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On the volume of hyperbolic polyhedra

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Introduction

The problem of calculating volumes of convex polytopes in euclidean, spherical or hyperbolic space is a very difficult one. However, every convex polytope admits a simplical subdivision, and for n-simplices S $(n \geqq 3)$ the above problem is considerably simpler. In the euclidean case the volume is explicitly given by

$$\mathrm{Vol}_n(S) = \frac{1}{n!}|\det(p_0, \ldots, p_n)|,$$

where the vectors $p_0, \ldots, p_n$ "generate" the simplex S.

In the non-euclidean case we do not have such an elementary formula for the volume of S. Nevertheless, in 1852, Schläfli gave a simple description for the vôlume differential $d\,\mathrm{Vol}_n(S)$ as a function of the dihedral angles w_{jk} $(0 \leqq j < k \leqq n)$ formed by the faces S_j, S_k of S (see [16, p. 227ff]):

$$d\,\mathrm{Vol}_n(S) = \frac{1}{n-1} \sum_{\substack{j,k=0 \\ j<k}}^{n} V_{jk}\, dw_{jk}, \tag{1}$$

where V_{jk} is the volume $\mathrm{Vol}_{n-2}(S_{jk})$ of the apex $S_{jk} := S_j \cap S_k$ to w_{jk}. Schläfli proved this formula for spherical simplices. In 1936, H. Kneser gave a second, very skilful proof of (1) (see [11] and [4, Sect. 5.1]) for both the spherical and hyperbolic cases (up to a change of sign in the latter case).

But even for a three-dimensional non-euclidean simplex, the integration of this Schläfli differential is practically impossible. In fact, the most basic objects in polyhedral geometry are orthoschemes (or orthogonal-simplices) first introduced by Schläfli: An n-orthoscheme R is an n-simplex with vertices $P_0, \ldots, P_n$ such that

$$\mathrm{span}(P_0, \ldots, P_i) \perp \mathrm{span}(P_i, \ldots, P_n) \quad \text{for} \quad i = 1, \ldots, n-1.$$

The two vertices P_0, P_n will play a special role (see 1.2); they are called the principal vertices of R. Further it can be shown that R has at most n dihedral angles $w_0, \ldots, w_{n-1}$, which are not right angles; they are called the essential angles of R.

In the non-euclidean case they determine R uniquely up to isometry, and form a system of independent parameters for R, i.e., $R = R(w_0, \ldots, w_{n-1})$.

For three-dimensional spherical orthoschemes, Schläfli was able to integrate the volume differential (1). Independently of him, in 1836, Lobachevsky found a volume formula for three-dimensional hyperbolic orthoschemes $R = R(w_0, w_1, w_2)$ [13]; as a function of three essential angles and a further angle θ, which is related to w_0, w_1, w_2 by trigonometric relations, he obtained the following expression for the volume of R:

$$\mathrm{Vol}_3(R) = \tfrac{1}{4}\left\{L(w_0+\theta) - L(w_0-\theta) + L\left(\frac{\pi}{2}+w_1-\theta\right) + L\left(\frac{\pi}{2}-w_1-\theta\right)\right.$$
$$\left. + L(w_2+\theta) - L(w_2-\theta) + 2L\left(\frac{\pi}{2}-\theta\right)\right\}, \tag{2}$$

where

$$L(\alpha) := -\int_0^\alpha \log|2\sin t|\,dt$$

denotes the Lobachevsky function and

$$0 \leqq \theta := \arctan\frac{\sqrt{\cos^2 w_1 - \sin^2 w_0 \sin^2 w_2}}{\cos w_0 \cos w_2} < \frac{\pi}{2}.$$

About 1935 Coxeter revived interest in the work of these two mathematicians by developing an integration method for non-euclidean orthoschemes of dimension three [6], which led to the combination of their results. In Coxeter's method, θ- the so-called principal parameter of R-plays a fundamental role, because θ relates the measures of the dihedral angles to the corresponding apices in Schläfli's differential. This relation was discovered by W. Maier in the year 1954 [14]. Then, considering the essential angles w_0, w_1, w_2 and the principal parameter θ as four independent parameters, Coxeter extends Schläfli's differential by $d\theta$ such that a complete differential form arises which reduces to the volume differential for $\theta = \theta(w_0, w_1, w_2)$. This integration method for orthoschemes of dimension three was generalized by Böhm in 1962 [4] to spaces of constant nonvanishing curvature of arbitrary dimension.

The purpose of this paper is to derive a volume formula for a new class of hyperbolic polytopes, the complete orthoschemes [9], by generalizing the method of Coxeter-Böhm appropriately. Complete orthoschemes arise by taking ordinary orthoschemes and allowing one or both of the principal vertices (and with them possibly further vertices) to lie outside the absolute quadric in real projective n-space defining hyperbolic space H^n as its interior. By cutting off the ideal vertices along the polar hyperplanes corresponding to the principal vertices (inasmuch as they lie outside the quadric), we obtain convex polytopes in H^n which in general are no longer simplices, but have metrical properties analogous to those of ordinary orthoschemes.

Every convex polytope with acute dihedral angles, in particular complete orthoschemes, admits a subdivision into finitely many ordinary orthoschemes (see

[4, p. 80–81] and [17, Proposition 3.1], or see [16, p. 246]). Since the volume functional Vol_n operates additively, with (2) one immediately obtains a volume formula for every three-dimensional complete orthoscheme $\tilde{R}$ as a function of the essential angles w_k^l and principal parameters θ^l of the subdividing orthoschemes R_l $(0 \leqq k \leqq 2, 0 \leqq l \leqq N)$. Using hyperbolic trigonometry, we can reduce this volume formula to a relation which depends only on the essential angles of $\tilde{R}$, but which is very complicated in view of (2).

Our aim is to show that (2) holds even in the more general context of truncated orthoschemes $\tilde{R}$ (after a small modification in one particular case). At the end we shall evaluate (2) for complete Coxeter orthoschemes, i.e., for the complete orthoschemes with natural dihedral angles $\frac{\pi}{p}$, $p \in \mathbf{N}$, $p \geqq 2$ (see Appendix). Complete Coxeter orthoschemes were classified by Im Hof in 1983 [9]; they are the fundamental polytopes of a particular class of hyperbolic Coxeter groups, i.e., of certain discrete groups generated by the reflections in finitely many hyperplanes of H^n.

1. Complete orthoschemes

1.1 *Hyperbolic space*

Let H^n denote the real n-dimensional hyperbolic space. There are various models for H^n. Here we shall use the following descriptions:

Let $R^{1,n}$ be the $(n+1)$-dimensional real vector space R^{n+1}, together with the following bilinear form of signature $(1, n)$

$$\langle x, y\rangle := -x_0y_0 + x_1y_1 + \cdots + x_ny_n,$$
$$\forall x = (x_0, \ldots, x_n),\ y = (y_0, \ldots, y_n) \in R^{n+1}.$$

Then we can identify

$$\begin{aligned} H^n &= \{x \in R^{1,n} \mid \langle x, x\rangle = -1, x_0 > 0\} \\ &= \{x \in R^{n+1} \mid -x_0^2 + x_1^2 + \cdots + x_n^2 = -1, x_0 > 0\}. \end{aligned}$$

If we interpret this in real projective n-space P^n, we see that H^n is the interior of P^n with respect to the quadric

$$\begin{aligned} Q_{1,n} &:= \{[x] \in P^n \mid \langle x, x\rangle = 0\}: \\ H^n &=: IQ_{1,n} = \{[x] \in P^n \mid \langle x, x\rangle < 0\}. \end{aligned}$$

The closure $\overline{H^n}$ of H^n in P^n represents the natural compactification of H^n. Points of the boundary $\partial H^n = \overline{H^n} - H^n$ are called *points at infinity* of H^n. We shall also consider points $[x] \in P^n$ with $\langle x, x\rangle > 0$ which lie outside the absolute quadric. These points are called *ideal points* of H^n relative to $Q_{1,n}$, and the set of all such points is denoted by $AQ_{1,n}$. A projective k-plane is a k-dimensional projective subspace of P^n. For $k = n-2$ or $n-1$ we use the terms hyperlines or hyperplanes. A hyperbolic k-plane E^k is the intersection of a projective k-plane Γ^k with H^n. We call Γ^k the projectively closed or *hyperbolic* k-plane in P^n corresponding to E^k and use the notation $\Gamma^k = \hat{E}^k$.

To every point in P^n corresponds a hyperplane in P^n and vice versa: Let $P=[x]\in P^n$. A point $[y]\in P^n$ is said to be *conjugate to* $[x]$ *relative to* $Q_{1,n}$ iff $\langle x,y\rangle=0$ holds. The set of all points which are conjugate to $P=[x]$ form a projective hyperplane

$$\Gamma_P := \{[y]\in P^n \mid \langle x,y\rangle = 0\},$$

the *polar hyperplane to P*. P is called the *pole to* Γ_P.

Notations

pol(P):= polar hyperplane Γ_P to P,
pol(Γ_P):= pole P to Γ_P.

The quadric $Q_{1,n}$ induces a bijection from the points of P^n onto its hyperplanes, the polar hyperplanes to the points of P^n relative to $Q_{1,n}$. This map *pole*$\mapsto$*polar hyperplane* realizes the duality principle of the projective space P^n (see [8, Sect. 4E]).

Properties (see [8, Sect. 4])

(a) The polar hyperplane Γ_P to $P\in P^n$ respectively intersects, touches or avoids the quadric $Q_{1,n}$, iff $P\in AQ_{1,n}$, $P\in Q_{1,n}$ or $P\in IQ_{1,n}$.
(b) If two lines g, h in P^2 intersect at $S:=g\cap h$, then pol(S) is the line determined by pol(g), pol(h).
(c) If a line g in P^2 contains the point pol(h) of the line h, then $g\perp h$ holds.

