ON SPONTANEOUS SURGERY ON KNOTS AND LINKS*

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Abstract Geometrical properties of cone-manifolds obtained by orbifold and spontaneous Dehn surgeries of knots and links are investigated. Explicit hyperbolic volume formulae for the Figure-eight knot, Whitehead link, and Borromean rings link cone-manifolds are obtained.

1. Introduction

In 1975 R. Riley ([16]) discovered the existence of complete hyperbolic structure on some knots and links complements in the 3-sphere. Later W. P. Thurston showed that a complement of a simple knot (except torical and spherical) admits a hyperbolic structure. It follows from this result ([17]) that almost all Dehn surgeries on a hyperbolic knot complement produce a hyperbolic manifold.

A number of works in the last 20 years has been devoted to precise descriptions of manifolds, orbifolds, and cone-manifolds obtained in this manner (see for example [2, 3, 4, 6, 11, 18]).

This paper is a review of recent results pertaining to the geometrical properties of cone-manifolds obtained by spontaneous Dehn surgeries on knots and links. This work is part of a talk given by the first named author on the "János Bolyai Conference on Hyperbolic Geometry" held in Budapest on 8-12 July, 2002. Mostly, it contains a survey of results

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obtained by the authors and their collaborators, but some new results are also given.

We remind the reader of some basic definitions:

Definition 1.1. *A 3-dimensional hyperbolic cone-manifold is a Riemannian 3-dimensional manifold of constant negative sectional curvature with cone-type singularity along simple closed geodesics. To each component of a singular set we associate a real number $n \geq 1$ such that the cone-angle around the component is $\alpha = 2\pi/n$. The concept of the hyperbolic cone-manifold generalizes the hyperbolic manifold which appears in the partial case when all cone-angles are 2π. The hyperbolic cone-manifold is also a generalization of the hyperbolic 3-orbifold which arises when all associated numbers n are integers. Euclidean and spherical cone–manifolds are defined similarly.*

We identify the group of orientation preserving isometries of $\mathbb{H}^3$ with the group $PSL(2,\mathbb{C})$ consisting of linear fractional transformations

$$A : z \in \mathbb{C} \to \frac{az+b}{cz+d}.$$

By the canonical procedure the linear transformation A can be uniquely extended to the isometry of $\mathbb{H}^3$. We prefer to deal with the matrix

$$\widetilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$$

rather than the element $A \in PSL(2,\mathbb{C})$. The matrix $\widetilde{A}$ is uniquely determined by the element A up to a sign. If there is no confusion we shall use the same letter A for both A and $\widetilde{A}$.

Let C be a hyperbolic cone–manifold with singular set $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_k$ being a link consisting of components $\Sigma_j = \Sigma_{\alpha_j}$, $j = 1, 2, \ldots, k$ with cone-angles $\alpha_1, \ldots, \alpha_k$ respectively. Then C defines a nonsingular but incomplete hyperbolic manifold $N = C - \Sigma$. Denote by Φ the fundamental group of the manifold N.

The hyperbolic structure of N defines, up to conjugation in $PSL(2,\mathbb{C})$, a holonomy homomorphism

$$\hat{h} : \Phi \to PSL(2,\mathbb{C}).$$

It is shown in [19] that the monodromy homomorphism of an orientable cone-orbifold can be lifted to $SL(2,\mathbb{C})$ if all cone angles are less than π. Denote by $h : \Phi \to SL(2,\mathbb{C})$ this lifting homomorphism. Choose an orientation on the link $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_k$ and fix a meridian-longitude pair $\{m_j, l_j\}$ for each component $\Sigma_j = \Sigma_{\alpha_j}$. Then the matrices

$M_j = h(m_j)$ and $L_j = h(l_j)$ satisfy the following properties:

$$M_j L_j = L_j M_j,\ j = 1, 2, \ldots, k.$$

Definition 1.2. *Cone-manifold C is said to be obtained by orbifold Dehn surgery with cone angle $\alpha_j = \frac{2\pi}{m}$ (or with a slope $\frac{m}{0}$) on the component Σ_j if $tr(M_j) = 2\cos(\frac{\alpha_j}{2})$.*

Definition 1.3. *Cone-manifold C is said to be obtained by spontaneous Dehn surgery with cone angle $\alpha_j = \frac{2\pi}{m}$ (or with a slope $\frac{0}{m}$) on the component Σ_j if $tr(L_j) = 2\cos(\frac{\alpha_j}{2})$.*

See [3] for details.

