Generalized volume and geometric structure of 3-manifolds

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Introduction

For several hyperbolic knots, a relation between certain quantum invariants and the volume of their complements are discovered by R. Kashaev in [2]. In [6], it is shown that Kashaev's invariants are specializations of the colored Jones polynomials. Kashaev used the saddle point method to obtain certain limit of invariants, and Y. Yokota proved that the equations to determine the saddle points correspond to the equations defining the hyperbolic structure of the knot complement. He introduce a simplicial decomposition of the complement associated to a knot diagram, and show that the equations to give the hyperbolic structure of each simplex coincide with the equations for saddle points. For 3-manifolds obtained by surgeries along a figure-eight knot, H. Murakami [5] follows Kashaev's computation for the Witten-Reshetikhin-Turaev invariants and found that a value at certain saddle point relates to the volume.

Trying to extend these works to the Turaev-Viro invariant [9], a formula for the volume of a hyperbolic tetrahedron is obtained in [7]. The Turaev-Viro invariant is defined from a simplicial decomposition of a 3-manifold, and use a state sum associating the quantum 6j-symbol to each tetrahedron, and the formula for the volume of a hyperbolic tetrahedron comes from the quantum 6j-symbols. Moreover, extending Yokota's theory to this case, we may get some relation between the volume and the geometric structure of the manifold, which is the main subject of this note.

1. Fluctuating structure

In this report, I would like to speculate a method to determine the geometric structure of a 3-manifold from its simplicial decomposition by using a *generalized volume function*, which is called a *potential function* in [12]. To define the generalized volume, we introduce a *fluctuating structure* of a simplicial decomposition.

1.1. **Fluctuating simplicial decomposition.** Let $\mathcal{T}$ be a simplicial decomposition of a 3-manifold M. Assume that $\mathcal{T}$ consists of k 3-simplices T_1, T_2, $\cdots$, T_k. Each 3-simplex has 6 corners corresponding to its two faces (2-simplices), and we associate real numbers to each corners, say θ^i_1, θ_2, $\cdots$, θ^i_6 for the 3-simplex T_i. These numbers are called the *dihedral angles* of the corners. Let Θ be the correction of the dihedral angles

$$\Theta = \{\theta^i_j \mid i = 1, 2, \cdots, k, j = 1, 2, \cdots, 6\}.$$

Let E be an edge (1-simplex) of $\mathcal{T}$, T_{i_1}, T_{i_2}, $\cdots$, T_{i_ℓ} be the tetrahedra containing the edge E, and $\theta^{i_1}_{j_1}$, $\theta^{i_2}_{j_2}$, $\cdots$, $\theta^{i_\ell}_{j_\ell}$ be the dihedral angles of the corners of T_{i_1}, T_{i_2}, $\cdots$, T_{i_ℓ} corresponding to E. Then the following relation is called the *angle relation* corresponding to E.

$$\text{angle relation:} \qquad \sum_{p=1}^{\ell} \theta^{i_p}_{j_p} = 2\,\pi. \tag{1}$$

The correction of the dihedral angles Θ is called a *fluctuating structure* of $\mathcal{T}$ if the angles of Θ satisfy the edge relations for all the edges of $\mathcal{T}$. A simplicial decomposition with a fluctuating structure Θ is denoted by $\mathcal{T}_\Theta$ and is called a *fluctuating simplicial decomposition.*

1.2. **Generalized volume.** The generalized volume is introduced to a fluctuating simplicial decomposition $\mathcal{T}_\Theta$. This is denoted by $\mathcal{V}(\mathcal{T}_\Theta)$ and it is defined as a sum of generalized volumes of each fluctuating 3-simplex, i.e.

$$\mathcal{V}(\mathcal{T}_\Theta) = \sum_{T \in \mathcal{T}^3} \mathcal{V}(T_\Theta), \tag{2}$$

where $\mathcal{T}^3$ be the set of 3-simplices of $\mathcal{T}$ and T_Θ means the fluctuating 3-simplex T with the fluctuating structure given by the restriction of Θ to T, in other words, the six corners of T are assigned dihedral angles given by Θ.

The generalized volume $\mathcal{V}$ for a fluctuating 3-simplex T_Θ is defined by the following formula. Let A, B, C be the three angles touching the same vertex of T, and D, E, F be the angles of T at the opposite position of A, B, C respectively as in Figure 1. Let $a = \exp\sqrt{-1}\theta_1$, $b = \exp\sqrt{-1}\theta_2$, $\cdots$ $f = \exp\sqrt{-1}\theta_6$,

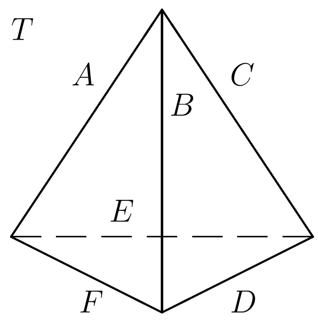


FIGURE 1. The six dihedral angles A, B, $\cdots$, F of T.

and $\mathrm{Li}_2(z)$ be the dilogarithm function defined as an analytic continuation of the following function.

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-x)}{x}\,dx = \sum_{k=1}^{\infty} \frac{x^k}{k^2}. \tag{3}$$

For the detail of Li_2, see, for example, [4]. Note that the dilogarithm $\mathrm{Li}_2(z)$ is a multi-valued function as the logarithm function $\log(z)$. We put

$$U(z,T) = \frac{1}{2}\left(\mathrm{Li}_2(z) + \mathrm{Li}_2(z\,a\,b\,d\,e) + \mathrm{Li}_2(z\,a\,c\,d\,f) + \mathrm{Li}_2(z\,b\,c\,e\,f) - \mathrm{Li}_2(-z\,a\,b\,c) - \mathrm{Li}_2(-z\,a\,e\,f) - \mathrm{Li}_2(-z\,b\,d\,f) - \mathrm{Li}_2(-z\,c\,d\,e)\right). \tag{4}$$

Let z_1, z_2 be the two non-trivial solutions of the equation

$$\frac{d}{dz}U(z,T) = \frac{2\,\pi\,k}{z}, \qquad (k \in \mathbf{Z}) \tag{5}$$

which is equivalent to

$$(1-z)\,(1-a\,b\,d\,e\,z)\,(1-a\,c\,d\,f\,z)\,(1-b\,c\,e\,f\,z) - (1+a\,b\,c\,z)\,(1+a\,e\,f\,z)\,(1+b\,d\,f\,z)\,(1+c\,d\,e\,z) = 0. \tag{6}$$

Note that a solution of (6) may be a solution of (5) in some branch because the function log is a multi-valued function. In the following formulas, we take an adequate branch of logarithm and dilogarithm functions corresponding to the solutions z_1 and z_2. Let k_1 and k_2 be the integers satisfying the following.