The *hyperbolic distance* $d(P,Q)$ between two points $P=[x]$ and $Q=[y]\in H^n$ with $\langle x,x\rangle=\langle y,y\rangle=-1$ is given by

$$\cosh d(P,Q) = |\langle x,y\rangle|, \quad 0\leq d(P,Q)<\infty.$$

Let H_e, H_f be two hyperplanes in H^n represented in the form (see [16] and [8, Sect. 14B])

$$H_e=\{[x]\in H^n \mid \langle x,e\rangle=0\}, \quad \langle e,e\rangle=1,$$
$$H_f=\{[x]\in H^n \mid \langle x,f\rangle=0\}, \quad \langle f,f\rangle=1.$$

Then the corresponding hyperplanes $\hat{H}_e$, $\hat{H}_f$ in P^n intersect in a hyperline

$$\alpha := \hat{H}_e \cap \hat{H}_f.$$

If $\alpha\cap IQ_{1,n}\neq\emptyset$, then the *hyperbolic angle* $\varphi(H_e,H_f)$ is determined by

$$\cos\varphi(H_e,H_f)=|\langle e,f\rangle|, \quad 0\leq\varphi\leq\frac{\pi}{2}.$$

If $\alpha\cap IQ_{1,n}=\emptyset$, then the *hyperbolic length* $\rho(H_e,H_f)$ of the common perpendicular is given by

$$\cosh\rho(H_e,H_f)=|\langle e,f\rangle|, \quad 0\leq\rho<\infty.$$

The formulas of hyperbolic trigonometry can be derived from these definitions.

Those relations which we shall need later can be found in [4, Sect. 4.6–4.7], or in [15, Sect. 4].

1.2 *Orthoschemes*

We denote by X^n either the n-dimensional euclidean space E^n, the n-dimensional sphere S^n or the hyperbolic space H^n.

A convex polytope (or simply a polytope) in X^n is the convex hull of finitely many points in E^n, S^n or $\overline{H^n}$. Usually, a three-dimensional polytope is called a polyhedron. In particular, a polytope in X^n is called a k-simplex ($2 \leq k \leq n$) if it is the convex hull of $k+1$ points of E^n, S^n or $\overline{H^n}$ not lying in a single $(k-1)$-plane.

Combinatorically, simplices are the most basic objects, since every polytope in X^n admits a simplicial subdivision (see [4, p. 36]). However, the geometrically simplest polytopes are the orthoschemes (see [4, Sect. 4], as reference for this paragraph):

Definition. A k-simplex R in X^n ($2 \leqq k \leqq n$) is a *k-orthoscheme* iff the $k+1$ vertices of R can be labelled by $P_0, \ldots, P_k$ in such a way that

$$\operatorname{span}(P_0, \ldots, P_i) \perp \operatorname{span}(P_i, \ldots, P_k) \quad \text{for} \quad 1 \leqq i \leqq k-1.$$

Remarks

The pairwise orthogonal edges P_iP_{i+1} of a k-orthoscheme R ($0 \leq i \leq k-1$) form an *orthogonal edge-path.* The initial point P_0 and the final point P_k are distinguished in the following way: Apart from certain spherical degenerations there are no right planar angles at P_0 and P_k; at all the other vertices $P_l, l \neq 0, k$, there are always right angles in a prescribed manner. P_0 and P_k are called *principal vertices* of R. In the hyperbolic case, at most the principal vertices may be points at infinity. In this case, R is called *simply* or *doubly asymptotic.*

Every l-face ($2 \leq l \leq k-1$) of R is itself an orthoscheme. In particular we denote the $(k-1)$-face opposite to the vertex P_i by R_i ($0 \leq i \leq k$). If two l-faces ($1 \leq l \leq k-1$) intersect in a $(l-1)$-face F, then the dihedral angle between them is said to be *of order l with apex F.* An orthoscheme R has at most k dihedral angles of order $k-1$ which are not right angles. Let $w_{i,j} := \angle(R_i, R_j)$ denote the dihedral angle of order $k-1$ of R between the faces R_i and R_j. Then we have

$$w_{i,j} = \frac{\pi}{2}, \quad \text{if} \quad 0 \leqq i < j-1 \leqq k.$$

The k remaining dihedral angles $w_{i,i+1}$ ($0 \leq i \leq k-1$) are called the *essential* angles of R. As an abbreviation we write $w_i := w_{i,i+1}$ for $i = 0, \ldots, k-1$. Apart from the spherical case, the essential angles are always acute:

$$w_i < \frac{\pi}{2} \quad \text{for} \quad i = 0, \ldots, k-1.$$

Every dihedral angle of order $k-1$ of R can be reduced to an angle of first or second order with the same measure. Moreover, let us consider a sufficiently small ball with center at each vertex P_l, $0 \leqq l \leqq k$, such that its surface can be interpreted as spherical space of curvature $+1$. Then, the intersection of this sphere with the cone $\operatorname{cn}(P_l, R_l)$, the vertex figure of P_l, is a spherical $(k-1)$-orthoscheme, which is called the *vertex orthoscheme* of P_l. Every dihedral angle of order $k-1$ with P_l belonging to its apex appears as dihedral angle of order $k-2$ of the vertex orthoscheme of P_l.

In the non-euclidean case, the essential angles determine the orthoscheme R up to a motion and form a complete system of invariants for R. In the euclidean case, the essential angles determine the orthoscheme only up to similarity.

Knowing the measures of the essential angles w_k $(0 \leqq k \leqq n-1)$ of an n-orthoscheme R in X^n, we can decide whether R lies in E^n, S^n or H^n. In particular, the following theorem holds for $n=3$ (see [4, p. 166]):

Theorem. *Let R be a 3-orthoscheme with acute essential angles w_0, w_1, w_2. Then*

$$\cos w_1 - \sin w_0 \sin w_2 \begin{cases} <0, & \text{if } \ R \text{ spherical}, \\ =0, & \text{if } \ R \text{ euclidean}, \\ >0, & \text{if } \ R \text{ hyperbolic}. \end{cases}$$

1.3 *Complete orthoschemes*

From now on let R be a hyperbolic n-orthoscheme with vertices $P_0, \ldots, P_n$. We denote by h_i, $i=0,\ldots,n$, the $n+1$ supporting hyperplanes of the faces R_i of R in H^n. Then one has

$$h_i \perp h_j \quad \text{for} \quad j \neq i-1, i, i+2 \quad (0 \leqq i, j \leqq n).$$

Every such hyperplane h_i can be written as

$$h_i = h_{e_i} := \{[x] \in H^n \mid \langle x, e_i \rangle = 0\},$$

where e_i is a vector in $R^{1,n}$ with $\langle e_i, e_i \rangle > 0$ (see 1.1). Let

$$h_i^{\pm} := \{[x] \in H^n \mid \langle x, e_i \rangle \gtrless 0\}$$

denote the two closed half-spaces bounded by h_i. After a suitable choice of orientation of the vectors e_i, it follows that

$$R = \bigcap_{i=0}^{n} h_i^{+}.$$

According to 1.2, only the principal vertices may be points at infinity. Now we extend the class of orthoschemes by allowing one or both of the principal vertices P_0, P_n (and with them possibly further vertices) to lie outside the absolute quadric $Q_{1,n}$, and by cutting off the ideal vertices by means of the polar hyperplanes $\operatorname{pol}(P_0)$ and $\operatorname{pol}(P_n)$ to P_0 and P_n (inasmuch as they lie outside the quadric):

Let $e_0, \ldots, e_{n+2} \in R^{1,n}$ be $n+3$ non-isotropic vectors with

$$\langle e_i, e_j \rangle = 0 \quad \text{for} \quad j \neq i-1, i, i+1 \quad (\text{Indices mod.}\, n+3), \tag{3}$$

and such that

$$C := \{x \in R^{1,n} \mid \langle x, e_i \rangle \geqq 0, 0 \leqq i \leqq n+2\}$$

contains a non-empty open subset of H^n.

For the projectively closed hyperplanes $H_i := H_{e_i} = \{[x] \in P^n \mid \langle x, e_i \rangle = 0\}$ belonging to e_i, we find that

$$H_i \perp H_j \quad \text{for} \quad h \neq i-1, i, i+1 \quad \text{(Indices mod.} n+3).$$

Relative to the quadric $Q_{1,n}$ there are the following possibilities [9]:

1. One hyperplane, say H_{n+2}, lies outside the quadric $Q_{1,n}$.
2. Two successive hyperplanes, say H_{n+1}, H_{n+2}, lie outside the quadric $Q_{1,n}$.
3. All hyperplanes intersect $Q_{1,n}$ and are hyperbolic.

Other cases do not occur, since out of two perpendicular hyperplanes at least one must intersect the quadric $Q_{1,n}$.

Now, the *hyperbolic* hyperplanes $H_0, \ldots, H_{n+d}$, $0 \leqq d \leqq 2$, in the cases 1.-3. bound a convex polytope in H^n (see [17, Sect. 2, Proposition 2.2]). Choosing a suitable orientation of their normal vectors e_i, $\langle e_i, e_i \rangle > 0$, we define:

Definition. The hyperbolic polytope

$$\tilde{R} := \bigcap_{i=0}^{n+d} H_i^+, \quad 0 \leqq d \leqq 2,$$

is called a *complete (hyperbolic) orthoscheme of dimension n and degree d.*

Remark. For $d = 0$ the notions of ordinary and complete orthoscheme coincide.

For $d > 0$, i.e., for truncated orthoschemes, we define (also [17, Sect. 4]):

Definition. The convex polytope in P^n

$$\hat{R} := \bigcap_{i=0}^{n} H_i^+ = \bigcap_{i=0}^{n} \{[x] \in P^n \mid \langle x, e_i \rangle \geqq 0\}$$

is called the *ideal orthoscheme* to $\tilde{R}$.

Remarks

(a) A complete orthoscheme $\tilde{R}$ of degree 2 can be interpreted as an ideal orthoscheme $\hat{R}$ with two ideal principal vertices P_0, P_n which is truncated by the polar hyperplanes $\mathrm{pol}(P_0)$, $\mathrm{pol}(P_n)$. Therefore, $\tilde{R}$ is also called *doubly truncated with continuation* $\hat{R}$.

(b) A complete orthoscheme $\tilde{R}$ of degree 1 can be interpreted as an ideal orthoscheme $\hat{R}$ with one ideal principal vertex, say P_0, which is truncated by $\mathrm{pol}(P_0)$. $\tilde{R}$ is called *simply truncated with ideal vertex* P_0 *and continuation* $\hat{R}$.

(c) The notions "dihedral angle of order k", "apex", etc. translate to complete orthoschemes according to 1.2.

(d) Every l-face $(2 \leqq l \leqq n-1)$ of a complete orthoscheme is itself a complete orthoscheme. This follows easily from the above definition using the corresponding properties of the associated ideal orthoscheme according to 1.2.