Definition 1.4. *A complex length γ_j of the singular component Σ_j of the cone-manifold C is defined as displacement of the isometry L_j of $\mathbb{H}^3$, where $L_j = h(l_j)$ is represented by the longitude l_j of Σ_j.*

Immediately from the definition we get [1, p.46]

$$2\cosh\gamma_j = \mathrm{tr}(L_j^2).$$

We note that the meridian-longitude pair $[m_j, l_j]$ is uniquely determined up to a common conjugating element of the group Φ. Hence the complex length $\gamma_j = l_j + i\phi_j$ is uniquely determined up to sign and $(mod\, 2\pi i)$ by the above definition. Since $\mathrm{tr}({L_j}^2) = \mathrm{tr}^2(L_j) - 2$ we have also $\mathrm{tr}^2(L_j) = 4\cosh^2(\frac{\gamma_j}{2})$

The main tool for volume calculation is the following Schläfli formula [4].

Theorem 1.5. *Suppose that C_t is a smooth 1-parameter family of (curvature K) cone–manifold structures on an n-manifold, with singular locus Σ of a fixed topological type. Then the derivative of volume of C_t satisfies*

$$(n-1)KdV(C_t) = \sum_{\sigma} V_{n-2}(\sigma)d\theta(\sigma)$$

where the sum is over all components σ of the singular locus Σ and $\theta(\sigma)$ is the cone angle along σ.

In the present paper we will deal with three-dimensional cone-manifolds of negative constant curvature $K = -1$. The Schläfli formula in this case reduces to

$$dV = -\frac{1}{2}\sum_{i} l_{\alpha_i} d\alpha_i,$$

where the sum is taken over all components of the singular set Σ with lengths l_{α_i} and cone angles α_i.

2. Figure-eight knot

2.1 Orbifold surgery on the figure-eight knot

Denote by E the compliment to the figure-eight knot in a 3-sphere (see Figure 1).

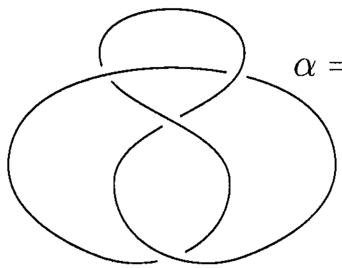

Figure 1. The figure-eight cone-manifold $E(\frac{m}{0})$.

The following theorem was obtained by Mednykh and Rasskazov in [9].

Theorem 2.1. *Let $E(\frac{m}{0})$ be a cone-manifold obtained by orbifold surgery on the figure-eight knot with cone angle $\alpha = \frac{2\pi}{m}$. Then $E(\frac{m}{0})$ is hyperbolic for $0 \leq \alpha < \frac{2\pi}{3}$, Euclidean for $\alpha = \frac{2\pi}{3}$, and spherical for $\frac{2\pi}{3} < \alpha < \frac{4\pi}{3}$. The hyperbolic volume of the cone-manifold is given by the formula:*

$$\mathrm{Vol}(E(\frac{m}{0})) = \int_{\alpha}^{\frac{2\pi}{3}} \operatorname{arcosh}(1 + \cos t - \cos 2t)\, dt\,.$$

2.2 Spontaneous surgery on the figure-eight knot

Recall that the first example of a complete hyperbolic 3-manifold of finite volume was constructed by Gieseking in 1912. This manifold can be obtained by identification of faces of regular ideal tetrahedra by orientation reversing isometries of $\mathbb{H}^3$ (see [12] for details).