$$\frac{d}{dz}U(z_1,T) = \frac{2\,\pi\,k_1}{z_1}, \qquad \frac{d}{dz}U(z_2,T) = \frac{2\,\pi\,k_2}{z_2}.$$

By using z_1 and z_2, let

$$\mathcal{V}(T_\Theta) = \frac{1}{2}\left(U(z_1,T_\Theta) - U(z_2,T_\Theta) - k_1\,\log z_1 + k_2\,\log z_2\right). \tag{7}$$

It is known in [7] that $|\sqrt{-1}\,\mathcal{V}(T_\Theta)|$ is equal to the actual volume of T_Θ if T_Θ is realized as a hyperbolic tetrahedron,

$$\mathrm{Volume}(T_\Theta), = |\sqrt{-1}\,\mathcal{V}(T_\Theta)| \tag{8}$$

and $|\mathcal{V}(T_\Theta)|$ is equal to the actual volume of T_Θ if T_Θ is realized as a tetrahedron in S^3,

$$\mathrm{Volume}(T_\Theta). = |\mathcal{V}(T_\Theta)| \tag{9}$$

This formula is proved by comparing with the formula given in [1]. Let z_1 be a solution of (6) such that z_1 goes to 1 if we deform T_Θ continuously to a tetrahedron with a ideal vertex. For this case, $-\sqrt{-1}\,\mathcal{V}(T_\Theta)$ is positive and we have

$$\mathrm{Volume}(T_\Theta). = -\sqrt{-1}\,\mathcal{V}(T_\Theta) \tag{10}$$

for a hyperbolic tetrahedron T_Θ.

1.3. **Some special cases.** First consider the case that the angles A, B, C satisfy

$$A + B + C = \pi.$$

Hence $a\,b\,c = -1$ and so the equation (6) is the following.

$$(1-z)\,(1-a\,b\,d\,e\,z)\,(1-a\,c\,d\,f\,z)\,(1-b\,c\,e\,f\,z) - (1-z)\,(1+a\,e\,f\,z)\,(1+b\,d\,f\,z)\,(1+c\,d\,e\,z) = 0. \tag{11}$$

Therefore $z = 1$ is one of the solution of (6). This implies that, if one of the vertex of the tetrahedron is an idal vertex, then $z = 1$ is one of the non-trivial solutions of (6).

Next, consider the case that

$$A = 0, \quad B + C = \pi, \quad E + F = \pi.$$

This tetrahedron is degenerated to a face since $A = 0$. The parameters $a, \cdots, f$ satisfy $a = 1$, $b\,c = -1$, $e\,f = -1$ and the equation (6) is the following.

$$\begin{aligned}(1-z)\,(1-b\,d\,e\,z)\,(1-a\,c\,d\,f\,z)\,(1-z)& \\ -(1-z)\,(1-z)\,(1+b\,d\,f\,z)\,(1+c\,d\,e\,z) &= 0.\end{aligned} \tag{12}$$

Hence the non-trivial solutions of (6) are $z_1 = z_2 = 1$, which is a multiple root.

1.4. **Gram matrix.** For a fluctuating 3-simplex T_Θ, let $\mathrm{Gram}(T)$ be the *Gram matrix* of T_Θ defined by

$$\mathrm{Gram}(T_\Theta) = \begin{pmatrix} 1 & -\cos\theta_1 & -\cos\theta_2 & -\cos\theta_6 \\ -\cos\theta_1 & 1 & -\cos\theta_3 & -\cos\theta_5 \\ -\cos\theta_2 & -\cos\theta_3 & -1 & -\cos\theta_4 \\ -\cos\theta_6 & -\cos\theta_5 & -\cos\theta_4 & 1 \end{pmatrix}.$$

The condition of realization of T_Θ as a tetrahedron in a hyperbolic space or a spherical space is known and is given in terms of the elements of $\mathrm{Gram}(T)$.

1.5. **Stationary decomposition and geometric decomposition.** Let M be a 3-manifold and let $\mathcal{T}_\Theta$ a simplicial decomposition of M with a fluctuating structure Θ. A fluctuating decomposition $\mathcal{T}_{\Theta_0}$ is called a *stationary decomposition* if Θ_0 is a stationary point of $\mathcal{V}(\mathcal{T}_\theta)$. In other words, the partial derivative of $\mathcal{V}(\mathcal{T}_\theta)$ with respect to any independent parameters of Θ vanishes.

A fluctuating decomposition $\mathcal{T}_\Theta$ is called a *geometric decomposition* if all the faces of $\mathcal{T}$ are parts of planes and the dihedral angles given by Θ coincide with the angle of faces with respect to the hyperbolic structure of M.

1.6. **Schläfli's formula and stationary point of volume.** Each tetrahedron T of a geometric decomposition satisfies Schänfli's formula, which is stated as follows. Let θ_1, θ_2, $\cdots$, θ_6 be the dihedral angles of T and let E_1, E_2, $\cdots$, E_6 be the corresponding edges. Then, for hyperbolic case,

$$d\ \mathrm{Volume}(T) = -\frac{1}{2}\sum_{i=1}^{6} \mathrm{Length}(E_i)\, d\theta_i. \tag{13}$$

Hence, by using (7), we have

$$d\,\mathcal{V}(T) = -\frac{\sqrt{-1}}{2}\sum_{i=1}^{6} \mathrm{Length}(E_i)\, d\theta_i. \tag{14}$$

Now, we vary the fluctuating structure Θ. Then Schänfli's formula implies the following theorem, which is equivalent to the argument to determine the hyperbolic structure given by Casson.

Theorem 1. *If $\mathcal{T}_{\theta_0}$ is a geometric decomposition of a hyperbolic 3-manifold M, then Θ_0 is a stationary point of $\mathcal{V}(\mathcal{T}_\Theta)$, i.e. $\mathcal{V}(\mathcal{T}_\Theta)$ is stable at Θ_0.*

Proof. We use Lagrange's multiplier method. The stationary point of $\mathcal{V}(\mathcal{T}_\Theta)$ for the parameter set Θ satisfying the angle relation (1) is given by the solution of the following set of equations. Let $E_1, E_2, \cdots, E_k$ be the edges of $\mathcal{T}$, and let $\theta^i_1, \theta^i_2, \cdots, \theta^i_{j_i}$ be the dihedral angles given by Θ around E_i. Then the equations for a stationary point of $\mathcal{V}(\mathcal{T}_\Theta)$ under the edge relations are given by

$$\frac{\partial \mathcal{V}(\mathcal{T})}{\partial \theta^i_1} = \frac{\partial \mathcal{V}(\mathcal{T})}{\partial \theta^i_2} = \cdots = \frac{\partial \mathcal{V}(\mathcal{T})}{\partial \theta^i_{j_i}} = \lambda_i \qquad (i = 1, 2, \cdots, k) \tag{15}$$