(e) Let $\tilde{R} = \cap_{i=0}^{n+d} H_i^+$ be a complete orthoscheme of degree d, $0 \leqq d \leqq 2$. Then, for two non-orthogonal bounding hyperplanes H_i, H_{i+1} one has one of the following cases (see (3) and [17, Sect. 1]):

$$\text{In } H^n,\ H_i \text{ and } H_{i+1} \begin{cases} \text{intersect on } \tilde{R} \text{ at an angle } < \pi/2. \\ \text{are parallel.} \\ \text{admit a common perpendicular.} \end{cases}$$

Since the dihedral angles of order $n-1$ of an orthoscheme are reducable to angles of second order at edges (see 1.2), for the ideal orthoscheme $\hat{R}$ corresponding to $\tilde{R}$ we make the following

Definition. $\hat{R}$ has *finite edges* (and $\tilde{R}$ *is of type A*) iff every edge emanating from the principal vertices P_0, P_n intersects the absolute quadric $Q_{1,n}$, hence contains a hyperbolic segment. In the other case, $\tilde{R}$ is said to be *of type B*.

Remark. If $\tilde{R}$ is a complete orthoscheme of dimension n and of type B (no continuation with finite edges), then, for $n \geqq 4$, $\tilde{R}$ has m essential angles with $n \leqq m \leqq n+3$. For $n=3$, however, we have $m=n=3$, since the Euler equation for compact polyhedrons implies that the number of degrees of freedom equals the number of dihedral angles. Furthermore, it is easy to show that for three-dimensional complete orthoschemes $\tilde{R}$ only three configurations of type A and one configuration of type B can occur (see also [1]):

A1. $\tilde{R} = R$ is an ordinary orthoscheme.
A2. $\tilde{R}$ is a *simple frustum with ideal vertex* P_0, i.e., $\tilde{R}$ is simply truncated with ideal vertex P_0 and with continuation $\hat{R}$ that has finite edges.
A3. $\tilde{R}$ is a *double frustum*, i.e., $\tilde{R}$ is doubly truncated with continuation $\hat{R}$ that has finite edges.
B. $\tilde{R}$ is a *Lambert cube*, i.e., $\tilde{R}$ is doubly truncated with continuation $\hat{R}$ whose hypotenuse (edge connecting the two principal vertices) lies outside the quadric $Q_{1,3}$. Hence, $\tilde{R}$ is a polyhedron with six Lambert quadrilaterals (i.e. quadrilaterals with one acute and three right angles) as bounding faces and with three essential angles at prescribed edges (opposite faces are congruent).

Hence, to calculate volumes of three-dimensional complete orthoschemes, it suffices to consider the above four types A1–A3 and B.

1.4 *Complete Coxeter orthoschemes*

Let X^n be E^n, S^n or H^n. A polytope in X^n is called a *Coxeter polytope* iff it has natural dihedral angles, i.e., angles of the form $\frac{\pi}{p}$, $p \in \mathbb{N}$, $p \geqq 2$.

To every Coxeter polytope P_C corresponds a Coxeter graph $\Sigma(P_C)$ ([17, Sect. 5]). In particular, a complete Coxeter orthoscheme $\tilde{R}_C$ of dimension n and degree d can be characterized as follows [9]:

1. If $\tilde{R}_C$ is an ordinary orthoscheme, the graph $\Sigma(\tilde{R}_C)$ consists of a linear chain of length $n+1$.
2. If $\tilde{R}_C$ is of degree 1, $\Sigma(\tilde{R}_C)$ consists of a linear chain of length $n+2$.
3. If $\tilde{R}_C$ is of degree 2, $\Sigma(\tilde{R}_C)$ consists of a cycle of length $n+3$.

2. The Schläfli differential formula

Let $\mathscr{P}^n_\kappa$ $(n \geqq 3)$ denote the set of all convex, compact n-polytopes in H^n of combinatorial type κ (see [1]) and dihedral angles not exceeding $\frac{\pi}{2}$. In particular, let $\mathscr{S}^n$ denote the set of all compact n-simplices in H^n.

It is known that the congruence class of an element of $\mathscr{P}^n_\kappa$ is uniquely determined by its dihedral angles ([1], Sect. 3, Uniqueness Theorem). Therefore the volume function $\mathrm{Vol}_n = \mathrm{Vol}_n | \mathscr{P}^n_\kappa$ on $\mathscr{P}^n_\kappa$ is a function of the dihedral angles.

On the set of spherical n-simplices, Schläfli established a formula for the differential of the volume depending on the dihedral angles [16]. Kneser proved the validity of this formula also in the case of hyperbolic n-simplices [11]:

Theorem (Schläfli's differential formula). *Let $S \in \mathscr{S}^n$ $(n \geqq 2)$ have vertices $P_0, \ldots, P_n$ and dihedral angles $w_{jk} = \angle(S_j, S_k)$, $0 \leqq j < k \leqq n$, of order $n-1$ with apex $S_{jk} := S_j \cap S_k$. Then the differential of the volume function* Vol_n *on $\mathscr{S}^n$ can be represented by*

$$d\,\mathrm{Vol}_n(S) = \frac{1}{1-n} \sum_{\substack{j,k=0 \\ j<k}}^{n} \mathrm{Vol}_{n-2}(S_{jk})\, dw_{jk} \quad (\mathrm{Vol}_0(S_{jk}) := 1).$$

Now, this theorem holds even on the set $\mathscr{P}^n_k$, i.e., if $P \in \mathscr{P}^n_\kappa$ is a polytope with dihedral angles w_j, $j \in J$, and corresponding apices F_j, $j \in J$, then the differential of the volume function Vol_n on $\mathscr{P}^n_k$ can be written as

$$d\,\mathrm{Vol}_n(P) = \frac{1}{1-n} \sum_{j \in J} \mathrm{Vol}_{n-2}(F_j)\, dw_j. \tag{4}$$

Schläfli stated this general proposition for spherical polytopes in [16, p. 272–273]. He there gives a sketch of the proof which is elementary, but very extensive. Schläfli's main idea consists in choosing a rule to subdivide (in the sense of elementary geometry) polytopes $P \in \mathscr{P}^n_\kappa$ into finitely many simplices. Then, taking the volume differential he applies the Theorem to the dissecting simplices separately. Finally, distinguishing the two cases of dimension $n = 3$ (here, apices are simplices) and $n > 3$, one can show that the differential expressions in the representation of $d\,\mathrm{Vol}_n(P)$ can be collected in such a way that they sum up to the desired formula.

The detailed proof can be carried out and translated for hyperbolic polytopes of arbitrary dimension. In the special case of complete orthoschemes of dimension three, we obtain the following result:

Theorem I. *Let $\mathscr{R}_\kappa$ be the set of compact complete orthoschemes $\tilde{R}$ in H^3 of combinatorial type κ with essential angles w_j and corresponding apices of length*

$V_j, j=0,1,2$. *Further, let* Vol_3 *denote the volume function restricted on* $\mathscr{R}_\kappa$. *Then*

$$d\,\mathrm{Vol}_3(\tilde{R}) = -\frac{1}{2}\sum_{j=0}^{2} V_j dw_j.$$

This formula for the differential of the volume function on $\mathscr{R}_\kappa$ will play an important role when calculating the volume of a three-dimensional complete orthoscheme.

3. The volume of three-dimensional complete orthoschemes

3.1 *The principal parameter*

To extend the notion of principal parameter to complete orthoschemes, we distinguish complete orthoschemes of type A and B (see 1.3).

A. Let $\tilde{R}$ be of type A, i.e. with continuation $\hat{R}$ having finite edges, and with essential angles w_k, $0 \leqq w_k \leqq \frac{\pi}{2}$, $k=0,1,2$.

Definition A. The *principal parameter* θ of $\tilde{R}$ is given by

$$\tan^2\theta := \frac{\cos^2 w_1 - \sin^2 w_0 \sin^2 w_2}{\cos^2 w_0 \cos^2 w_2},$$

and it is uniquely determined by $0 \leqq \theta < \frac{\pi}{2}$.

Remark. We have to check whether the definition of θ makes sense, i.e. whether

$$\cos w_1 \geqq \sin w_0 \sin w_2. \tag{5}$$

In the following we shall analyse this point and deduce further properties of θ.

Properties of θ

(a) From Definition A one derives the equations

$$\cos^2\theta = \frac{\cos^2 w_0 \cos^2 w_2}{\cos^2 w_0 - \sin^2 w_1 + \cos^2 w_2} \tag{6}$$

and

$$(\cos^2\theta - \cos^2 w_0)(\cos^2\theta - \cos^2 w_2) = \cos^2\theta(\cos^2\theta - \sin^2 w_1). \tag{7}$$

(b) θ describes the combinatorial character of $\tilde{R}$ as follows: Let $\hat{R}$ be the ideal orthoscheme associated to $\tilde{R}$ and denote the principal vertices of $\hat{R}$ by P_0, P_3 and the essential angles by w_0, w_1, w_2. If the principal vertex P_k, $k=0$ and/or 3, is a finite point of H^3, then the vertex figure to P_k (see 1.2) is a spherical 2-orthoscheme. If P_k is an ideal vertex (beyond infinity), then $\mathrm{pol}(P_k) \cap \hat{R}$ is a hyperbolic 2-orthoscheme (see 1.1). E.g. for $k=0$ we have the following picture:

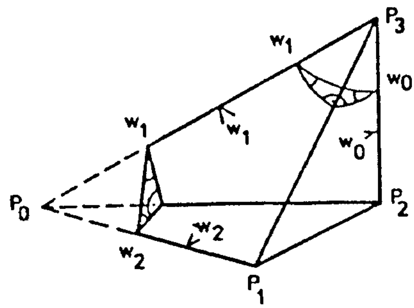


Fig. 1. Simple frustum with ideal vertex P_0

From the known area formulas for non-euclidean triangles we deduce (5), and using the notation

$$\bar{w}_k := \begin{cases} w_k, & k=1 \\ \frac{\pi}{2} - w_k, & k=0,2, \end{cases} \tag{8}$$

we obtain the following results for $0 < w_0, w_1, w_2 < \frac{\pi}{2}$ (see [10, Sect. 3.1]):

A1. For an ordinary orthoscheme $\tilde{R}$:

$$w_1 + w_k > \frac{\pi}{2}, \quad k=0,2; \quad 0 < \theta + \bar{w}_k < \frac{\pi}{2}, \quad k=0,1,2. \tag{9}$$

A2. For a simple frustum $\tilde{R}$ with ideal vertex P_0:

$$w_1 + w_2 < \frac{\pi}{2}, \quad 0 < \theta + \bar{w}_0 < \frac{\pi}{2} < \theta + \bar{w}_k < \pi, \quad k=1,2. \tag{10}$$

A3. For a double frustum $\tilde{R}$:

$$0 < \theta + w_1 < \frac{\pi}{2} < \theta + \bar{w}_k < \pi, \quad k=0,2. \tag{11}$$