Spontaneous Dehn surgery on a Gieseking manifold was considered in [13] by E. Molnár, I. Prok, and J. Szirmai with obvious and easily recoverable mistakes, noticed also by the second named author. In the improvements to the paper [14], sent to the reviewer J. Böhm (see Zbl pre01604732), it was proven that $G(\frac{0}{m})$ is hyperbolic if $0 \leq \alpha < 2\pi$, $\alpha = \frac{2\pi}{m}$ and the fundamental polyhedron was constructed in $\mathbb{H}^3$. Also the hyperbolic volume was obtained as a sum of three Lobachevsky functions. We give a more simple hyperbolic volume formula in the following:

Theorem 2.2. *Let $G(\frac{0}{m})$ be a hyperbolic cone-manifold obtained by spontaneous surgery on the Gieseking manifold with cone angle $\alpha = \frac{2\pi}{m}$. Then the volume of $G(\frac{0}{m})$ is given by the formula:*

$$\mathrm{Vol}(G(\frac{0}{m})) = \frac{1}{2}\int_{\pi/m}^{\pi} \mathrm{arcosh}(\frac{1+\sqrt{17-8\cos x}}{4})\, dx\,.$$

Proof. Denote by $V = \mathrm{Vol}(G(\alpha))$ the hyperbolic volume of $G(\alpha)$. Then by virtue of the Schläfli formula [4] we have

$$\frac{\partial V}{\partial \alpha} = -\frac{l_\alpha}{2} \tag{1}$$

where l_α is the length of a singular geodesic corresponding to cone angle α. Moreover, by [3] we note that

$$V \to 0 \text{ as } \alpha \to 2\pi\,. \tag{2}$$

We set

$$\begin{aligned}\tilde{V} &= \frac{1}{2}\int_{\pi/m}^{\pi} \mathrm{arcosh}(\frac{1+\sqrt{17-8\cos x}}{4})\, dx \\ &= \frac{1}{4}\int_{\alpha}^{2\pi} \mathrm{arcosh}(\frac{1+\sqrt{17-8\cos\frac{x}{2}}}{4})\, dx\end{aligned}$$

and show that $\tilde{V}$ satisfies conditions (1) and (2). Then $\tilde{V} = V$ and the theorem is proven.

To verify (1) we note that l_α can be found from the following equation (see [13] and [14] for a geometric basis):

$$l_\alpha = \log|z-1|\,,$$

where $(z-1)$ is derived from the equation ([14])

$$\frac{z}{(z-1)^2} = -e^{\frac{i\alpha}{2}}\,.$$

Hence l_α is represented by the expression

$$l_\alpha = \log|e^{\mathrm{arcosh}((-\frac{1}{2})e^{\frac{-i\alpha}{4}})}|\,.$$

Keeping in mind that $|e^{\zeta}| = e^{\Re(\zeta)}$ we get after simplifying

$$l_\alpha = \frac{1}{2}\mathrm{arcosh}(\frac{1+\sqrt{17-8\cos\frac{\alpha}{2}}}{4})\,,$$

hence $\dfrac{\partial \tilde{V}}{\partial \alpha} = -\dfrac{l_\alpha}{2}$.

The boundary condition

$$\tilde{V} = \frac{1}{4}\int_\alpha^{2\pi} \operatorname{arcosh}\Big(\frac{1+\sqrt{17-8\cos\frac{x}{2}}}{4}\Big)\,dx \to 0$$

as $\alpha \to 2\pi$ follows from the convergence of the integral. □

We recall that a double sheeted covering of the Gieseking manifold is a complement to the figure-eight knot (see [12]). Hence $E(\frac{0}{m})$ obtained by spontaneous surgery on the figure-eight knot is a double sheeted covering over the cone-manifold $G(\frac{0}{m})$ obtained by spontaneous surgery on the Gieseking manifold with cone angle $\alpha = \frac{2\pi}{m}$.

Hilden, Lozano, and Montesinos-Amilibia have shown (see [3]) that the cone-manifold $E(\frac{0}{m})$ is hyperbolic for $0 \le \alpha < 2\pi$, $\alpha = \frac{2\pi}{m}$. Some complicated formula for the hyperbolic volume was also obtained. We found a very simple version of this formula as a consequence of Theorem 2.2.