Here λ_i $(i = 1, 2, \cdots, k)$ are some numbers called Lagrange's multipliers. In this equation,

$$\frac{\partial \mathcal{V}(\mathcal{T})}{\partial \theta^i_p} = \frac{\partial \mathcal{V}(T^i_p)}{\partial \theta^i_p},$$

where T^i_p is the tetrahedron containing the dihedral angle θ^i_p, and so the above equations are equivalent to the following system of equations.

$$\frac{\partial \mathcal{V}(T^i_p)}{\partial \theta^i_p} = \lambda_i \qquad (i = 1, 2, \cdots, k, \quad p = 1, 2, \cdots, j_i) \tag{16}$$

If θ^i_p's are dihedral angles given by Θ_0 corresponding to a geometric decomposition, (16) is satisfied by putting

$$\lambda_i = -\frac{\sqrt{-1}}{2}\,\mathrm{Length}(E_i).$$

from (13). Hence Θ_0 correspond to a stationary point of $\mathcal{V}(\mathcal{T}_\Theta)$. □

Theorem 2. *If $\mathcal{T}_{\theta_0}$ is a geometric decomposition of a elliptic 3-manifold M, i.e. the universal cover of M is isomorphic to S^3, then Θ_0 is a stationary point of $\mathcal{V}(\mathcal{T}_\Theta)$, i.e. $\mathcal{V}(\mathcal{T}_\Theta)$ is stable at Θ_0.*

Proof. Theorem 2 is proved similarly as for Theorem 1. In this case, the constant $\lambda = i$ satisfy

$$\lambda_i = \frac{1}{2}\,\mathrm{Length}(E_i).$$ □

Remark. The relation (15) means that the length of the edge E_i is equal with respect to all the tetrahedra with the edge E_i. By this reason, we call (15) the *length relation*.

Definition (generalized length). We call $\dfrac{\partial \mathcal{V}(T_p^i)}{\partial \theta_p^i}$ the *generalized length* of the edge E_i with respect to the terahedron T_p^i. It is a real number if T_p^i is a elliptic tetrahedron, a pure imaginary number if T_p^i is a hyperbolic tetrahedron, and 0 if T_p^i is a Euclidean tetrahedron. If the fractuating structure Θ satisfies the length relation, then the generalized length does not depend on the choice of the tetrahedron containing E_i.

1.7. **Geometric structure of general case.** Generalizing Theorem 1, it may be not so bad to expect the following.

> **Working hypothesis:** Let M be a 3-manifold and $\mathcal{T}$ its simplicial decomposition. Then a stationary point of the fluctuating structure $\mathcal{V}(\mathcal{T}_\Theta)$ introduces a geometric structure of M.

In other words, a real soltion of the angle relation (1) and the length relation (15) may give the geometric structure.

2. Examples

In this section, we apply the formula and the working hypothesis to several examples.

2.1. **Regular tetrahedron.** The first example is the volume and edge length of a regular tetrahedron T. Let x be the dihedral angle of T. Then the volume function behaves as in Figure 2. The value explains the absolute value of the real part of the volume function if $x \geq \arccos 1/3 = 0.391827\,\pi$ and the absolute value of the imaginary part if $x < \arccos 1/3$. If $x = \arccos 1/3$, then T is the Euclidean regular tetrahedron, and if $x = \pi/3$ then T is the ideal regular tetrahedron. If $x < \pi/3$, then the absolute value of the imaginary part of the volume function explains the volume of the *truncated tetrahedron* as shown in [3]. Expecially, if $x = 0$, then the truncated tetrahedron is equal to the ideal octahedron, which is a union of eight tetrahedra with dihedral angles $\pi/2$, $\pi/2$, $\pi/2$, $\pi/4$, $\pi/4$ and $\pi/4$, whose volume is 0.457983.

2.2. **Lens spaces.** The lens space $L_{p,q}$ is known to have a spherical structure, and so there is a geometric decomposition. Such fluctuating structure is given by one of the stationary decompositions. A simple geometric decomposition is given by p tetrahedra of the same shape as in Figure 4 with dihedral angles A, B, C, D, E, F. Then

$$A = \frac{2\,\pi}{p}, \quad B = \frac{\pi}{2}, \quad C = \frac{\pi}{2}, \quad D = \frac{2\,\pi}{p}, \quad E = \frac{\pi}{2}, \quad F = \frac{\pi}{2}$$

satisfy the equation (15).

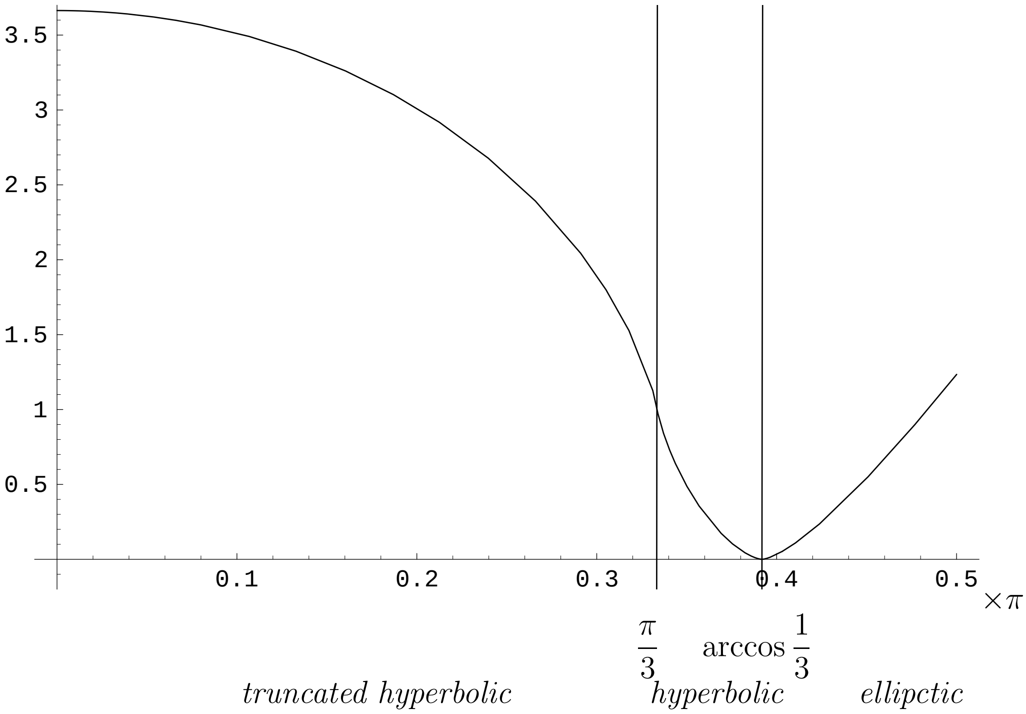


FIGURE 2. Volume of a regular polyhedron.