B. Let $\tilde{R}$ be of type B, i.e., $\tilde{R}$ is a Lambert cube (see 1.3). Denote by $\hat{R}$ the continuation of $\tilde{R}$ with vertices $P_0, \ldots, P_3$. Then the principal vertices P_0, P_3 are ideal points such that the polar planes $\mathrm{pol}(P_0)$, $\mathrm{pol}(P_3)$ intersect at an angle, say σ, on a edge of $\tilde{R}$ of length l. This edge is the common perpendicular of the faces $\hat{R}_1$ and $\hat{R}_2$ (see 1.1). The angle σ and its apex of length l replace the ideal hypotenuse P_0P_3 and the purely imaginary dihedral angle w_1 as follows (see Fig. 2 and Remark (c)):

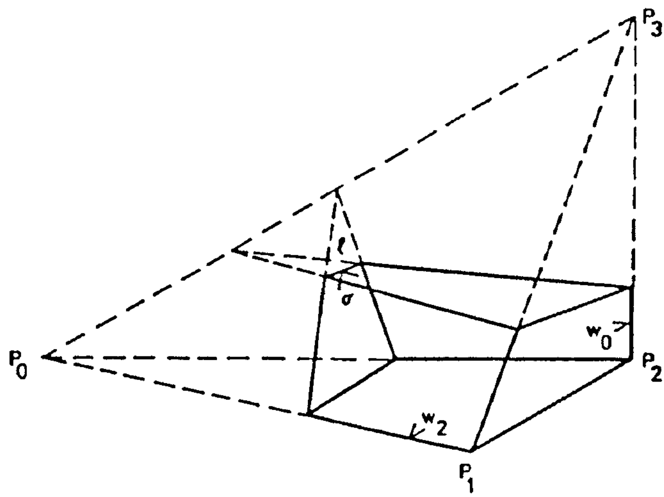


Fig. 2. Lambert cube

For a Lambert cube $\tilde{R}$ with essential angles $0 \leqq w_0, \sigma, w_2 \leqq \frac{\pi}{2}$ as above we now define:

Definition B. The *principal parameter* θ of a Lambert cube $\tilde{R}$ is given by

$$\tan^2 \theta := \frac{\cosh^2 l - \sin^2 w_0 \sin^2 w_2}{\cos^2 w_0 \cos^2 w_2} \quad \text{with} \quad 0 < \theta \leqq \frac{\pi}{2},$$

where l denotes the length of the apex belonging to σ.

Remarks

(a) The use of the same symbols and names for θ in the cases A and B is legitimate, since hyperbolic geometry admits a complex continuation to the space $AQ_{1,n}$ of ideal points of H^n relative to $Q_{1,n}$ ([15, Sect. 5]). Using this concept the dihedral angle w_1 of $\hat{R}$ at the hypotenuse P_0P_3 is purely imaginary and corresponds to the perpendicular of length l according to

$$iw_1 = l \quad \text{or} \quad \cos w_1 = \cosh l.$$

(b) Since $\cosh l > \sin w_0 \sin w_2$, the quantity θ is well-defined and satisfies the inequalities

$$\theta > w_0, w_2. \tag{12}$$

We shall show later, that $\theta > \sigma$.
(c) Analogously to (6) we have

$$\cos^2 \theta = \frac{\cos^2 w_0 \cos^2 w_2}{\cos^2 w_0 - \sinh^2 l + \cos^2 w_2}. \tag{13}$$

3.2 *The principal parameter in the asymptotic case*

The asymptotic complete orthoschemes (see 1.2) form the limiting cases between the different types A1–A3, B of complete orthoschemes, as will now be seen.

A. Let $\tilde{R}$ be simply or doubly asymptotic. Then the vertex figure to the principal vertex P_k, $k=0$ or 2, at infinity is a euclidean 2-orthoscheme ([4, p. 190]). Hence it follows:

$$\tilde{R} \text{ simply asymptotic} \Leftrightarrow \theta = w_k = \frac{\pi}{2} - w_1, \quad k = 0 \text{ or } 2. \tag{14}$$

$$\tilde{R} \text{ doubly asymptotic} \Leftrightarrow \theta = w_0 = w_2 = \frac{\pi}{2} - w_1. \tag{15}$$

B. The transition from a double frustum (type A3) to a Lambert cube (type B) is characterized by $w_1 = l = 0$.

3.3 *The fundamental relations*

The principal parameter θ relates the essential angles w_k, $0 < w_k < \frac{\pi}{2}$, to the lengths of the corresponding apices as follows:

A. Suppose $\tilde{R}$ is a complete orthoscheme of type A, i.e., the ideal orthoscheme $\hat{R}$ with vertices $P_0, \ldots, P_3$ has finite edges. The following cases can occur:

A1. For ordinary orthoschemes $\tilde{R} = R$ the relationship mentioned above is already known in the form (see [4, p. 229, (11)])

$$\tanh V_k = \tan\theta \cdot \tan\bar{w}_k \quad \text{or} \quad V_k = \tfrac{1}{2}\log\frac{\cos(\theta - \bar{w}_k)}{\cos(\theta + \bar{w}_k)}, \quad k = 0, 1, 2, \tag{16}$$

with the standard notation (8) for $\bar{w}_k$.

A2. Let $\tilde{R}$ be a simple frustum with ideal vertex P_0. Then, for $1 \leqq i \leqq 3$, the faces $\tilde{R}_i$ of $\tilde{R}$ opposite to P_i are Lambert quadrilaterals. Furthermore, let u_k and v_k $(l = 1, 2)$ denote the lengths of the edges of the hyperbolic polar orthoscheme $\mathrm{pol}(P_0) \cap \hat{R}$ to P_0, and of the spherical vertex orthoscheme, according to the following figure:

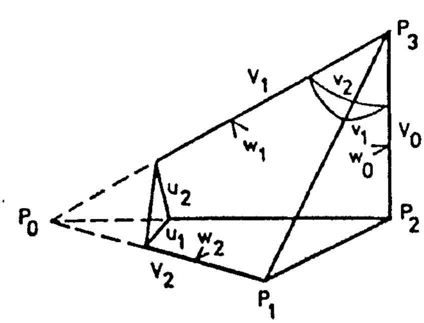


Fig. 3

By means of non-euclidean trigonometry, we deduce the following equalities:

$$\cos v_1 = \frac{\cos w_1}{\sin w_0} \quad \text{and} \quad \cos v_2 = \cot w_0 \cdot \cot w_1,$$

$$\cosh V_0 = \frac{\cos w_2}{\sin v_1},$$

as well as

$$\cosh u_1 = \frac{\cos w_1}{\sin w_2} \quad \text{and} \quad \sinh V_2 = \frac{\cos w_0}{\sinh u_1},$$

$$\cosh u_2 = \frac{\cos w_2}{\sin w_1} \quad \text{and} \quad \sinh V_1 = \frac{\cot v_2}{\tanh u_2}.$$

Hence it follows that

$$\tanh V_0 = \tan\theta \cdot \cot w_0, \quad \coth V_k = \tan\theta \cdot \tan \bar{w}_k \quad \text{resp.}$$

$$V_0 = \tfrac{1}{2}\log\frac{\cos(\theta - \bar{w}_0)}{\cos(\theta + \bar{w}_0)}, \quad V_k = \tfrac{1}{2}\log\frac{\cos(\theta - \bar{w}_k)}{-\cos(\theta + \bar{w}_k)} \quad \text{for} \quad k = 1, 2.$$

Putting

$$\bar{V}_k := \begin{cases} V_k & \text{for} \quad k = 0 \\ V_k + i\dfrac{\pi}{2} & \text{for} \quad k = 1, 2, \end{cases}$$

we finally obtain, for $k = 0, 1, 2$,

$$\tanh \bar{V}_k = \tan\theta \cdot \tan \bar{w}_k \quad \text{or} \quad V_k = \tfrac{1}{2}\log\left|\frac{\cos(\theta - \bar{w}_k)}{\cos(\theta + \bar{w}_k)}\right|. \tag{17}$$

A3. Let $\tilde{R}$ be a double frustum. Here, we derive the relations [10, p. 24]

$$\tanh V_1 = \tan\theta \cdot \tan w_1, \quad \coth V_k = \tan\theta \cdot \tan \bar{w}_k \quad \text{resp.}$$

$$V_1 = \tfrac{1}{2}\log\frac{\cos(\theta - \bar{w}_1)}{\cos(\theta + \bar{w}_1)}, \quad V_k = \tfrac{1}{2}\log\frac{\cos(\theta - \bar{w}_k)}{-\cos(\theta + \bar{w}_k)} \quad \text{for} \quad k = 0, 2.$$

Putting

$$\bar{V}_k = \begin{cases} V_k + i\pi & k = 1 \\ V_k + i\dfrac{\pi}{2} & k = 0, 2, \end{cases}$$

we have again, for $k = 0, 1, 2$,

$$\tanh \bar{V}_k = \tan\theta \cdot \tan \bar{w}_k \quad \text{or} \quad V_k = \tfrac{1}{2}\log\left|\frac{\cos(\theta - \bar{w}_k)}{\cos(\theta + \bar{w}_k)}\right|. \tag{18}$$

B. Let $\tilde{R}$ be a Lambert cube with essential angles $0 < w_0, \sigma, w_2 < \dfrac{\pi}{2}$ and corresponding apices of length V_0, l, V_2 (see Fig. 2.). Then we obtain the relations [10, p. 25]):

$$\tan\sigma = \tan\theta\cdot\tanh l$$

$$\coth V_k = \tan\theta\cdot\cot w_k \quad \text{for} \quad k = 0, 2,$$

where

$$\tan\theta = \frac{\sqrt{\cosh^2 l - \sin^2 w_0 \sin^2 w_2}}{\cos w_0 \cos w_2}.$$

The first relation leads to the inequality $\sigma < \theta$ asserted in 3.1, B, (b).

