponds to* $\mathbf{b}$ *via an appropriate holomorphic map* $f\colon D^2 \to MP_n$.
(2) *There exist* n *and* $\mathbf{b}$ *such that no cusp curve in* $\mathbb{C}^2$ *corresponds to* $\mathbf{b}$ *via a polynomial map* $f\colon D^2 \to MP_n$. □

127(1) is of course a special case of **73**, as proved (in much greater generality) by Orevkov (1998). **127**(2) was proved by Moishezon (1994), in contradiction to "a statement of Chisini" (1954).

There is an enormous, interesting, and constantly growing literature on braid monodromy and its many offsprings, including applications to line arrangements, the Alexander invariants of complex plane curves, and so on; see Libgober (1986); Loeser and Vaquié (1990); Dung (1994); Dũng (1994); Dung and Hà (1995); Cohen and Suciu (1997); Dung (1999); Artal et al. (2001); Kharlamov and Kulikov (2001); Dung (2002); Kulikov and Kharlamov (2003); Yetter (2003).

6.5 Stein surfaces

By **41**, **69**, **70**, and **71**, a branched covering of the bidisk branched over a quasipositive braided surface is diffeomorphic to a Stein surface. Loi and Piergallini (2001) proved the following converse.

128. Theorem. *If* X *is a Stein surface, then* X *is diffeomorphic to a branched covering of* $D^2 \times D^2$ *branched over a quasipositive braided surface.* □

7 The future of the knot theory of complex plane curves

This section suggests some possible new directions for the knot theory of complex plane curves.

7.1 Transverse $\mathbb{C}$-links and their Milnor maps.

As observed before **114**, the Milnor map $\varphi_{F,r}$ at radius $r > m(F)$ of a non-constant reduced polynomial $F \in \mathbb{C}[z, w]$ can have degenerate critical points. However, this can happen for only finitely many $r \in \mathbb{R}_{>m(F)}$. Every other $\varphi_{F,r}$ is a Morse map, and typically *not* a fibration (even if $\mathcal{V}_F \cap rS^3$ is a fibered link; see Hirasawa and Rudolph, 2003). These Morse maps deserve further study.

129. Motto. *A Milnor map is interesting even if it is not a fibration.*

130. Question. Here are a couple of specific questions about Milnor maps that are Morse. Is such a map necessarily topless? Under what conditions is such a map minimal (that is, possessed of exactly the minimum number necessary for any Morse map on its domain; see Pajitnov, Weber, and Rudolph, 2001; Hirasawa and Rudolph, 2003, and references cited therein)?

Rational functions and non-reduced polynomials also have Milnor maps. Cimasoni's study (2004) of Alexander invariants of links of infinity of (reduced) polynomials leads him to investigate, in passing, the Milnor maps at infinity of non-reduced polynomials, and Pichon (2003) has begun the study of the Milnor map of rational functions at their points of indeterminacy, but a general study of these maps remains to be undertaken.

7.2 Transverse $\mathbb{C}$-links as links at infinity in the complex hyperbolic plane.

To date, with the exception of Rudolph (2002), all investigations of links at infinity have implicitly or explicitly concerned themslves with behavior at "affine infinity", that is, in affine space $\mathbb{C}^2$. Is that an oversight?

131. Motto. *A transverse $\mathbb{C}$-link is a complex hyperbolic link at infinity.*

The meaning of **131** is this. As in the case of the real hyperbolic plane $\mathbb{H}^2(\mathbb{R})$, there is a Klein model of the complex hyperbolic plane $\mathbb{H}^2(\mathbb{C})$ which displays $\mathbb{H}^2(\mathbb{C})$ as holomorphically equivalent to $\operatorname{Int} D^4 \subset \mathbb{C}^2$. From a purely topological—even complex-analytic—viewpoint, therefore, a link at infinity in the complex plane is nothing more (nor less) than a transverse $\mathbb{C}$-link in S^3.

But, like $\mathbb{H}^2(\mathbb{R})$, $\mathbb{H}^2(\mathbb{C})$ has a geometry which makes it very different from the affine plane over the same field. Rudolph (2002) exploited (in a very minor way) the linear structure in $\mathbb{H}^2(\mathbb{C})$. What about more subtle geometric features, like any of several different variants on curvature?

132. Question. One way to interpret the main theorem of Osserman (1981) is that, if $f \in \mathcal{O}(\mathbb{C}^2)$ is entire, then $\mathcal{V}_f$ has a well-defined link at infinity (i.e., is an algebraic curve) if and only if $\mathcal{V}_f$ has finite total absolute curvature. Given $f \in \mathcal{O}(\mathbb{H}^2(\mathbb{C}))$, possibly subject to further conditions (e.g., finite topological type), are there reasonable geometric conditions—for instance, integral curvature conditons—on $\mathcal{V}_f$ that are equivalent to the existence of a well-defined transverse $\mathbb{C}$-link "at infinity" for $\mathcal{V}_f$?