Theorem 2.3. *Let $E(\frac{0}{m})$ be a hyperbolic cone-manifold obtained by spontaneous surgery on the figure-eight knot manifold with cone angle $\alpha = \frac{2\pi}{m}$. Then the volume of $E(\frac{0}{m})$ is given by the formula:*

$$\operatorname{Vol}(E(\frac{0}{m})) = \int_{\pi/m}^{\pi} \operatorname{arcosh}\Big(\frac{1+\sqrt{17-8\cos x}}{4}\Big)\,dx.$$

3. Whitehead link cone-manifold

3.1 Orbifold surgery on the Whitehead link

We denote by W the Whitehead link shown on Figure 2. Recall ([17]) that $\mathbb{S}^3 \setminus W$ is a hyperbolic manifold. Denote by $h : \Phi = \pi_1(\mathbb{S}^3 \setminus W) \to SL(2,\mathbb{C})$ the lifting of its holonomy homomorphism.

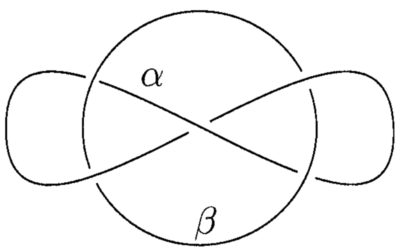

Figure 2. The Whitehead link cone-manifold $W(\alpha,\beta)$.

By slight modification of arguments from [8] we obtain the following two propositions:

Proposition 3.1. *Up to conjugation in* $SL(2,\mathbb{C})$ *the matrices* $M_\alpha = h(m_\alpha)$ *and* $M_\beta = h(m_\beta)$ *can be represented in the following form:*

$$M_\alpha = \begin{pmatrix} \cos\frac{\hat\alpha}{2} & ie^{\frac{\rho}{2}}\sin\frac{\hat\alpha}{2} \\ ie^{-\frac{\rho}{2}}\sin\frac{\hat\alpha}{2} & \cos\frac{\hat\alpha}{2} \end{pmatrix},$$

$$M_\beta = \begin{pmatrix} \cos\frac{\hat\beta}{2} & ie^{\frac{\rho}{2}}\sin\frac{\hat\beta}{2} \\ ie^{-\frac{\rho}{2}}\sin\frac{\hat\beta}{2} & \cos\frac{\hat\beta}{2} \end{pmatrix},$$

where $\hat\alpha$ *and* $\hat\beta$ *satisfy relations* $\mathrm{tr}(M_\alpha) = 2\cos\frac{\hat\alpha}{2}$, $\mathrm{tr}(M_\beta) = 2\cos\frac{\hat\beta}{2}$, *and* ρ *is a complex distance between axes of* M_α *and* M_β. *Moreover,* $u = \cosh(\rho)$ *is a complex root of equation*

$$u^3 - \hat A\hat B u^2 + \frac{\hat A^2\hat B^2 + \hat A^2 + \hat B^2 - 1}{2}u + \hat A\hat B = 0\,,$$

where $\hat A = \cot\frac{\hat\alpha}{2}$ *and* $\hat B = \cot\frac{\hat\beta}{2}$.

Setting $z = \dfrac{\hat A\hat B}{u}$ and multiplying the obtained polynomial equation by $(z+1)$ we have (see also [10])

$$2(z^2 + \hat A^2)(z^2 + \hat B^2) = (1 + \hat A^2)(1 + \hat B^2)(z^2 - z^3)\,. \tag{3}$$

Proposition 3.2. *Let* $W(S,T) = STS^{-1}T^{-1}ST^{-1}S^{-1}T$, M_α *and* M_β *be the same matrices as in Proposition 3.1,* $L_\alpha = W(M_\alpha, M_\beta)$, $L_\beta = W(M_\beta, M_\alpha)$, *and the condition* $M_\alpha L_\alpha = L_\alpha M_\alpha$ *be satisfied. Then*

$$\frac{\coth\frac{i\hat\alpha}{2}}{\coth\frac{\hat l_\alpha}{4}} = \frac{\coth\frac{i\hat\beta}{2}}{\coth\frac{\hat l_\beta}{4}} = z\,,$$

where z *is a root of Equation (3),* $\Im(z) > 0$ *and* $\hat l_\alpha$ *and* $\hat l_\beta$ *can be derived from* $2\cosh(\hat l_\alpha) = \mathrm{tr}({L_\alpha}^2)$ *and* $2\cosh(\hat l_\beta) = \mathrm{tr}({L_\beta}^2)$.