Putting

$$w_1 := \sigma, \quad V_1 := l,$$

we find that

$$\tanh V_k = \cot\theta\cdot\tan w_k \quad \text{resp.} \quad V_k = \tfrac{1}{2}\log\frac{\cos(\theta - \bar{w}_k)}{-\cos(\theta + \bar{w}_k)}, \quad k = 0, 1, 2.$$

More conveniently, using the notations ($k = 0, 1, 2$)

$$\bar{w}_k := \frac{\pi}{2} - w_k, \quad \bar{V}_k := V_k + i\frac{\pi}{2}:$$

$$\tanh \bar{V}_k = \tan\theta\cdot\tan\bar{w}_k, \quad \text{resp.} \quad V_k = \tfrac{1}{2}\log\left|\frac{\cos(\theta - \bar{w}_k)}{\cos(\theta + \bar{w}_k)}\right|, \tag{19}$$

where

$$\tan\theta = \frac{\sqrt{\cosh^2 l - \sin^2 w_0 \sin^2 w_2}}{\cos w_0 \cos w_2}.$$

We summarize these fundamental relations between the measures of essential angles, apices and principal parameters of complete orthoschemes of dimension three as follows:

Lemma 1. *Let $\tilde{R}$ be a three-dimensional complete hyperbolic orthoscheme. If the essential angles $\bar{w}_k$, $0 < w_k < \frac{\pi}{2}$, and the lengths $\bar{V}_k (k = 0, 1, 2)$ of the apices belonging to w_k are determined according to the formulas appropriate to the type A1–A3 or B of $\tilde{R}$, then*

$$\tanh \bar{V}_k = \tan\theta\cdot\tan\bar{w}_k \quad \textit{resp.} \quad V_k = \tfrac{1}{2}\log\left|\frac{\cos(\theta - \bar{w}_k)}{\cos(\theta + \bar{w}_k)}\right|, \quad k = 0, 1, 2.$$

Remarks

(a) In the case of classical orthoschemes the invariance property

$$\tan\theta = \tanh \bar{V}_k\cdot\cot\bar{w}_k, \quad k = 0, 1, 2,$$

of θ led to the notion of *principal parameter* or *invariant of the orthoscheme.*

(b) From Lemma 1 we deduce the following relation which will be of importance later

$$\frac{\partial V_k}{\partial \theta}=\frac{\sin \tilde{w}_k \cos \tilde{w}_k}{\cos^2 \theta-\sin^2 \tilde{w}_k} \quad \text{for} \quad k=0,1,2, \tag{20}$$

where $\tilde{w}_k$ is again determined according to the actual type A1–A3 or B of $\tilde{R}$.

3.4 *The cases* $\theta=0$ *and* $\theta=\frac{\pi}{2}$

The values $\theta=0$ and $\frac{\pi}{2}$ describe degenerated complete orthoschemes of zero 3-volume:

Lemma 2. *Let* $\tilde{R}$ *denote a complete orthoscheme of dimension three with principal parameter* θ.

A. *If* $\tilde{R}$ *is of type A, then* $\theta=0$ *implies* $\mathrm{Vol}_3(\tilde{R})=0$.

B. *If* $\tilde{R}$ *is of type B, then* $\theta=\frac{\pi}{2}$ *implies* $\mathrm{Vol}_3(\tilde{R})=0$.

Proof. A1. If $\tilde{R}$ is an ordinary orthoscheme with vertices $P_0,\ldots,P_3$, it follows from Lemma 1 (see 3.3) for $0<w_k<\frac{\pi}{2}$, $k=0,1,2$, that:

$$\theta=0 \Rightarrow V_k=0 \quad \text{and} \quad \cos w_1=\sin w_0 \sin w_2. \tag{21}$$

In this case, $\tilde{R}$ degenerates to a point-shaped orthoscheme in H^3 with edges of length zero and euclidean angle configurations (see 1.2, Theorem). In the limiting case $\tilde{w}_k=\frac{\pi}{2}$, $k=0,1,2$, the dimension decreases (and therefore $\mathrm{Vol}_3(\tilde{R})=0$), since then the vertices P_0, P_1 resp. P_2, P_3 coincide.
A2. If $\tilde{R}$ is a simple frustum with ideal vertex P_0, we derive from 3.1, (10), and 3.2:

$$0 \leqq \frac{\pi}{2}-w_1, w_2 \leqq \theta \leqq w_0<\frac{\pi}{2}.$$

It follows that

$$\theta=0 \Rightarrow \frac{\pi}{2}-w_1=w_2=0, \quad 0 \leqq w_0<\frac{\pi}{2}. \tag{22}$$

Moreover, Lemma 1 shows that

$$\theta=0, \quad 0<w_0<\frac{\pi}{2} \Rightarrow V_0=0. \tag{23}$$

When $\theta=0$ and $w_0=0$ the claim follows from case A1.

A3. If $\tilde{R}$ is a double frustum, we have, according to 3.1, (6) and (11):

$$\theta = 0 \Rightarrow \bar{w}_k = \frac{\pi}{2} \quad (k = 0, 1, 2). \tag{24}$$

Hence $\theta = 0$ again implies a decrease in the dimension of $\tilde{R}$ (see A1).

B. Finally, if $\tilde{R}$ is a Lambert cube with essential angles $0 \leqq w_k \leqq \frac{\pi}{2}$, by 3.2, (13), we see that $\theta = \frac{\pi}{2}$ iff at least one essential angle, say w_0, is equal to $\frac{\pi}{2}$. But in this case, the two opposite faces of $\tilde{R}$ being Lambert quadrilaterals with angle $w_0 = \frac{\pi}{2}$ degenerate to a point. Hence, we have again decrease in the dimension of $\tilde{R}$.

QED

3.5 *The Lobachevsky function*

We shall see that the volume formula for a complete orthoscheme of dimension three is an expression involving the Lobachevsky function or related functions ([12, Sect. 4]).

Let $\omega \in \mathbf{R}$:

Definition. The function

$$L(\omega) := \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(2n\omega)}{n^2} = -\int_0^{\omega} \log|2 \sin t|\, dt = \left(\frac{\pi}{2} - \omega\right) \log 2 + \int_0^{(\pi/2)-\omega} \log|\cos t|\, dt$$

is called *Lobachevsky function.*

The Lobachevsky function $L(\omega)$ is closely related to the Clausen function [12, Sect. 4]

$$\mathrm{Cl}(\theta) := \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2} = -\int_0^{\theta} \log\left|\sin\frac{t}{2}\right| dt$$

according to

$$L(\omega) = \tfrac{1}{2}\mathrm{Cl}(2\omega), \quad \forall \omega \in \mathbb{R},$$

and has the following properties:

Properties [12, Sect. 4].

(a) $L(\omega)$ is well-defined and continuous for all $\omega \in \mathbb{R}$. $L(\omega)$ is odd and π-periodic. It assumes its maximum value at $\omega_k = \frac{\pi}{6} + k\pi$, $k \in \mathbb{Z}$.

(b) $L(\omega)$ satisfies the following functional equation:

$$L(n\omega) = n \sum_{k \bmod n} L\left(\omega + \frac{k\pi}{n}\right), \quad \forall n \in \mathbb{N}, \forall \omega \in \mathbb{R}.$$

In particular for $n=2$, this relation yields

$$L(2\omega)=2L(\omega)+2L\left(\frac{\pi}{2}+\omega\right),\quad \forall\omega\in\mathbb{R}.$$

(c) For actual computation we use the following representation of $L(\omega)$ for $|\omega|<\frac{\pi}{2}$:

$$L(\omega)=\omega\left(1-\log|2\omega|+\sum_{n=1}^{\infty}\frac{B_n\cdot(2\omega)^{2n}}{2n(2n+1)!}\right),$$

where $B_n, n\geqq 1$, denote the Bernoulli numbers.

3.6 *The volume in case A*

Let $\tilde{R}$ be a complete orthoscheme of type A with essential angles w_k and corresponding apices of length V_k $(0\leqq k\leqq 2)$. Then, the Schläfli differential formula for the volume $\mathrm{Vol}_3(\tilde{R})$ of $\tilde{R}$ yields (see Theorem I, 2):

$$d\,\mathrm{Vol}_3(\tilde{R})=-\frac{1}{2}\sum_{k=0}^{2}V_k dw_k.$$

Here, the coefficients V_k are given by (see Lemma 1, 3.3):

$$V_k=\tfrac{1}{2}\log\left|\frac{\cos(\theta-w_k)}{\cos(\theta+\bar{w}_k)}\right|,\quad k=0,1,2.$$

Hence, they are very complicated expressions in the essential angles of $\tilde{R}$. Extending the above volume differential by the differential of θ, we can carry out the integration and identify the result as a formula for the volume of $\tilde{R}$. For this purpose, we interpret the coefficients as functions of the *four* parameters $w_0, w_1, w_2, w_2:=\theta$, now regarded as independent from each other. With this in mind, we define

$$\hat{V}_k(w_0,\dots,w_3):=\tfrac{1}{2}\log\left|\frac{\cos(w_3-\bar{w}_k)}{\cos(w_3+\bar{w}_k)}\right|,\quad k=0,1,2. \tag{25}$$

Hence, it follows that

$$\hat{V}_k|_{w_3=\theta(w_0,w_1,w_2)}=V_k(w_0,w_1,w_2),\quad k=0,1,2. \tag{26}$$

Consider the region

$$G:=\left\{(w_0,\dots,w_3)\in\mathbb{R}^4\,|\,0<w_0,\dots,w_3<\frac{\pi}{2};\,w_3\neq\frac{\pi}{2}-\bar{w}_k, k=0,1,2\right\},$$

and on G, the following differential form

$$\Omega:=\sum_{k=0}^{3}W_k dw_k,$$

with

$$W_k(w_0,\dots,w_3):=-\tfrac{1}{2}\hat{V}_k(w_0,\dots,w_3),\quad k=0,1,2. \tag{27}$$

We determine $W_3 \in C^1(G)$ such that

(I) the integrability conditions hold: $\frac{\partial W_i}{\partial w_k} = \frac{\partial W_k}{\partial w_i}$ for $0 \leqq i, k \leqq 3, i \neq k$.

(II) $W_3 = 0$ for $w_3 = \theta(w_0, w_1, w_2)$.

The definitions (25) and (27) imply that $W_k(w_0, \ldots, w_3)$ depends only on w_k and w_3, i.e.:

$$\frac{\partial W_k}{\partial w_i} = \frac{\partial W_i}{w_k} = 0 \quad \text{for} \quad 0 \leqq i, k \leqq 2, i \neq k.$$

On the other hand, 3.3, (20), yields

$$\frac{\partial W_k}{\partial w_3} = -\tfrac{1}{2} \frac{\sin \bar{w}_k \cos \bar{w}_k}{\cos^2 w_3 - \sin^2 \bar{w}_k} \quad \text{for} \quad k = 0, 1, 2.$$

Hence, W_3 is given by (see [2] for the case of ordinary orthoschemes)

$$W_3 = \tfrac{1}{4} \log \frac{(\cos^2 w_3 - \sin^2 \bar{w}_1) \cos^2 w_3}{(\cos^2 w_3 - \sin^2 \bar{w}_0)(\cos^2 w_3 - \sin^2 \bar{w}_2)}. \tag{28}$$

It is obvious that $W_3 \in C^1(G)$, and that W_3 satisfies (I), (II) (see 3.1, (7)).

