A formula for the volume of a hyperbolic tetrahedon

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Introduction. Calculating the volume of a polyhedron in three-dimensional space is a very old and difficult problem. The first known result in this direction is due to Tartaglia (1494), who found the volume of a Euclidean tetrahedron, a result now known as the Cayley–Menger formula. It has recently been shown [1], [2] that the volume of any Euclidean polyhedron is a root of an algebraic equation whose coefficients depend on the lengths of the edges and are completely determined by the combinatorial type of the polyhedron.

In hyperbolic and spherical spaces the situation is more complicated. Formulae for the volume of a biorthogonal tetrahedron in these spaces have been known since the time of Lobachevskii and Schläfli [3]. The volume of a regular tetrahedron in hyperbolic space was found in [4], and the case of a hyperbolic tetrahedron with some ideal vertices was studied in [3]. A formula for the volume of an arbitrary tetrahedron remained unknown for a long time. A general algorithm for finding such a formula was outlined in [5]. A complete solution of the problem was obtained fairly recently by several authors [6]–[8]. All the results are expressed in terms of a combination of 16 dilogarithmic or Lobachevskii functions depending on the dihedral angles of the tetrahedron and certain auxiliary parameters that are roots of a rather complicated quadratic equation with complex coefficients. A geometric interpretation of the Murakami–Yano formula [7] is explained in [9] from the point of view of the so-called Regge symmetry. An excellent account of these ideas and a full geometric proof of the formula can be found in [10].

We remark that the formula for the volume is much simpler in the case of a symmetric tetrahedron, that is, one with opposite dihedral angles equal. This was first discovered by Milnor [11] for an ideal tetrahedron. It was later shown [12] that a fairly simple formula for the volume also exists for an arbitrary symmetric tetrahedron with proper vertices.

In this paper we present an elementary integral formula for the volume of a hyperbolic tetrahedron, expressed in terms of simple geometric quantities depending on the dihedral angles. The formula is suitable for implementation on a computer, and the results of [6]–[8] are obtained as a corollary.

The volume of a hyperbolic tetrahedron. Let $T(A,B,C,D,E,F)$ be a compact hyperbolic tetrahedron with dihedral angles A, B, C, D, E, F chosen in such a way that A, B, C lie at one vertex and D, E, F are opposite to A, B, C, respectively.

Theorem 1. *The volume of the hyperbolic tetrahedron $T = T(A,B,C,D,E,F)$ is equal to*

$$\mathrm{Vol}(T) = -\frac{1}{4}\int_{z_1}^{z_2} \log \frac{\cos\frac{A+B+C+z}{2}\cos\frac{A+E+F+z}{2}\cos\frac{B+D+F+z}{2}\cos\frac{C+D+E+z}{2}}{\sin\frac{A+B+D+E+z}{2}\sin\frac{A+C+D+F+z}{2}\sin\frac{B+C+E+F+z}{2}\sin\frac{z}{2}}\,dz,$$

where z_1 and z_2 are the roots of the integrand, given by

$$z_1 = \tan^{-1}\frac{k_2}{k_1} - \tan^{-1}\frac{k_4}{k_3}, \qquad z_2 = \tan^{-1}\frac{k_2}{k_1} + \tan^{-1}\frac{k_4}{k_3},$$

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with

$$\begin{aligned}
k_1 &= -(\cos S + \cos(A+D) + \cos(B+E) + \cos(C+F) + \cos(D+E+F) \\
&\qquad + \cos(D+B+C) + \cos(A+E+C) + \cos(A+B+F)), \\
k_2 &= \sin S + \sin(A+D) + \sin(B+E) + \sin(C+F) + \sin(D+E+F) + \sin(D+B+C) \\
&\qquad + \sin(A+E+C) + \sin(A+B+F), \\
k_3 &= 2\,(\sin A \sin D + \sin B \sin E + \sin C \sin F), \\
k_4 &= \sqrt{k_1^2 + k_2^2 - k_3^2},
\end{aligned}$$

and $S = A + B + C + D + E + F$.

Remark 1. The sums $V_1 = A + B + C$, $V_2 = A + E + F$, $V_3 = B + D + F$, $V_4 = C + D + E$ in the numerator and $H_1 = A + B + D + E$, $H_2 = A + C + D + F$, $H_3 = B + C + E + F$ in the denominator have a simple geometric interpretation: the former are the sums of the dihedral angles at the vertices of T, while the latter are the angle-sums along the Hamiltonian cycles of T.

Remark 2. The limits z_1 and z_2 of integration satisfy the equation $k_1 \cos z + k_2 \sin z = k_3$, and $k_4^2 = -4\det(G)$, where G is the Gram matrix of T. A geometric interpretation of the quantities z_1 and z_2 can be extracted from the results of [10]. They occur as parameters of the partition of an ideal octahedron into four ideal tetrahedra with a common edge. The octahedron is canonically determined by T and its dihedral angles are linear combinations of those of T.

Consider the dilogarithmic function

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt,$$

where $x \in \mathbb{C} \setminus [1,\infty)$, and the continuous branch $\log\xi = \log|\xi| + i\arg\xi$ is determined by the conditions $-\pi < \arg\xi < \pi$. Put $l(z) = \mathrm{Li}_2(e^{iz})$.

An immediate consequence of Theorem 1 is the following result, obtained earlier in [7], [8].

Corollary 1. *The hyperbolic volume of the tetrahedron T is equal to* $\mathrm{Im}(U(z_1,T) - U(z_2,T))/2$, *where*

$$\begin{aligned}
U(z,T) = (&l(z) + l(A+B+D+E+z) + l(A+C+D+F+z) + l(B+C+E+F+z) \\
&- l(\pi + A + B + C + z) - l(\pi + A + E + F + z) \\
&- l(\pi + B + D + F + z) - l(\pi + C + D + E + z))/2.
\end{aligned}$$

For the simplification of the Murakami–Yano formula, we note that $\mathrm{Im}(l(z)) = \mathrm{Im}(\mathrm{Li}_2(e^{iz})) = 2\Lambda(z/2)$, where $\Lambda(z)$ is the Lobachevskii function given by

$$\Lambda(z) = -\int_0^z \log|2\sin t|\,dt.$$

The volume of the tetrahedron can thus be written as a linear combination of 16 Lobachevskii functions.

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