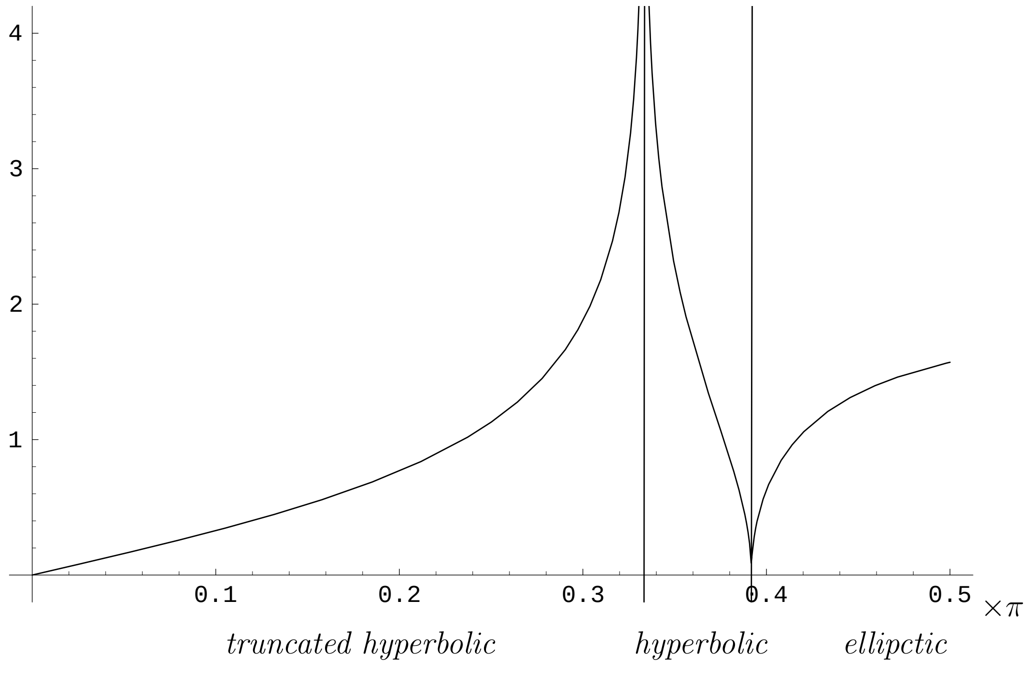


FIGURE 3. Edge length of a regular polyhedron.

The volume of the tetrahedron with the above dihedral angles are $\dfrac{2\pi^2}{p^2}$ and the volume of $L_{p,q}$ is $\dfrac{2\pi^2}{p}$. The length of the edges corresponding to A and D

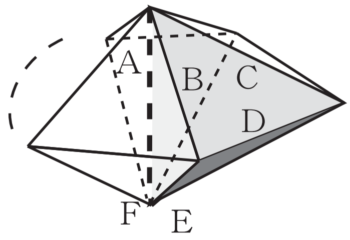


FIGURE 4. Thetrahedral decomposition of a Lens space.

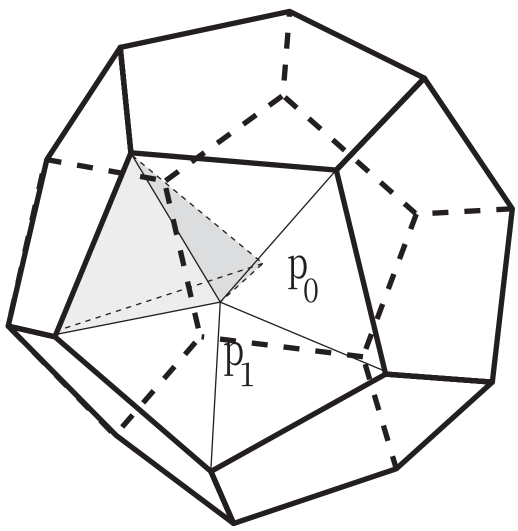


FIGURE 5. The tetrahedron T in the dodecahedron.

is equal to $\dfrac{2\pi}{p}$ and that corresponding to B, C, E and F is equal to $\dfrac{\pi}{2}$. The above volume and lengths can be obtained by using the formula $\mathcal{V}(T)$.

2.3. **Poincarè homology sphere.** In the book of Thurston [8], examples of two 3-manifolds obtained by glueing the faces of a dodecahedron are given. One has a spherical structure and the another one has a hyperbolic structure, and the first one is known to be the Poincarè homology sphere.

For this case, the dihedral angle of the dodecahedron is equal to $\dfrac{2\,pi}{3}$. Let T be a tetrahedron obtained by subdividing the dodecahedron by using the center of the dodecahedron p_0 and the center of the pentagon p_1 as in Figure 5. Then the dihedral angles of T assigned as in Figure 10 are given as follows.

$$A = \frac{2\pi}{5}, \quad B = \frac{\pi}{3}, \quad C = \frac{\pi}{3}, \quad D = \frac{\pi}{3}, \quad E = \frac{\pi}{2}, \quad F = \frac{\pi}{2}.$$

The volume of T is equal to $\dfrac{\pi^2}{3600}$. Recall that the volume of S^3 is equal to $2\pi^2$, the Poincarè homology sphere is the quotient of S^3 by a finite group of order 120 (the binary icosahedral group), and the volume of the dodecahedron is 60 times the volume of T. The lengths of the edges corresponding to $A, \cdots, F$ are $0.086236\,\pi$, $0.123549\,\pi$, $0.123549\,\pi$, $\dfrac{\pi}{10}$, $0.123549\,\pi$, $0.123549\,\pi$ respectively.

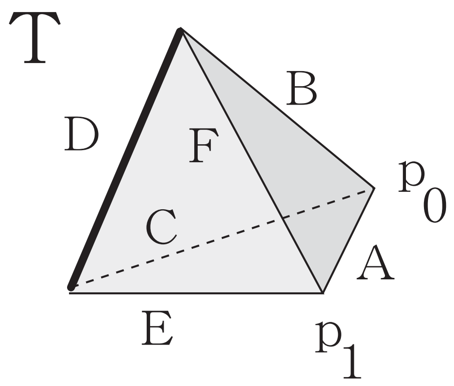


FIGURE 6. The angles of T.

These results suggest that the distances of the vertex v of the dodecahedron from p_0 and p_1 are equal.