The differential form Ω of (26) restricted to the hypersurface

$$w_3 = \theta(w_0, w_1, w_2) \quad \text{in } \mathbb{R}^4$$

is identical with the Schläfli volume differential $d\,\mathrm{Vol}_3(\tilde{R})$ and is called the *extended Schläfli differential form.*

According to (I), the extended Schläfli differential form satisfies the integrability conditions (I). Hence, following the Theorem of Poincaré [5, Sect. 2]), it is exact and path-independent in every connected component of G. We now perform the integration for the different types A1–A3 separately.

A1. An ordinary orthoscheme R is realizable for essential angle $0 < w_0, w_1, w_2 < \frac{\pi}{2}$ with (see 3.1, (9), and 3.4)

$$w_0 + w_1 > \frac{\pi}{2}, \quad w_1 + w_2 > \frac{\pi}{2}, \quad \text{i.e.:} \quad 0 \leqq \theta < \frac{\pi}{2} - \bar{w}_k < \frac{\pi}{2}, \quad k = 0, 1, 2.$$

Therefore we assign to R the convex region of $\bar{G}$

$$G_1 := \left\{ (w_0, \ldots, w_3) \in \bar{G} \,\middle|\, w_0 + w_1, w_1 + w_2 > \frac{\pi}{2}, 0 \leqq w_3 < w_0, w_2, \frac{\pi}{2} - w_1 < \frac{\pi}{2} \right\}.$$

Next we integrate the extended Schläfli differential form

$$\Omega = \sum_{k=0}^{3} W_k dw_k$$

over G_1. Let $P := (w_0, \ldots, w_3) \in G_1$ be arbitrary and $P_0 := (w_0, w_1, w_2, 0) \in G_1$. Then the line integral from P_0 to P

$$\hat{V}_1(w_0, \ldots, w_3) := \int_0^{w_3} W_3(w_0, \ldots, w_3)\, dw_3$$

is an antiderivative of Ω in G_1. From (28) and 3.5, we deduce that

$$\begin{aligned}\hat{V}_1 &= \frac{1}{4}\int_0^{w_3} \log \frac{(\cos^2 w_3 - \sin^2 \bar{w}_1)\cos^2 w_3}{(\cos^2 w_3 - \sin^2 \bar{w}_0)(\cos^2 w_3 - \sin^2 \bar{w}_2)}\, dw_3 \\ &= \frac{1}{4}\int_0^{w_3} \log \frac{\cos(w_3+\bar{w}_1)\cos(w_3-\bar{w}_1)\cos^2 w_3}{\cos(w_3+\bar{w}_0)\cos(w_3-\bar{w}_0)\cos(w_3+\bar{w}_2)\cos(w_3-\bar{w}_2)}\, dw_3 \\ &= \sum_{k=0}^{2} \frac{(-1)^k}{4}\left\{L\left(\frac{\pi}{2}+w_3+\bar{w}_k\right)+L\left(\frac{\pi}{2}+w_3-\bar{w}_k\right)\right\}+\tfrac{1}{2}L\left(\frac{\pi}{2}-w_3\right). \qquad (29)\end{aligned}$$

Restricting to $w_3=\theta(w_0,w_1,w_2)$, this formula represents the volume $\mathrm{Vol}_3(R)$, since: (i) For $w_3=\theta(w_0,w_1,w_2)$, Leibniz's Rule yields together with (I), (II) and Lemma 1, 3.3:

$$\begin{aligned}\frac{\partial \hat{V}_1}{\partial w_k} &= \frac{\partial}{\partial w_k}\int_0^{\theta} W_3(w_0,\dots,w_3)\, dw_3 \\ &= W_3(w_0,w_1,w_2,\theta)\cdot\frac{\partial\theta}{\partial w_k}+\int_0^{\theta}\frac{\partial W_3}{\partial w_k}\, dw_3 \\ &= \int_0^{\theta}\frac{\partial W_k}{\partial w_3}\, dw_3 = W_k(w_0,w_1,w_2,\theta)-W_k(w_0,w_1,w_2,0) \\ &= W_k(w_0,w_1,w_2,\theta) = -\tfrac{1}{2}V_k(w_0,w_1,w_2)=\frac{\partial \mathrm{Vol}_3(R)}{\partial w_k}, \quad k=0,1,2.\end{aligned}$$

(ii) For $w_3=\theta=0$, both $\hat{V}_1$ and $\mathrm{Vol}_3(R)$ vanish according to (29) and Lemma 2, 3.4. Hence the volume of an ordinary orthoscheme R is given by the formula (see (2) in the Introduction)

$$\begin{aligned}\mathrm{Vol}_3(R) = \frac{1}{4}\Big\{&L(w_0+\theta)-L(w_0-\theta)+L\left(\frac{\pi}{2}+w_1-\theta\right)+L\left(\frac{\pi}{2}-w_1-\theta\right) \\ &+L(w_2+\theta)-L(w_2-\theta)+2L\left(\frac{\pi}{2}-\theta\right)\Big\}, \qquad (30)\end{aligned}$$

where

$$0 \leqq \theta = \arctan\frac{\sqrt{\cos^2 w_1 - \sin^2 w_0 \sin^2 w_2}}{\cos w_0 \cos w_2} < \frac{\pi}{2}.$$

A2. A simple frustum $\tilde{R}$ with ideal vertex P_0 is realizable for essential angles w_0, w_1, w_2 with (see 3.1, (10))

$$0 \leqq w_2 < \frac{\pi}{2} - w_1 < \theta < w_0 < \frac{\pi}{2}.$$

Hence we consider the convex region

$$G_2 := \left\{(w_0,\dots,w_3)\in\bar{G} \,\middle|\, 0 \leqq w_2 < \frac{\pi}{2} - w_1 < w_3 < w_0 < \frac{\pi}{2}\right\}$$

in $\bar{G}$ and integrate Ω from $P_0 := (w_0, \frac{\pi}{2}, 0, 0) \in \partial G_2$ to $P := (w_0,\dots,w_3)\in G_2$ along the path P_0P_1, P_1P_2, P_2P, where

$$P_1 := (w_0, \frac{\pi}{2}, 0, w_3), \quad P_2 := (w_0, w_1, 0, w_3).$$

Then according to (25)–(28), the function

$$\hat{V}_2 := \int_{\pi/2}^{w_1} W_1(w_0, w_1, 0, w_3)\, dw_1 + \int_0^{w_2} W_2(w_0, w_1, w_2, w_3)\, dw_2$$
$$+ \int_0^{w_3} W_3(w_0, \frac{\pi}{2}, 0, w_3)\, dw_3$$

is well-defined and an antiderivative of Ω in G_2. We deduce that

$$\hat{V}_2 = -\frac{1}{4}\int_{\pi/2}^{w_1} \log\left|\frac{\cos(w_3 - \bar{w}_1)}{\cos(w_3 + \bar{w}_1)}\right| dw_1 - \frac{1}{4}\int_0^{w_2} \log\left|\frac{\cos(w_3 - \bar{w}_2)}{\cos(w_3 + \bar{w}_2)}\right| dw_2$$
$$+ \frac{1}{4}\int_0^{w_3} \log\frac{\cos^2 w_3}{\cos^2 w_3 - \sin^2 \bar{w}_0}\, dw_3$$
$$= \sum_{k=0}^{2} \frac{(-1)^k}{4}\left\{L\left(\frac{\pi}{2} + w_3 + \bar{w}_k\right) + L\left(\frac{\pi}{2} + w_3 - \bar{w}_k\right)\right\} + \tfrac{1}{2}L\left(\frac{\pi}{2} - w_3\right). \tag{31}$$

For $w_3 = \theta(w_0, w_1, w_2)$, we have again $\hat{V}_2 = \mathrm{Vol}_3(\tilde{R})$, since:
(i) According to (29) and (31) $\hat{V}_1 = \hat{V}_2$. Hence, it follows that

$$\frac{\partial \hat{V}_2}{\partial w_k} = -\tfrac{1}{2} V_k(w_0, w_1, w_2) = \frac{\partial\, \mathrm{Vol}_3(\tilde{R})}{\partial w_k} \quad \text{for} \quad k = 0, 1, 2.$$

(ii) It follows from (31) that $\hat{V}_2(w_0, \frac{\pi}{2}, 0, 0) = 0$. On the other hand, Lemma 2, 3.4, implies that $\mathrm{Vol}_3(\tilde{R}) = 0$ for $\theta(w_0, \frac{\pi}{2}, 0) = 0$.

Hence, for $w_3 = \theta$ (31) yields a volume formula for $\tilde{R}$.

A3. For a double frustum $\tilde{R}$, we have the angle conditions (see 3.1, (11)):

$$0 \leqq w_0, w_2 < \theta < \frac{\pi}{2} - w_1 \leqq \frac{\pi}{2}.$$

Therefore we consider in $\bar{G}$ the convex region

$$G_3 := \left\{(w_0, \ldots, w_3) \in \bar{G} \,\middle|\, 0 \leqq w_0, w_2 < w_3 < \frac{\pi}{2} - w_1 \leqq \frac{\pi}{2}\right\}.$$

Let $(w_0, \ldots, w_3) \in G_3$. Then (see A2)

$$\hat{V}_3 := \int_0^{w_0} W_0(w_0, w_1, 0, w_3)\, dw_0 + \int_{\pi/2}^{w_1} W_1(0, w_1, 0, 0)\, dw_1$$
$$+ \int_0^{w_2} W_2(w_0, w_1, w_2, w_3)\, dw_2 + \int_0^{w_3} W_3(0, w_1, 0, w_3)\, dw_3$$

is a (well-defined) antiderivative of Ω in G_3. It follows (see also A1, A2) that

$$\hat{V} = -\frac{1}{4}\int_0^{w_0} \log\left|\frac{\cos(w_3 - \bar{w}_0)}{\cos(w_3 + \bar{w}_0)}\right| dw_0 - \frac{1}{4}\int_0^{w_2} \log\left|\frac{\cos(w_3 - \bar{w}_2)}{\cos(w_3 + \bar{w}_2)}\right| dw_2$$
$$+ \frac{1}{4}\int_0^{w_3} \log\frac{(\cos^2 w_3 - \sin^2 \bar{w}_1)\cos^2 w_3}{(\cos^2 w_3 - 1)(\cos^2 w_3 - 1)} dw_3$$
$$= \sum_{k=0}^{2} \frac{(-1)^k}{4}\left\{L\left(\frac{\pi}{2} + w_3 + \bar{w}_k\right) + L\left(\frac{\pi}{2} + w_3 - \bar{w}_k\right)\right\} + \tfrac{1}{2}L\left(\frac{\pi}{2} - w_3\right). \qquad (32)$$

Hence we have $\hat{V}_3 = \hat{V}_2 = \hat{V}_1$, and for $w_3 = \theta(w_0, w_1, w_2)$:

(i) According to A1: $\dfrac{\partial \hat{V}_3}{\partial w_k} = \dfrac{\partial \operatorname{Vol}_3(\tilde{R})}{\partial w_k}$ for $k = 0, 1, 2$.