The following result was obtained in [7] and [10].

Theorem 3.3. *Let* $W(\frac{m}{0}, \frac{n}{0})$ *be a hyperbolic cone-manifold obtained by orbifold surgeries on the components of the Whitehead link with cone angles* $\alpha = \frac{2\pi}{m}$ *and* $\beta = \frac{2\pi}{n}$. *Then*

$$\mathrm{Vol}(W(\frac{m}{0}, \frac{n}{0})) = i\int_{\zeta_1}^{\zeta_2} \log\left[\frac{2(\zeta^2 + A^2)(\zeta^2 + B^2)}{(1 + A^2)(1 + B^2)(\zeta^2 - \zeta^3)}\right]\frac{d\zeta}{\zeta^2 - 1}$$

where $A = \cot\frac{\alpha}{2}$, $B = \cot\frac{\beta}{2}$, $\zeta_1 = \overline{z}$, $\zeta_2 = z$, $\Im(z) > 0$ *and* z *is a root of the cubic equation*

$$z^3 + \frac{1}{2}(A^2B^2 + A^2 + B^2 - 1)z^2 - A^2B^2z + A^2B^2 = 0.$$

3.2 Spontaneous surgery on the Whitehead link

Proposition 3.4. *Let* $W(\frac{0}{m}, \frac{0}{n})$ *be a hyperbolic cone-manifold obtained by spontaneous surgery on the components of the Whitehead link with cone angles equal to* $\frac{2\pi}{m}$ *and* $\frac{2\pi}{n}$ *respectively. Denote by* l_α *and* l_β *the complex lengths of singular geodesics of* $W(\frac{0}{m}, \frac{0}{n})$ *with cone angles* $\alpha = \frac{2\pi}{m}$ *and* $\beta = \frac{2\pi}{n}$ *respectively. Then*

$$\frac{\coth\frac{l_\alpha}{2}}{\coth\frac{i\alpha}{4}} = \frac{\coth\frac{l_\beta}{2}}{\coth\frac{i\beta}{4}} = z\,,$$

where $\Im(z) > 0$, *and* z *is a root of the equation*

$$2z^2(1+\hat{A}^2)(1+\hat{B}^2) = (z^2+\hat{A}^2)(z^2+\hat{B}^2)(1-z)\,,$$

$\hat{A} = \tan\frac{\alpha}{4}$, $\hat{B} = \tan\frac{\beta}{4}$.

Proof. The result follows from Proposition 3.2 for $l_\alpha = i\hat{\alpha}$, $\hat{l}_\alpha = i\alpha$, $l_\beta = i\hat{\beta}$, and $\hat{l}_\beta = i\beta$. □

Theorem 3.5. *Let* $W(\frac{0}{m}, \frac{0}{n})$ *be a hyperbolic cone-manifold obtained by spontaneous surgery on the components of the Whitehead link with cone angles* $\alpha = \frac{2\pi}{m}$ *and* $\beta = \frac{2\pi}{n}$. *Then*

$$\mathrm{Vol}(W(\frac{0}{m}, \frac{0}{n})) = i\int_{\zeta_1}^{\zeta_2} \log\left[\frac{1+\hat{A}^2}{\zeta^2+\hat{A}^2}\frac{1+\hat{B}^2}{\zeta^2+\hat{B}^2}\frac{2\zeta^2}{1-\zeta}\right]\frac{d\zeta}{\zeta^2-1}\,,$$

where $\hat{A} = \tan\frac{\alpha}{4}$, $\hat{B} = \tan\frac{\beta}{4}$, $\zeta_1 = \overline{z}$, $\zeta_2 = z$, $\Im(z) > 0$, *and*

$$2z^2(1+\hat{A}^2)(1+\hat{B}^2) = (z^2+\hat{A}^2)(z^2+\hat{B}^2)(1-z)\,. \qquad (4)$$