2.4. **Seifert-Weber dodecahedral space.** A hyperbolic 3-manifold is obtained by glueing the faces of a dodecahedron and it is called the Seifert-Weber dodecahedral space. Let T be a tetrahedron as in Figure 5. Then the dihedral angles of T corresponding to the Seifert-Weber space are given as follows.

$$A = \frac{2\pi}{5}, \quad B = \frac{\pi}{3}, \quad C = \frac{\pi}{3}, \quad D = \frac{\pi}{5}, \quad E = \frac{\pi}{2}, \quad F = \frac{\pi}{2}.$$

Then the volume of T is equal to 0.186651, and so the volume of the Seifert-Weber space is 11.1991. The lengths of the edges corresponding to $A, \cdots, F$ are 1.99277, 1.43911, 1.43911, 0.996384, 1.90285, 1.90285 respectively. These results suggest that the length of an edge of the dodecahedron is twice the length of p_0p_1.

2.5. **3-dimensional torus.** A 3-dimensional torus has a simplicial decomposition with six tetrahedra as in Figure 7. Assume that the dihedral angles at the edges of the cube are all equal to $\pi/2$, the right angle, and the length of the edges are all equal.

Now consider about the three tetrahedra in the triangle cilynder ABC-EFG. Let T_1 be the tetrahedron AEFG, T_2 be the tetrahedron ABFG and T_3 be the tetrahedron ABCG. Then the dihedral angles of T_1 at the edges AE, AF, AG, EF, EG, FG are $\pi/4$, $\pi/2$, $\pi/3$, $\pi/2$, $\pi/2$, $\pi/4$ respectively, the dihedral angles of T_2 at the edges AB, AF, AG, BF,BG, FG are $\pi/4$, $\pi/2$, $\pi/3$, $\pi/2$, $\pi/2$, $\pi/4$ respectively, and the dihedral angles of T_3 at the edges AB, AC, AG, BC, BG, CG are $\pi/4$, $\pi/2$, $\pi/3$, $\pi/2$, $\pi/2$, $\pi/4$ respectively. Such angles are obtained by solving the equation that the determinant of the Gram matrix of each tetrahedron is equal to 0.

Remark. Let T_1 and T_2 be two adjacent Equclidean tetrahedra at an edge E. Then the generalized length of E with respect to T_1 and T_2 are both 0 and so the length relation is always satisfied.

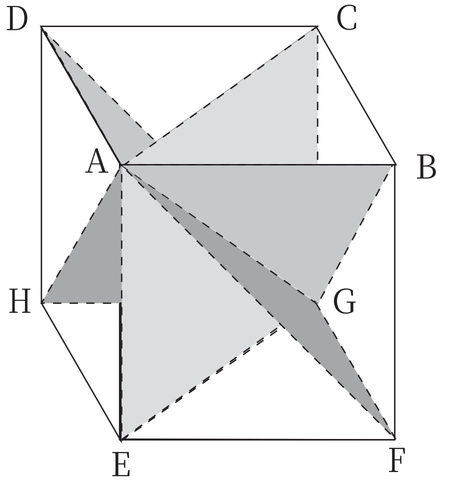


FIGURE 7. Decomposition of a cube by six tetrahedra.

2.6. **Subdivision of a hyperbolic hexadron.** Let A-BCD-E be the hexadron H_1 given in Figure 8, whose dihedral angles at the edges AB, AC, AD, BE, CE, DE are all $\pi/2$, and those at the edges BC, BD, CD are all $3\,\pi/5$. Then H_1 is realized in the hyperbolic space. Let T_1, T_2 be the tetrahedra ABCD and BCDE. Now obtain the dihedral angles of T_1 and T_2 at the edges BC, CD, and BD from the working hypothesis.

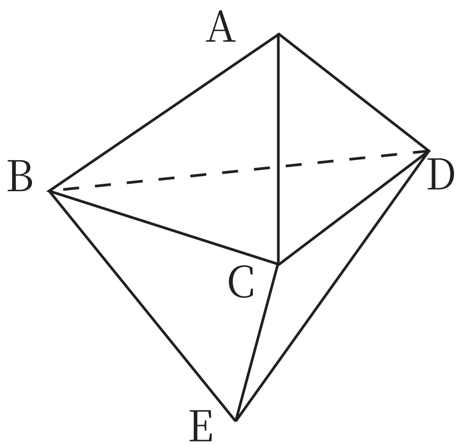


FIGURE 8. A hexadron with 6 triangle faces.

Let A_1, B_1, C_1 be the dihedral angles at BC, CD, DB of T_1 and A_2, B_2, C_2 be the dihedral angles at BC, CD, DB of T_2. Let $a_1 = \exp\sqrt{-1}\,A_1$, $\cdots$, $c_2 = \exp\sqrt{-1}\,C_2$. Then the equations comes from the working hypothesis are the following.

$$a_1\,a_2 = b_1\,b_2 = c_1\,c_2 = \exp\sqrt{-1}\,\pi/3,$$

$$\begin{aligned}
&(1+\sqrt{-1}\,z\,a_1\,b_1)\,(1+\sqrt{-1}\,z\,a_1\,c_1)\,(1+z'\,a_1\,b_1)\,(1+z'\,a_1\,c_1)\times\\
&(1+\sqrt{-1}\,u'\,a_2\,b_2)\,(1+\sqrt{-1}\,u'\,a_2\,c_2)\,(1+u\,a_2\,b_2)\,(1+u\,a_2\,c_2)=\\
&\qquad(1+\sqrt{-1}\,z'\,a_1\,b_1)\,(1+\sqrt{-1}\,z'\,a_1\,c_1)\,(1+z\,a_1\,b_1)\,(1+z\,a_1\,c_1)\times\\
&\qquad(1+\sqrt{-1}\,u\,a_2\,b_2)\,(1+\sqrt{-1}\,u\,a_2\,c_2)\,(1+u'\,a_2\,b_2)\,(1+u'\,a_2\,c_2),\\
&(1+\sqrt{-1}\,z\,a_1\,b_1)\,(1+\sqrt{-1}\,z\,b_1\,c_1)\,(1+z'\,a_1\,b_1)\,(1+z'\,b_1\,c_1)\times\\
&(1+\sqrt{-1}\,u'\,a_2\,b_2)\,(1+\sqrt{-1}\,u'\,b_2\,c_2)\,(1+u\,a_2\,b_2)\,(1+u\,b_2\,c_2)=\\
&\qquad(1+\sqrt{-1}\,z'\,a_1\,b_1)\,(1+\sqrt{-1}\,z'\,b_1\,c_1)\,(1+z\,a_1\,b_1)\,(1+z\,b_1\,c_1)\times\\
&\qquad(1+\sqrt{-1}\,u\,a_2\,b_2)\,(1+\sqrt{-1}\,u\,b_2\,c_2)\,(1+u'\,a_2\,b_2)\,(1+u'\,b_2\,c_2),\\
&(1+\sqrt{-1}\,z\,a_1\,c_1)\,(1+\sqrt{-1}\,z\,b_1\,c_1)\,(1+z'\,a_1\,c_1)\,(1+z'\,b_1\,c_1)\times\\
&(1+\sqrt{-1}\,u'\,a_2\,c_2)\,(1+\sqrt{-1}\,u'\,b_2\,c_2)\,(1+u\,a_2\,c_2)\,(1+u\,b_2\,c_2)=\\
&\qquad(1+\sqrt{-1}\,z'\,a_1\,c_1)\,(1+\sqrt{-1}\,z'\,b_1\,c_1)\,(1+z\,a_1\,c_1)\,(1+z\,b_1\,c_1)\times\\
&\qquad(1+\sqrt{-1}\,u\,a_2\,c_2)\,(1+\sqrt{-1}\,u\,b_2\,c_2)\,(1+u'\,a_2\,c_2)\,(1+u'\,b_2\,c_2),
\end{aligned}\tag{17}$$