7.3 Spaces of $\mathbb{C}$-links

The sets of function defined in (**K**) can be used to describe sets of singular transverse and totally tangential $\mathbb{C}$-links, as follows. In fact, it is obvious that the quotient set $(\mathcal{O}(D^4) \setminus \{0\})/\mathcal{U}(D^4)$ is naturally identified with the set of "singular transverse $\mathbb{C}$-links" (with positive multiplicities), and likewise that the quotient set $\mathcal{W}(D^4)/\mathcal{U}(D^4)$ is naturally identified with the set of "singular totally tangential $\mathbb{C}$-links". Moreover, both quotient sets are (more or less naturally) partitioned by the severity of the "singularity" of the $\mathbb{C}$-link. It is perhaps less obvious what topologies are most appropriate to turn these partitions (or suitable refinements thereof) into good stratifications of some infinite-dimensional manifolds, in which (for instance) the codimension-0 strata represent restricted isotopy types of ordinary non-singular transverse, or totally tangential, $\mathbb{C}$-links. Nonetheless, I proclaim the following.

133. Motto. *Naturally stratified spaces of singular $\mathbb{C}$-links exists and should be studied* à la *Vassiliev.*

A significant distinction between "singular links" in the present style, and in the style of Vassiliev, is that because these singular links are the 0-loci of (restrictions of holomorphic) functions $S^3 \to \mathbb{C}$ rather than the images of functions $S^1 \times \mathbf{k} \to S^3$, at a generic bifurcation (i.e., a codimension-1 stratum in one of the proposed spaces of singular links) the number of components changes (by one).

7.4 Other questions.

Here are two last open questions ranging somewhat further afield than the earlier ones.

134. Question. It is a long-standing conjecture (closely related to the Whitehead Conjecture; see Howie, 1985) that, if $S \subset D^4$ is a ribbon 2-disk, then $D^4 \setminus S$ is aspherical. A simple homological argument shows that this conjecture is equivalent to the conjecture that $H^2(\widetilde{D^4 \setminus S};\mathbb{Z}) = 0$, where for any M, $\widetilde{M}$ here denotes the universal covering space of M. Now, as Rudolph (1983b) points out, if $S \subset D^4$ is any ribbon 2–disk, then there is a piece of complex plane curve $\Gamma \subset D^4$ (diffeomorphic to D^2) such that $D^4 \setminus S$ is homotopy-equivalent to $D^4 \setminus \Gamma$. If a piece of complex plane curve $\Gamma \subset D^4$ is diffeomorphic to D^2, is $D^4 \setminus \Gamma$ aspherical? Of course $D^4 \setminus \Gamma$ is homotopy equivalent to $\mathrm{Ext}(\Gamma \subset D^4)$. For an appropriate choice of $\mathrm{Nb}(\Gamma \subset D^4)$, $\mathrm{Ext}(\Gamma \subset D^4)$ is a Stein domain. By **41** and **38**, to answer this question affirmatively it would suffice to show that every complex curve in the universal cover X of $\mathrm{Int}\, D^4\Gamma$ is $\mathcal{V}_f$ for some $f \in \mathcal{O}(X)$.

135. Question. Call $K \subset S^3$ a *counter-Smith knot* in case K is a non-trivial knot and, for some $p > 1$, the p-sheeted cyclic branched cover of S^3 branched along K is again S^3. Assuming the Poincaré Conjecture, the Smith Conjecture is equivalent to the assertion that there are no counter-Smith knots. A 3-dimensional proof of the Smith Conjecture has of course been given by Thurston *et al.*; it is beautiful but lengthy, and depends on separate analyses of a number of different cases. In 1983 I noted that a simple appeal to Donaldson's results from 4-dimensional gauge theory shows that, if the knot K is a non-slice transverse $\mathbb{C}$-link, then K is not counter-Smith. Can techniques that have been developed in the meantime be used to give a 4-dimensional proof of the Smith Conjecture (modulo the Poincaré Conjecture) that avoids a case-by-case analysis?

Acknowledgements

It is a pleasure to be able to thank Claude Weber for his support and encouragement since my first trip to Switzerland in 1982. My continuing attempts to understand the knot theory of complex plane curves have been greatly furthered over the years by much work and talk with Michel Boileau, Walter Neumann, and Stepan Orevkov, and more recently Mikami Hirasawa. I have also had many useful conversations with Norbert A'Campo, Thomas Fiedler, Francoise Michel, and Andrei Pajitnov. A number of graduate students (several of them advised by mathematicians already mentioned), having written theses contributing to the knot theory of complex plane curves, have been kind enough to explain their work to me; these include Laurence Fourrier, Laure Eichenberger, Arnaud Bodin, and others. Many other workers in this and related fields have helped me keep reasonably in touch with developments; I thank them all.