Proof. Denote by $V = \mathrm{Vol}(W(\alpha,\beta))$ the hyperbolic volume of $W(\alpha,\beta)$. Then by virtue of the Schläfli formula [4] we have

$$\frac{\partial V}{\partial \alpha} = -\frac{\Re(l_\alpha)}{2}, \quad \frac{\partial V}{\partial \beta} = -\frac{\Re(l_\beta)}{2}\,, \qquad (5)$$

where l_α and l_β are complex lengths of singular geodesics corresponding to cone angles α and β respectively. Moreover, by [10] we note that

$$V \to Vol(W(0,0)) \text{ as } \alpha \to 0 \text{ and } \beta \to 0, \tag{6}$$

where

$$\mathrm{Vol}(W(0,0)) = i\int_{1-i}^{1+i} \log\left[\frac{2}{z^2 - z^3}\right]\frac{dz}{z^2-1} = 3.66386\ldots$$

is the hyperbolic volume of the Whitehead link complement $W(0,0) = S^3 \setminus W$.

We set $\tilde{V} = i\int_{\zeta_1}^{\zeta_2} \log\left[\frac{1+\hat{A}^2}{\zeta^2+\hat{A}^2}\frac{1+\hat{B}^2}{\zeta^2+\hat{B}^2}\frac{2\zeta^2}{1-\zeta}\right]\frac{d\zeta}{\zeta^2-1}$ and show that $\tilde{V}$ satisfies conditions (5) and (6). Then $\tilde{V} = V$ and the theorem is proven.

To verify (5) we introduce the function

$$F(\zeta, \hat{A}, \hat{B}) = \frac{i}{\zeta^2 - 1}\log\left[\frac{1+\hat{A}^2}{\zeta^2+\hat{A}^2}\frac{1+\hat{B}^2}{\zeta^2+\hat{B}^2}\frac{2\zeta^2}{1-\zeta}\right].$$

Then by the Leibniz formula we get

$$\frac{\partial \tilde{V}}{\partial \alpha} = F(\zeta_2, \hat{A}, \hat{B})\frac{\partial \zeta_2}{\partial \alpha} - F(\zeta_1, \hat{A}, \hat{B})\frac{\partial \zeta_1}{\partial \alpha} + \int_{\zeta_1}^{\zeta_2} \frac{\partial F(\zeta, \hat{A}, \hat{B})}{\partial \hat{A}}\frac{\partial \hat{A}}{\partial \alpha}\, d\zeta. \tag{7}$$

We note that $F(\zeta_1, \hat{A}, \hat{B}) = F(\zeta_2, \hat{A}, \hat{B}) = 0$ if ζ_1, ζ_2, $\hat{A}$, and $\hat{B}$ are as stated in the theorem. Moreover, since $\alpha = 4\arctan \hat{A}$ we have $\frac{\partial \hat{A}}{\partial \alpha} = \frac{1+\hat{A}^2}{4}$ and

$$\frac{\partial F(\zeta, \hat{A}, \hat{B})}{\partial \hat{A}}\frac{\partial \hat{A}}{\partial \alpha} = \frac{i\hat{A}}{2(\zeta^2 + \hat{A}^2)}.$$

Hence, by Proposition 3.4 we obtain from Equation (7)

$$\frac{\partial \tilde{V}}{\partial \alpha} = \frac{i}{2}\int_{\zeta_1}^{\zeta_2} \frac{\hat{A}\, d\zeta}{\zeta^2 + \hat{A}^2} = \frac{i}{2}\arctan\frac{\zeta_2}{\hat{A}} - \frac{i}{2}\arctan\frac{\zeta_1}{\hat{A}} = -\frac{\Re(l_\alpha)}{2}.$$

The equation $\frac{\partial \tilde{V}}{\partial \beta} = -\frac{\Re(l_\beta)}{2}$ can be obtained in the same way.