where z, z' be the non-trivial solutions of

$$\begin{aligned}
&(1-z)\,(1+z\,a_1\,b_1)\,(1+z\,a_1\,c_1)\,(1+z\,b_1\,c_1)=\\
&\quad(1+\sqrt{-1}\,z\,a_1\,b_1)\,(1+\sqrt{-1}\,z\,a_1\,c_1)\,(1+\sqrt{-1}\,z\,b_1\,c_1)\,(1-\sqrt{-1}\,z),
\end{aligned}\tag{18}$$

and u, u' be the non-trivial solutions of

$$\begin{aligned}
&(1-u)\,(1+u\,a_2\,b_2)\,(1+u\,a_2\,c_2)\,(1+u\,b_2\,c_2)=\\
&\quad(1+\sqrt{-1}\,u\,a_2\,b_2)\,(1+\sqrt{-1}\,u\,a_2\,c_2)\,(1+\sqrt{-1}\,u\,b_2\,c_2)\,(1-\sqrt{-1}\,u).
\end{aligned}\tag{19}$$

Then

$$a_1=b_1=c_1=a_2=b_2=c_2=\pi/6$$

satisfies (17). Then the volumes of T_1 and T_2 are equal to 0.00610257 and the length of the edges of the hexadron are computed numerically as follows.

$$\begin{aligned}
\mathrm{AB}=\mathrm{AC}=\mathrm{AD}=\mathrm{BE}=\mathrm{CE}=\mathrm{DE}&=0.481212,\\
\mathrm{BC}=\mathrm{BD}=\mathrm{CD}&=0.337138.
\end{aligned}$$

2.7. **Subdivision of an elliptic hexadron.** Let A-BCD-E be the hexadron H_1 given in Figure 8, whose dihedral angles at the edges AB, AC, AD, BE, CE, DE are all $\pi/2$, and those at the edges BC, BD, CD are all $2\,\pi/3$. Then H_1 is realized in S^3. Let T_1, T_2 be the tetrahedra ABCD and BCDE. Then the dihedral angles corresponding to the edges of BC, BD, CD of T_1 and T_2 are all equal to $\pi/3$. Hence the volumes of T_1 and T_2 are equal to 0.102808 and the lengths of edges are computed numerically as follows.

$$\begin{aligned}
\mathrm{AB}=\mathrm{AC}=\mathrm{AD}=\mathrm{BE}=\mathrm{CE}=\mathrm{DE}&=1.0472,\\
\mathrm{BC}=\mathrm{BD}=\mathrm{CD}&=0.785398.
\end{aligned}$$

2.8. **Subdivision of a Euclidean hexadron.** Let A-BCD-E be the hexadron H_1 given in Figure 8, whose dihedral angles at the edges AB, AC, AD, BE, CE, DE are all $\pi/2$, and those at the edges BC, BD, CD are all $2\arccos 1/\sqrt{3}$. Then H_1 is realized in the Euclidean space. Let T_1, T_2 be the tetrahedra ABCD and BCDE. Now obtain the dihedral angles of T_1 and T_2 at the edges BC, CD, and BD from the working hypothesis.

Let A_1, B_1, C_1 be the dihedral angles at BC, CD, DB of T_1 and A_2, B_2, C_2 be the dihedral angles at BC, CD, DB of T_2. Let

$$g_1 = g_1(A_1, B_1, C_1) = \det \mathrm{Gram}(T_1),$$
$$g_2 = g_2(A_2, B_2, C_2) = \det \mathrm{Gram}(T_2).$$

Since T_1 and T_2 should be tetrahedra in the Euclidean space, we should have

$$g_1(A_1, B_1, C_1) = g_2(A_2, B_2, C_2) = 0.$$

The equations corresponding to the edges are all satisfied for such Euclidean case because the length of the 'edges' are all equal to 0. Actual computaion show that

$$g_1(A_1, B_1, C_1) = 1 - \cos^2 A_1 - \cos^2 B_1 - \cos^2 C_1,$$
$$g_2(A_2, B_2, C_2) = 1 - \cos^2 A_2 - \cos^2 B_2 - \cos^2 C_2.$$

Since $A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = 2\arccos 1/\sqrt{3}$,

$$\cos^2 A_1 + \cos^2 A_2 \geq \frac{2}{3}, \quad \cos^2 B_1 + \cos^2 B_2 \geq \frac{2}{3}, \quad \cos^2 C_1 + \cos^2 C_2 \geq \frac{2}{3}.$$

The equalities hold for $A_1 = A_2 = B_1 = B_2 = C_1 = C_2 = \arccos 1/\sqrt{3}$. Therefore,

$$g_1(A_1, B_1, C_1) + g_2(A_2, B_2, C_2) \leq 0,$$

and $g_1(A_1, B_1, C_1) + g_2(A_2, B_2, C_2) = 0$ if $A_1 = A_2 = B_1 = B_2 = C_1 = C_2 = \arccos 1/\sqrt{3}$.

Note that the six unknowns are determined from the five relations.

We explain another example of a Euclidean hexadron. Let A-BCD-E be the hexadron H_2 given in Figure 8, whose dihedral angles at the edges AB, AC, AD, BE, CE, DE are all $\arccos 1/3$, and those at the edges BC, BD, CD are all $2\arccos 1/3$. Then H_2 is realized in the Euclidean space. Let T_1, T_2 be the tetrahedra ABCD and BCDE.