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A And now a few words from our inspirations

A.1 Hilbert's 16th Problem (1891, 1900)

> 16. Problem der Topologie algebraischer Curven und Flächen.
> Die Maximalzahl der geschlossenen und getrennt liegenden Züge, welche eine ebene algebraische Curve n ter Ordnung haben kann, ist von Harnack ... bestimmt worden; es ensteht die weitere Frage nach der gegenseitigen Lage der Curvenzüge in der Ebene. ... *Eine gründliche Untersuchung der gegenseitigen Lage bei der Maximalzahl von getrennten Zügen scheint mir ebenso sehr von Interesse zu sein ....*
>
> (Hilbert, 1900, p. 283)

> 16. Problem of the topology of algebraic curves and surfaces
> The maximum number of closed and separate branches which a plane algebraic curve of the nth order can have has been determined by Harnack.... There arises the further question as to the relative position of the branches in the plane. ... *A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest ....* (Hilbert, 1902, p. 23)

A.2 Zariski's Conjecture (1929)

> **Theorem 7.** The fundamental group of an irreducible curve f of order n, possessing ordinary double points only, is cyclic of order n. ... To prove the theorem 7, we prove first the following lemma:
>
> **Lemma.** *The fundamental group of n lines in arbitrary position is abelian.*
>
> ... Now let f be an irreducible curve of order n with ordinary double points only. The continuous system $\{f\}$ ... contains in particular curves which degenerate into n arbitrary lines.[*]...
>
> [*] This follows from a noted principle, announced by F. Enriques in 1904, "Sulla proprietá caratteristica delle superficie algebriche irregolari," *Rendiconto delle*

sessioni della R. Accademia delle Science dell'Instituto di Bologna, nuova serie, Vol. 9 (1904–1905), pp. 5–13, and completed later by F. Severi, *Vorlesungen über algebraische Geometrie:* Anhang F, No. 7. Leipzig, Teubner, 1921. ....

(Zariski, 1929, pp. 316, 317, & 318)

A.3 Keller's Jacobian Problem (1939)

> Es sind nur die Faelle 6., 7. noch unentschieden, also die Frage, ob Polynome mit der Funktionaldeterminante 1 sich stets durch Polynome umkehren lassen. ... Mir scheint die Frage eine Untersuchung sehr zu lohnen, sie scheint jedoch bereits im ebenen Fall sehr schwierig zu sein.

(Keller, 1939, p. 301)

[It is only cases 6., 7. that remain undecided, that is, the question of whether a polynomial mapping with Jacobian determinant 1 always has a polynomial inverse. ... To me it seems that this is a very worthwhile question, but it seems to be very difficult already in the case of the plane.]

A.4 Chisini's Statement (1954)

> É naturale che la esperienza abbia indotto i ricercatori ad ammettere (si pure nella sola fase di ricerca) che le trecce dedotte dalla treccia canonica con modifica dell'ordine dei tratti e con fusione di questi (curve che la treccia stessa mostra se irriducibili o non) corrispondano a curve algebriche esistenti, cioè siano trecce davvero algebriche. La dimostrazione del fatto indicato è lo scopo di questo lavoro... .

(Chisini, 1954, p. 144)

[Naturally, experience has led researchers to hypothesize (if only at this stage of research) that the braids derived from the canonical braid by modification of the order of the strings and by the fusion thereof (curves which the braid itself shows to be irreducible or not) correspond to existing algebraic curves, that is, they are truly algebraic braids. The proof of the indicated fact is the purpose of this work... .]

A.5 Milnor's Question (1968)

> Now let us look at the algebraic geometry of our singular point. To any singular point z of a curve $V \subset \mathbb{C}^2$ there is associated an integer $\delta_z > 0$ which intuitively measures the number of double points of V concentrated at z. ...
>
> **Remark 10.9.** It would be nice to have a better topological interpretation of the integer δ. It can be shown that the link $K = V \cap S_\epsilon$ bounds a collection of r smooth 2-cells in the disk D_ϵ having no singularities other than δ ordinary double points. *Question:* Is δ perhaps equal to the "Überschneidungszahl" of K: the smallest number of times which K must be allowed to cross itself during a smooth deformation so as to transform K into a collection of r unlinked and unknotted circles? ...
>
> Milnor (1968, pp. 85 & 92)

A.6 The Thom Conjecture (c. 1977)

> **Problem 4.36.** (A) *Conjecture*: The minimal genus of a smooth imbedded surface in CP^2 representing $n \in H_2(CP^2; Z) = Z$ is $(n-1)(n-2)/2$.
>
> Kirby (1978, p. 309)

According to Kirby (1997), "... when this problem was written in 1977, Thom's name was not generally associated with the conjecture; however the association developed later although it is not clear whether Thom ever made the conjecture." Indeed, Sullivan (1973) states the conjecture with no name attached. On the other hand, Massey (1973) calls it both "the Thom–Atiyah conjecture" and "the Atiyah–Thom conjecture", a joint attribution which seems to be otherwise unattested in the literature. Both Thom (1982) and Massey (1992) have disclaimed being the first to state this conjecture.

List of Figures

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Index of Definitions and Notations

Page references are to first, or defining, instances of terms and symbols.