Given $\Im(z) > 0$ we have $z \to 1 + i$, as $\alpha \to 0$ and $\beta \to 0$. Then the boundary condition (6) for the function $\tilde{V}$ follows from the integral formula. □

Remark 3.6. *There is exactly one root of Equation 4 such that $\Im(z) > 0$. Indeed, Equation 4 is equivalent to*

$$(z+1)(z^4 - 2z^3 + (\hat{A}^2 + \hat{B}^2 + 2)z^2 + 2\hat{A}^2\hat{B}^2 z - \hat{A}^2\hat{B}^2) = 0.$$

Let

$$P(z) = z^4 - 2z^3 + (\hat{A}^2 + \hat{B}^2 + 2)z^2 + 2\hat{A}^2\hat{B}^2 z - \hat{A}^2\hat{B}^2 ,$$

then by Mathematica we have

$$\begin{aligned} R = \text{Resultant}\left[P'(z)/2, P(z), z\right] &= -\hat{A}^2(1+\hat{A}^2)\hat{B}^2(1+\hat{B}^2)(8 + 12\hat{A}^2 \\ &+ 6\hat{A}^4 + \hat{A}^6 + 12\hat{B}^2 + 39\hat{A}^2\hat{B}^2 + 6\hat{B}^4 + 30\hat{A}^2\hat{B}^4 + 27\hat{A}^4\hat{B}^4 + \hat{B}^6) \\ &= -\hat{A}^2\hat{B}^2(1+\hat{A}^2)(1+\hat{B}^2)Q , \end{aligned}$$

where $Q \geq 8$.

Since $R < 0$ for all nonzero $\hat{A}$ and $\hat{B}$ one can easily deduce that equation $P(z) = 0$ always has a pair of real and a pair of complex conjugate roots for all $\hat{A}$ and $\hat{B}$.

Proposition 3.7. *Let $W(\frac{0}{m}, \frac{n}{0})$ be a hyperbolic cone-manifold obtained by a spontaneous surgery on the first component of the Whitehead link and an orbifold surgery on the second component with cone angles equal to $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$ respectively. Denote by l_α and l_β complex lengths of singular geodesics of $W(\frac{0}{m}, \frac{n}{0})$ with cone angles $\alpha = \frac{2\pi}{m}$ and $\beta = \frac{2\pi}{n}$ respectively. Then*

$$\frac{\coth\frac{l_\alpha}{2}}{\coth\frac{i\alpha}{4}} = \frac{\coth\frac{i\beta}{2}}{\coth\frac{-l_\beta}{4}} = z ,$$

where $\Im(z) > 0$ and z is a root of the equation

$$2(1+A^2)(z^2+B) = (1+A^2z^2)(1+B^2)(1-z) , \tag{8}$$

$A = \cot\frac{\alpha}{4}$ *and* $B = \cot\frac{\beta}{2}$.

Proof. The result follows from Proposition 3.2 for $l_\alpha = i\hat{\alpha}$, $\hat{l}_\alpha = i\alpha$, $\hat{\beta} = \beta$, and $\hat{l}_\beta = -l_\beta$. □

Following the plot of the proof of Theorem 3.5 and applying Proposition 3.7 we obtain the following:

Theorem 3.8. *Let $W(\frac{0}{m}, \frac{n}{0})$ be a cone-manifold obtained by spontaneous surgery on the first component and orbifold surgery on the second component of the Whitehead link with cone angles $\alpha = \frac{2\pi}{m}$ and $\beta = \frac{2\pi}{n}$ respectively. Then*

$$\text{Vol}(W(\frac{0}{m}, \frac{n}{0})) = i\int_{\zeta_1}^{\zeta_2} \log\left[2\frac{1+A^2}{1+A^2\zeta^2}\frac{\zeta^2+B^2}{1+B^2}\frac{1}{1-\zeta}\right]\frac{d\zeta}{\zeta^2-1} ,$$

where $A = \cot\frac{\alpha}{4}$, $B = \cot\frac{\beta}{2}$, $\zeta_1 = \overline{z}$, $\zeta_2 = z$, $\Im(z) > 0$, *and*

$$2(1 + A^2)(z^2 + B) = (1 + A^2z^2)(1 + B^2)(1 - z)\,.$$

4. Boromean rings cone-manifold

4.1 Orbifold surgery on the Boromean rings

In this subsection we study the geometrical properties of cone-manifolds $B(\alpha, \beta, \gamma)$ obtained by orbifold Dehn surgery on three components of the Borromean rings with cone angles α, β, and γ (see Figure 3).