Let A_1, B_1, C_1 be the dihedral angles at BC, CD, DB of T_1 and A_2, B_2, C_2 be the dihedral angles at BC, CD, DB of T_2. Let

$$g_1 = g_1(A_1, B_1, C_1) = \det \mathrm{Gram}(T_1),$$
$$g_2 = g_2(A_2, B_2, C_2) = \det \mathrm{Gram}(T_2).$$

Since T_1 and T_2 should be tetrahedra in the Euclidean space, we should have

$$g_1(A_1, B_1, C_1) = g_2(A_2, B_2, C_2) = 0. \tag{20}$$

The equations corresponding to the edges are all satisfied for such Euclidean case because the length of the 'edges' are all equal to 0. Actual computaion show that

$$g_1(A_1, B_1, C_1) = \frac{8}{9} \left(\frac{2}{3} - \cos^2 A_1 - \cos^2 B_1 - \cos^2 C_1 \right.$$
$$\left. - \cos A_1 \, \cos B_1 - \cos A_1 \, \cos C_1 - \cos B_1 \, \cos C_1 \right), \quad (21)$$

$$g_2(A_2, B_2, C_2) = \frac{8}{9} \left(\frac{2}{3} - \cos^2 A_2 - \cos^2 B_2 - \cos^2 C_2 \right.$$
$$\left. - \cos A_2 \, \cos B_2 - \cos A_2 \, \cos C_2 - \cos B_2 \, \cos C_2 \right). \quad (22)$$

Now let $\cos A_1 = 1/3 + x$, $\cos B_1 = 1/3 + y$, $\cos C_1 = 1/3 + z$, then $\cos A_2 = 1/3 - x$, $\cos B_2 = 1/3 - y$, $\cos C_2 = 1/3 - z$, since $A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = 2 \arccos 1/3$, and so

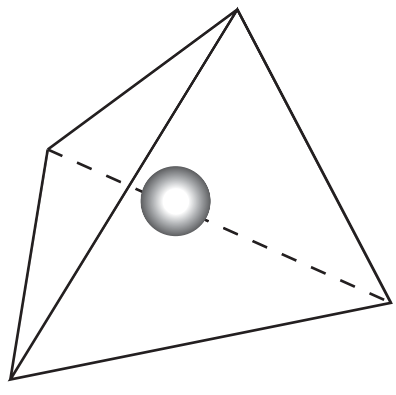
$$g_1(A_1, B_1, C_1) + g_2(A_2, B_2, C_2) = -\frac{16}{9} \left(x^2 + y^2 + z^2 + x\,y + x\,z + y\,z \right)$$
$$= -\frac{8}{9} \left((x+y)^2 + (z+z)^2 + (y+z)^2 \right) \le 0. \quad (23)$$

The equalities hold for $A_1 = A_2 = B_1 = B_2 = C_1 = C_2 = \arccos 1/3$. Therefore,

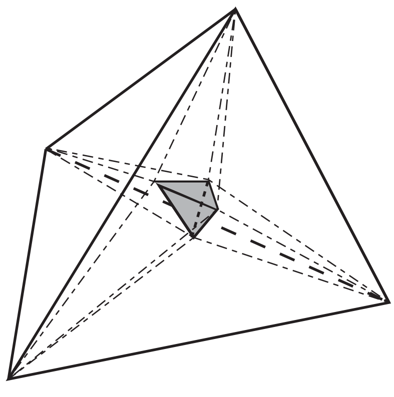
$$g_1(A_1, B_1, C_1) + g_2(A_2, B_2, C_2) \le 0,$$

and $g_1(A_1, B_1, C_1) + g_2(A_2, B_2, C_2) = 0$ if $x = y = z = 0$, i.e. $A_1 = A_2 = B_1 = B_2 = C_1 = C_2 = \arccos 1/3$.

2.9. **Tetrahedron with a hole.** Here we consider a tetrahedron with a small hole, which is homeomorphic to a tetrahedron with a removed ball. This object



A tetrahedron with a hole.



A subdivion of the tetrahedron.

FIGURE 9. Subdivision of a tetrahedron with a hole.

is useful for donsidering about the connected sum. For simplisity, we consider the symmetric case. Let T be a regular tegrahedron whose dihedral angles ar all equal to θ, let T' be a small tetrahedron at the center of T, and we subdivide $T \setminus T'$ into 14 tetrahedra of three kinds of shapes as in Figure 9. Let Type I be tetrahedra corresponding to the faces, Type II be those corresponding to the edges and Tye III be those corresonding to vertices. There are four Type I tetrahedra, six Type II tetahedra and four Type III tetrahedra.

We assume that the subdivision is symmetric and the angles of each tetrahedron is assigned as in Figure 10. Then

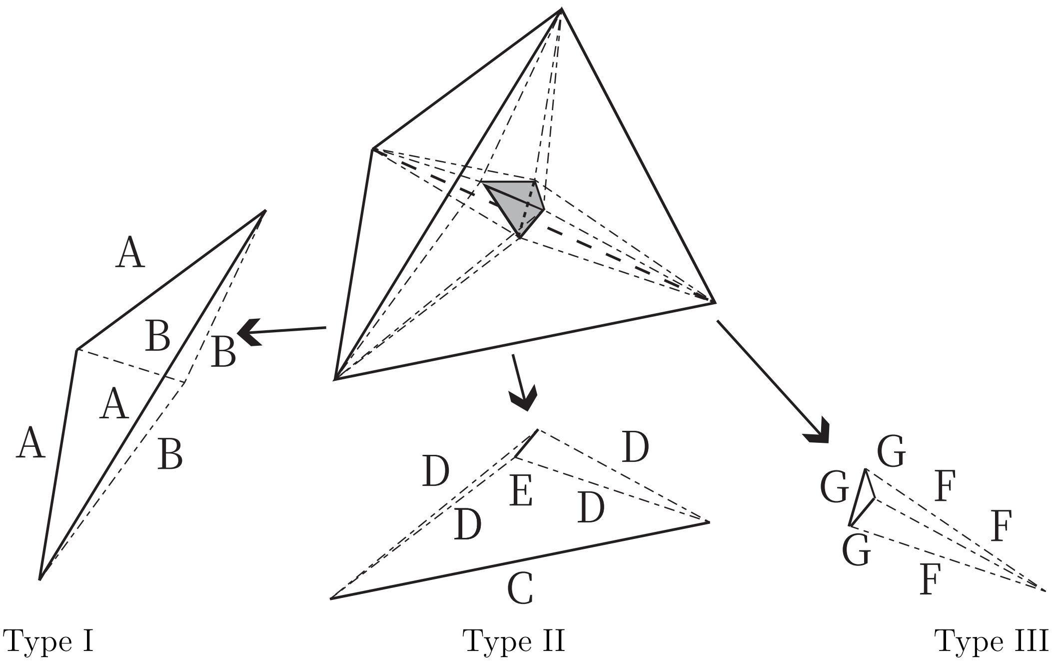


FIGURE 10. Angles of tetrahedra in the subdivision.

$$A = \frac{\theta}{2},\ B = \frac{2\pi}{3},\ C = 0,\ D = \frac{\pi}{4},\ E = \text{arbituraly}\ F = \frac{\pi}{3},\ G = \frac{\pi}{2}$$

are a solution of the length relation (15), and so if we put $E = 0$, then the sum of length around an edge of T' is equal to $E + 2G = \pi$. Therefore, we can glue two tetrahedra with a hole with such geometric structure at the boundary spheres of the holes.