(ii) From 32 we derive that $V_3(0, \frac{\pi}{2}, 0, 0) = 0$. But Lemma 2, 3.4, implies that $\operatorname{Vol}_3(\tilde{R}) = 0$ for $\theta(0, \frac{\pi}{2}, 0) = 0$.

Finally we proved the following

Theorem II. *Let $\tilde{R}$ be a complete orthoscheme of type A with essential angles w_k, $0 \leqq w_k \leqq \frac{\pi}{2}$, for $k = 0, 1, 2$. Then the volume* $\operatorname{Vol}_3(\tilde{R})$ *of $\tilde{R}$ is given by*

$$\operatorname{Vol}_3(\tilde{R}) = \frac{1}{4}\left\{L(w_0 + \theta) - L(w_0 - \theta) + L\left(\frac{\pi}{2} + w_1 - \theta\right) + L\left(\frac{\pi}{2} - w_1 - \theta\right)\right.$$
$$\left. + L(w_2 + \theta) - L(w_2 - \theta) + 2L\left(\frac{\pi}{2} - \theta\right)\right\}, \qquad (33)$$

where

$$0 \leqq \theta = \arctan\frac{\sqrt{\cos^2 w_1 - \sin^2 w_0 \sin^2 w_2}}{\cos w_0 \cos w_2} < \frac{\pi}{2}.$$

Combining Theorem II with the results of 3.2, A, we obtain the following

Corollary. (1) *If $\tilde{R}$ is a simply asymptotic complete orthoscheme of type A, then*

$$\operatorname{Vol}_3(\tilde{R}) = \tfrac{1}{4}\{L(w_0 + w_2) - L(w_0 - w_2) + 2L(w_1)\}. \qquad (34)$$

(2) *If $\tilde{R}$ is a doubly asymptotic complete orthoscheme of type A, then*

$$\operatorname{Vol}_3(\tilde{R}) = \tfrac{1}{2}L\left(\frac{\pi}{2} - \bar{w}_k\right), \quad \textit{where} \quad \bar{w}_k := \begin{cases} w_k, & k = 1 \\ \frac{\pi}{2} - w_k, & k = 0, 2 \end{cases}. \qquad (35)$$

Using Theorem II we can explicitly calculate volumes of complete Coxeter orthoschemes of type A (see 1.4). Results are listed in the Appendix.

Remark. It can be shown that every complete orthoscheme of dimension three admits a dissection into exactly three ordinary orthoschemes. In this way, we obtain volume formulas for complete orthoschemes using the classical volume formula derived by Lobachevsky. In the simplest case of an asymptotic simple frustum $\tilde{R}$ with ideal vertex P_0 this method leads to the following volume identity (see [10], p. 37ff):

$$\begin{aligned}\mathrm{Vol}_3(\tilde{R}) = \frac{1}{2}\left\{L(w_1+\psi) - L\left(\frac{\pi}{2}+w_1\right) - L\left(\frac{\pi}{2}+\psi\right)\right\} \\ + \frac{1}{4}\left\{L(\psi+\phi) + L(\psi-\phi) + L\left(\frac{\pi}{2}-\phi+w_1\right) + L\left(\frac{\pi}{2}+\phi+w_1\right)\right. \\ \left. + L\left(\frac{\pi}{2}+w_2-w_1-\psi\right) - L\left(\frac{\pi}{2}+w_2+w_1+\psi\right)\right\},\end{aligned} \tag{36}$$

where

$$0 \leqq \phi = \arccos\left(\frac{\sin w_1}{\cos w_2}\right), \quad \psi = \arctan\left(\frac{\cos^2 w_2 - \sin^2 w_1}{\sin w_1 \cos w_1}\right) \leqq \frac{\pi}{2}.$$

But the above Corollary, (34), yields:

$$\mathrm{Vol}_3(\tilde{R}) = \tfrac{1}{2}L(w_1) + \frac{1}{4}\left\{L\left(\frac{\pi}{2}-w_1+w_2\right) - L\left(\frac{\pi}{2}+w_1+w_2\right)\right\}!$$

This procedure can be applied to all three-dimensional complete orthoschemes. From the volume-theoretical point of view, the resulting formulas are not interesting. However, we obtain in this way functional equations for the Lobachevsky function $L(\omega)$. Despite the apparent new form (36) of (34), one can show in this case that (36) can be reduced to (34) by means of known formulas for the Lobachevsky function (see 3.5 and [12, (4.68), (4.69)]).* Although $L(\omega)$ is a very interesting function in itself, we do not pursue this direction of research.

3.7 *The volume in case B*

Let $\tilde{R}$ be a complete orthoscheme of type B, i.e., $\tilde{R}$ is a Lambert cube (see 1.3). We denote by w_k, $0 \leqq w_k \leqq \frac{\pi}{2}$, and $V_k = 0, 1, 2$, the measures of the essential angles and corresponding apices of $\tilde{R}$ according to 3.3, B. Then, Theorem I in 2 and Lemma 1 in 3.3 yield for the differential of $\mathrm{Vol}_3(\tilde{R})$:

$$d\,\mathrm{Vol}_3(\tilde{R}) = -\frac{1}{2}\sum_{k=0}^{2} V_k\, dw_k, \quad \text{where} \quad V_k = \tfrac{1}{2}\log\left|\frac{\cos(\theta-\tilde{w}_k)}{\cos(\theta+\tilde{w}_k)}\right|, \quad k = 0, 1, 2.$$

* The author is grateful to the referee for this remark

The parameter θ is given by (see Definition B in 3.1)

$$\tan^2\theta=\frac{\cosh^2 V_1-\sin^2 w_0\sin^2 w_2}{\cos^2 w_0\cos^2 w_2}$$

with the properties for $k=0,1,2$ (see 2, 3.1 and 3.3):

1. $0<\theta\leqq\frac{\pi}{2}, \quad \theta=\theta(w_0,w_1,w_2), \quad \theta\geqq w_k,$

2. $\tanh V_k=\tan(\frac{\pi}{2}-\theta)\cdot\tan w_k,$

3. $\dfrac{\partial V_k}{\partial\theta}=\dfrac{\sin w_k\cos w_k}{\cos^2\theta-\cos^2 w_k}.$

Interpreting $w_0,w_1,w_2,w_3:=\theta$ as four independent variables, we introduce the functions (see 3.6):

$$\hat{V}_k(w_0,\ldots,w_3):=\tfrac{1}{2}\log\left|\frac{\cos(w_3-\tilde{w}_k)}{\cos(w_3+\tilde{w}_k)}\right|, \quad k=0,1,2.$$

Consider the differential form

$$\Omega:=\sum_{k=0}^{3} W_k\,dw_k$$

over the region

$$G:=\left\{(w_0,\ldots,w_3)\in\mathbf{R}^4\,\middle|\,0\leqq w_k<w_3<\frac{\pi}{2}, \quad k=0,1,2\right\}$$

with (see 3.6)

$$W_k(w_0,\ldots,w_3):=-\tfrac{1}{2}\hat{V}_k(w_0,w_1,w_2,w_3)\in C^1(G), \quad k=0,1,2.$$

We determine $W_3\in C^1(G)$ such that (see 3.6):

(I) Ω satisfies the integrability conditions in G.
(II) $W_3|_{w_3=\theta(w_0,w_1,w_2)}=0.$

Since θ and the partial derivatives of W_j with respect to w_k (see 3.) have a different form than the corresponding ones in 3.6, the integration of the integrability conditions yields a coefficient W_3 which is different from (28). In fact, we get:

$$W_3=\tfrac{1}{4}\log\frac{\sin^2 w_3\cos^4 w_3}{(\cos^2 w_0-\cos^2 w_3)(\cos^2 w_1-\cos^2 w_3)(\cos^2 w_2-\cos^2 w_3)}. \tag{37}$$

It is obvious that $W_3\in C^1(G)$, and that (I) is satisfied. Furthermore, with 3.1 (13), one easily checkes (II). Hence Ω is exact over G. For the integration over G we choose a point $P:=(w_0,\ldots,w_3)\in G$ with $w_k>0$, $k=0,1,2$. Then we integrate Ω from $Q:=(w_0,w_1,w_2,\frac{\pi}{2})\in\partial G$ to P. Since

$$W_k(w_0,w_1,w_2,\frac{\pi}{2})=0, \quad k=0,1,2,$$

we obtain the following antiderivative of Ω in G (see (37) and 3.5):

$$\begin{aligned}\hat{V} &:= \int_{\pi/2}^{w_3} W_3(w_0, w_1, w_2, w_3)\, dw_3 \\ &= \frac{1}{4}\int_{\pi/2}^{w_3} \log \frac{\sin^2 w_3 \cos^4 w_3}{(\cos^2 w_0 - \cos^2 w_3)(\cos^2 w_1 - \cos^2 w_3)(\cos^2 w_2 - \cos^2 w_3)}\, dw_3 \\ &= \frac{1}{4}\sum_{k=0}^{2} \left\{ L(w_k + w_3) - L(w_k - w_3) \right\} + L\left(\frac{\pi}{2} - w_3\right) - \tfrac{1}{2} L(w_3). \qquad (38)\end{aligned}$$

Restricting to the hypersurface $w_3 = \theta(w_0, w_1, w_2)$ in $\mathbf{R}^4$, we can identify $\hat{V}$ with the volume $\mathrm{Vol}_3(\tilde{R})$ of the Lambert cube $\tilde{R}$, since (see 3.6):

(i) For $w_3 = \theta$:

$$\frac{\partial \hat{V}}{\partial w_k} = W_k(w_0, w_1, w_2, \theta) = -\tfrac{1}{2} V_k(w_0, w_1, w_2) = \frac{\partial \mathrm{Vol}_3(\tilde{R})}{\partial w_k}, \quad k = 0, 1, 2.$$

(ii) For $w_3 = \frac{\pi}{2}$ we have $\hat{V} = 0$. On the other hand, Lemma 2 in 3.4 shows that $\theta = \frac{\pi}{2}$ implies $\mathrm{Vol}_3(\tilde{R}) = 0$.

Using 3.5, (b), we derive the following

Theorem III. *Let $\tilde{R}$ be a Lambert cube with essential angles w_k, $0 \leq w_k \leq \frac{\pi}{2}$, $k = 0, 1, 2$. Then the volume $\mathrm{Vol}_3(\tilde{R})$ of $\tilde{R}$ is given by*

$$\begin{aligned}\mathrm{Vol}_3(\tilde{R}) = \frac{1}{4}\Big\{ & L(w_0 + \theta) - L(w_0 - \theta) + L(w_1 + \theta) - L(w_1 - \theta) \\ & + L(w_2 + \theta) - L(w_2 - \theta) - L(2\theta) + 2L\left(\frac{\pi}{2} - \theta\right) \Big\} \qquad (39)\end{aligned}$$

with

$$0 < \theta = \arctan \frac{\sqrt{\cosh^2 V_1 - \sin^2 w_0 \sin^2 w_2}}{\cos w_0 \cos w_2} \leq \frac{\pi}{2}.$$

Remarks. (a) By means of hyperbolic trigonometry, the quantity $\cosh^2 V_1$ in the definition of θ can be expressed as a function of the essential angles w_0, w_1, w_2 as follows:

$$\cosh^2 V_1 = 1 + \tfrac{1}{2}\left(\sqrt{A^2 + (2B \sin w_1)^2} - A\right)$$

with

$$A = \cos^2 w_0 + \cos^2 w_2 - B^2, \quad B = \frac{\cos w_0 \cos w_2}{\cos w_1}. \qquad (40)$$

(b) In the limiting case $w_1 = V_1 = 0$ (see 3.2, B), the formulas (33) and (39) for $\mathrm{Vol}_3(\tilde{R})$ of Theorems II and III coincide. Apart from this special case, these two formulas are conceptually different, i.e., they cannot be related to each other by means of suitable functional equations for $L(\omega)$. This can be proved by evaluating both abstract formulas for the values $w_0 = w_1 = w_2 = \frac{\pi}{4}$ using (40).

By means of Theorem III we can explicitly calculate volumes of complete Coxeter orthoschemes of type B (see 1.4). Results are listed in the Appendix.

Appendix

A. Let $\tilde{R}_C$ be a three-dimensional complete Coxeter orthoscheme of type A with essential angles $\frac{\pi}{p_k}$, $p_k > 2$, $k = 0, 1, 2$.

A1. There are exactly 10 realizations of ordinary Coxeter orthoschemes $\tilde{R}_C$ (see [7]) with graphs $\Sigma(\tilde{R}_C)$ and volumes $V(p_0, p_1, p_2)$:

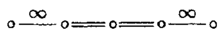
$\Sigma(\widetilde{R_C})$	$V(p_0, p_1, p_2)$
$\circ\!-\!\circ\!-\!\circ\overset{6}{-}\circ$	$V(3,3,6) = \frac{1}{8}\text{Л}(\frac{\pi}{3}) \simeq 0.0423$
$\circ\!-\!\circ\!=\!\circ\!=\!\circ$	$V(3,4,4) = \frac{1}{6}\text{Л}(\frac{\pi}{4}) \simeq 0.0763$
$\circ\!-\!\circ\overset{5}{-}\circ\!-\!\circ$	$V(3,5,3) \simeq 0.0391$
$\circ\!-\!\circ\overset{6}{-}\circ\!-\!\circ$	$V(3,6,3) = \frac{1}{2}\text{Л}(\frac{\pi}{3}) \simeq 0.1692$
$\circ\!=\!\circ\!-\!\circ\overset{5}{-}\circ$	$V(4,3,5) \simeq 0.0359$
$\circ\!=\!\circ\!-\!\circ\overset{6}{-}\circ$	$V(4,3,6) = \frac{5}{24}\text{Л}(\frac{\pi}{6}) \simeq 0.1057$
$\circ\!=\!\circ\!=\!\circ\!=\!\circ$	$V(4,4,4) = \frac{1}{2}\text{Л}(\frac{\pi}{4}) \simeq 0.2290$
$\circ\overset{5}{-}\circ\!-\!\circ\overset{5}{-}\circ$	$V(5,3,5) \simeq 0.0933$
$\circ\overset{5}{-}\circ\!-\!\circ\overset{6}{-}\circ$	$V(5,3,6) \simeq 0.1715$
$\circ\overset{6}{-}\circ\!-\!\circ\overset{6}{-}\circ$	$V(6,3,6) = \frac{1}{2}\text{Л}(\frac{\pi}{6}) \simeq 0.2537$

A2. The simple Coxeter frustums $\tilde{R}_C$ form an infinite class of polyhedra (see [9]):

$\circ\cdots\circ\overset{p_0}{-}\circ\overset{p_1}{-}\circ\overset{p_2}{-}\circ$ $\qquad \frac{1}{p_0} + \frac{1}{p_1} \geq \frac{1}{2}\,, \quad \frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{2}$

$\circ\overset{\infty}{-}\circ\overset{p_0}{-}\circ\overset{p_1}{-}\circ\overset{\infty}{-}\circ$ $\qquad \frac{1}{p_0} + \frac{1}{p_1} \geq \frac{1}{2}$

The volume is maximal for

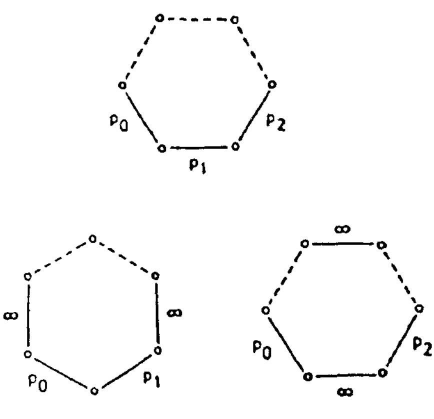


$$V(4,4,0) = Л\left(\frac{\pi}{4}\right) \simeq 0.4560$$

The volume is minimal for the asymptotic Coxeter frustum

$$V(3,3,6) = \frac{1}{8} Л\left(\frac{\pi}{3}\right) \simeq 0.0423 \quad .$$

A3. The double Coxeter frustums $\tilde{R}_C$ form an infinite class of polyhedra [9]:



$$\frac{1}{p_0} + \frac{1}{p_1} < \frac{1}{2}, \quad \frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{2}.$$

$$\frac{1}{p_0} + \frac{1}{p_k} < \frac{1}{2}, \quad k = 1 \quad \text{or} \quad 2.$$

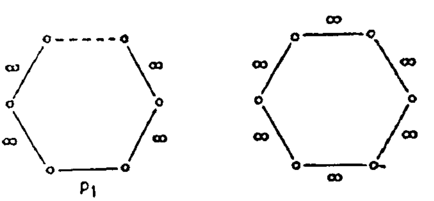
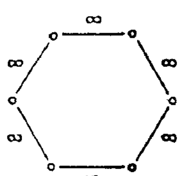


Fig. 4

The maximal volume is attained in the asymptotic limit case

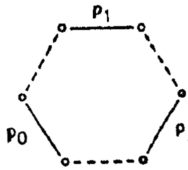


$$V(0,0,0) = 2L\left(\frac{\pi}{4}\right) \simeq 0.9160.$$

The minimal volume is attained in the asymptotic limit case

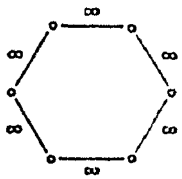
$$V(3,6,3) = \tfrac{1}{2} L\left(\frac{\pi}{3}\right) \simeq 0.1692.$$

B. Let $\tilde{R}_C$ be a three-dimensional complete Coxeter orthoscheme of type B with essential angles $\dfrac{\pi}{p_k}$, $p_k > 2$, $k = 0, 1, 2$. These Coxeter polyhedra form an infinite class (see [9]):



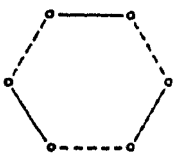
$$2 < p_0, p_1, p_2 < \infty.$$

We get maximal volume for the asymptotic Coxeter polyhedron



$$V(0,0,0) = 2L\left(\frac{\pi}{4}\right) \simeq 0.9160.$$

We get minimal volume for the compact Coxeter polyhedron



$$V(3,3,3) \simeq 0.3244.$$

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References

1. Andreev, E.M.: On convex polyhedra in Lobačevskii spaces. Math. USSR Sbornik **10** Vol. 3, 413–440 (1970)
2. Böhm, J.: Inhaltsmessung im $\mathbb{R}^5$ konstanter Krümmung. Arch. Math. **11**, 298–309 (1960)
3. Böhm, J.: Zu Coxeters Integrationsmethode in gekrümmten Räumen. Math. Nachr. **27**, 179–214 (1964)
4. Böhm, J., Hertel, E.: Polyedergeometrie in n-dimensionalen Räumen konstanter Krümmung. Basel: Birkhäuser 1981
5. Cartan, H.: Differentialformen. Zürich: Bibliographisches Institut 1967
6. Coxeter, H.S.M.: The Functions of Schläfli and Lobatschefsky. In: Twelve geometric essays. Carbondale: Southern Illinois Univ. Press 1968
7. Coxeter, H.S.M.: Regular honeycombs in elliptic space. Proc. Lond. Math. Soc. (3) **4**, 471–501 (1954)
8. Giering, O.: Vorlesungen über höhere Geometrie. Braunschweig Wiesbaden: Vieweg 1982
9. Im Hof, H.-C.: A Class of Hyperbolic Coxeter Groups. Expo. Math. **3**, 179–186 (1985)
10. Kellerhals, R.: Ueber den Inhalt hyperbolischer Polyeder in den Dimensionen drei und vier. Dissertation, Basel 1988
11. Kneser, H.: Der Simplexinhalt in der nichteuklidischen Geometrie. Deutsche Math. **1**, 337–340 (1936)
12. Lewin, L.: Dilogarithms and associated functions. New York Oxford: North Holland 1981
13. Lobatschefskij, N.I.: Imaginäre Geometrie und ihre Anwendung auf einige Integrale. Deutsche Übersetzung von H. Liebmann. Leipzig: Teubner 1904
14. Maier, W.: Inhaltsmessung im R_3 fester Krümmung. Arch. Math. **5**, 266–273 (1954)

15. Roeser, E.: Die nicht-euklidischen Geometrien und ihre Beziehungen untereinander. München: Oldenburg 1957
16. Schläfli, L.: Theorie der vielfachen Kontinuität. In: Gesammelte mathematische Abhandlungen **1**. Basel: Birkhäuser 1950
17. Vinberg, È.B.: Hyperbolic reflection groups. Russ. Math. Surv. **40**, 31–75 (1985)

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