Remark 1. It may be natural to suppose that $E = \arccos -1/3$. However, to satisfy the angle relation for the connect sum, the solution $E = 0$ is much better than $E = \arccos -1/3$.

Remark 2. The signature of the determinant of the Gram matrices of Type I tetrahera are equal to that of the original tetrahedron T. The determinant of the Gram matrices of Type II and Type III tetrahedra are equal to 0, and so the generalized length of the edges of these tetrahedra are equal to 0. However, by reformulating the length relation to an algebraic relation, then such algebraic version of the length relation is satisfied for the edge corresponding to B, D and F, and the edge corresponding to E and G.

3. Speculations

3.1. **The geometric structure of a fluctuating tetrahedron.** The geometric structure of a fluctuating tetrahedron T_Θ can be determined by the determinant of the Gram matrix $\det \mathrm{Gram}(T_\Theta)$. Assume that T_Θ can be realized in hyperbolic, Euclidean or spherical spaces. If $\det \mathrm{Gram}(T_\Theta)$ is negative, T_Θ can be realized as a hyperbolic tetrahedron. If $\det \mathrm{Gram}(T_\Theta)$ is positive, T_Θ can be realized as a spherical tetrahedron. If $\det \mathrm{Gram}(T_\Theta) = 0$, T_Θ can be realized as a Euclidean tetrahedron. For the actual realization of T_Θ, there are some conditions concerning to the minor determinants of $\mathrm{Gram}(T_\Theta)$ (see, e.g. [10]).

3.2. **Generalized tetrahedron.** Here, we consider the case that the conditions for dihedral angles to realize a tetrahedron are not satisfied. In this note, we would like to generalize the notion of tetrahedron so that it admits any dihedral angles.

One extension is to truncated tetrahedron in a hyperbolic space. If the solid angle at a vertex is less than π, this vertex can be realized in none of hyperbolic, Euclidean and spherical spaces. But it can be realized as a truncated tetrahedron in a hyperbolic space as in [3]. By this reason, we extend the notion of a tetrahedron to a truncated tetrahedron. Let T_Θ mean the corresponding truncated tetrahedron if T_Θ can be realized as a truncated tetrahedron in the hyperbolic space. In this case, [3] shows that the volume of the truncated tetrahedron is also given by

$$\mathrm{Volume}(T_\Theta) = |\operatorname{Im} \mathcal{V}(T_\Theta)|.$$

By this reason, we extend the notion of tetrahedron to such truncated tetrahedron.

Other extensions are tetrahedrons with negative volumes and edges with negative lengths. For some fluctuating structure, it may happen that the volume or the length of a edge is negative and so we would like to admit such tetrahedron by giving a suitable rule for cancellation of overlapped tetrahedra with positive and negative volumes.

With these generalizations of the notion of geometric tetrahedron, we can give a geometric structure to any fluctuating tetrahedron from its fluctuating structure.

3.3. **Geometric structure of a fluctuating simplicial decomposition.** For a fluctuating simplicial decomposition $\mathcal{T}_\Theta$, we can give a geometric structure to each tetrahedron of $\mathcal{T}$. But these structures of two tetrahedra sharing a face may not be compatible at this face. The lengths of an edge of this face given by the sharing tetrahedra may not be equal in general.

3.4. **Geometric structure of a stationary decomposition.**

3.4.1. **Homogeneous structure case.** Let $\mathcal{T}_{\Theta_0}$ be a stationary decomposition of a 3-manifold M. If the structures of all the tetrahedra of $\mathcal{T}$ given by Θ_0 are hyperbolic (resp. spherical), then these structures are all compatible at all the faces of $\mathcal{T}$. Hence these structures determine a hyperbolic (resp. spherical) structure of M.

3.4.2. **Non-homogenous structure case.** Let $\mathcal{T}_{\Theta_0}$ be a stationary decomposition of a 3-manifold M. Consider the case that the geometric structure of some two tetrahedron is different. Let T_1 and T_2 be adjacent tetrahedra with a common edge E. Let θ_1 and θ_2 be the dihedral angles of T_1 and T_2 respectively corresponding to the edge E. For a stationary decomposition, we have

$$\frac{\partial \mathcal{V}(T_{\Theta_0})}{\partial \theta_1} = \frac{\partial \mathcal{V}(T_{\Theta_0})}{\partial \theta_2} \tag{24}$$

If the determinant of the Gram matrix of T_1 is positive and that of T_2 is negative, then $\dfrac{\partial \mathcal{V}(T_{\Theta_0})}{\partial \theta_1}$ is non-negative real number and $\dfrac{\partial \mathcal{V}(T_{\Theta_0})}{\partial \theta_2}$ is $\sqrt{-1}$ times a real number, and the both are non-zero. This is a contradiction to (24) and so we have the following.

Observation. The product of the determinants of the Gram matrices of two adjacent tetrahedra is non-negative.

This observation is true if the two adjacent tetrahedra is actually realizable in hyperbolic, Euclidean, or spherical spaces. However, it is not proved yet for other generalized cases.

3.5. **Existence of a stationary point.** The volume formula (7) is given in terms of dilogarithm functions and so its partial derivatives with respect to the dihedral angles are given in terms of logarithm functions. Hence, the set of equations to get a stationary point is deformed to a system of algebraic equations by taking exponential of the original equations of indeterminates $x_j^i = \exp \sqrt{-1}\,\theta_i^j$. Since the number of equations and the numbers of the independent parameters are equal, this system should have at least one solution. However, the above solution may not be a complex number of unit length. If they are all of unit length, it is still not clear that their arguments satisfy the edge relation, if we take the appropriate choices of branches. The sum of dihedral angles around an edge may be an integral multiple of 2π instead of 2π. Therefore, it is not so clear that there always exist a stationary decomposition.

On the other hand, a geometric decomposition is a stationary decomposition. Hence a simplicial decomposition $\mathcal{T}_\Theta$ is realizable by a geometric decomposition, such geometric decomposition can be obtained as a stationary decomposition.

In this note, we generalized the notion of tetrahedron to admit negative volume and negative lengths of edges. If any simplicial decomposition of a 3-manifold M is realizable as a geometric decomposition with such generalized tetrahedron, the geometric structure of M should be given by one of the stationary decomposition.

4. Conclusion

A method to get a geometric structure of 3-manifolds from Schänfli's formula is already considered by Casson. His method uses the lengths of edges as parameters, while our method uses the dihedral angles as parameters. From the view point of the generalized volume function, dihedral angles seems to be very natural parameters. For degenerate case, the length may vary from 0 to infinity, while any dihedral angle is bounded. By this reason, I expect that the working hypothesis proposed in this note works not only for the hyperbolic case but also for generalized case.

The obstruction for our working hypothesis is that we don't know that there is a solution of the algebraic version of the equations for stationary points such that each parameters of the solution are of unit length.

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