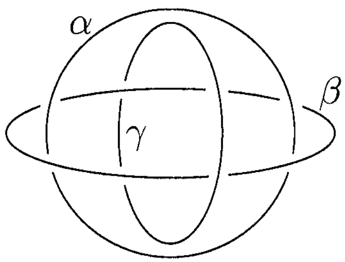

Figure 3. The Borromean cone–manifold $B(\alpha, \beta, \gamma)$.

The following result was essentially obtained by R. Kellerhals [5] (see [7] for details of the proof):

Theorem 4.1. *Let* $B(\frac{k}{0}, \frac{l}{0}, \frac{m}{0})$ *be a cone-manifold obtained by orbifold surgery on the components of the Borromean rings with cone angles* $\alpha = \frac{2\pi}{k}$, $\beta = \frac{2\pi}{l}$ *and* $\gamma = \frac{2\pi}{m}$. *Then* $B(\frac{k}{0}, \frac{l}{0}, \frac{m}{0})$ *is hyperbolic for* $0 < \alpha, \beta, \gamma < \pi$ *and its volume is given by the formula:*

$$\mathrm{Vol}(B(\frac{k}{0}, \frac{l}{0}, \frac{m}{0})) = 2\int_T^\infty \log\left[\frac{(t^2 - A^2)(t^2 - B^2)(t^2 - C^2)}{(1 + A^2)(1 + B^2)(1 + C^2)t^2}\right]\frac{dt}{t^2 + 1}\,,$$

where T *is a positive root of the equation*

$$T^4 - (A^2 + B^2 + C^2 + 1)T^2 - A^2B^2C^2 = 0\,,$$

$A = \tan\frac{\alpha}{2}$, $B = \tan\frac{\beta}{2}$, *and* $C = \tan\frac{\gamma}{2}$.

4.2 Spontaneous surgery on the Boromean rings

The following three results were proved by M. Pashkevich in [15]:

Theorem 4.2. *Let* $B(\frac{0}{k}, \frac{l}{0}, \frac{m}{0})$ *be a hyperbolic cone-manifold obtained by a spontaneous surgery with cone angle* $\alpha = \frac{2\pi}{k}$ *on one component of* *the Borromean rings and an orbifold surgery with cone angles* $\beta = \frac{2\pi}{l}$, $\gamma = \frac{2\pi}{m}$ *on the other two components. Then*

$$\mathrm{Vol}(B(\frac{0}{k}, \frac{l}{0}, \frac{m}{0})) = 2\int_T^\infty \log\left|\frac{(1+A^2)(t^2-B^2)(t^2-C^2)}{(1-t^2A^2)(1+B^2)(1+C^2)}\right|\frac{dt}{t^2+1},$$

where T *is a positive root of the equation*

$$(1+A^2)T^2 - (1+B^2+C^2-A^2B^2C^2) = 0,$$

$A = \tan\frac{\pi}{2k}$, $B = \tan\frac{\pi}{l}$, $C = \tan\frac{\pi}{m}$.

Theorem 4.3. *Let* $B(\frac{0}{k}, \frac{0}{l}, \frac{m}{0})$ *be a hyperbolic cone-manifold obtained by spontaneous surgery with cone angles* $\alpha = \frac{2\pi}{k}$, $\beta = \frac{2\pi}{l}$ *on two components of the Borromean rings and an orbifold surgery with cone angle* $\gamma = \frac{2\pi}{m}$ *on the third component. Then*

$$\mathrm{Vol}(B(\frac{0}{k}, \frac{0}{l}, \frac{m}{0})) = -2\int_0^T \log\left|\frac{(1+A^2)(1+B^2)(t^2-C^2)t^2}{(1-t^2A^2)(1-t^2B^2)(1+C^2)}\right|\frac{dt}{t^2+1}$$

where T *is a positive root of the equation*

$$(1+A^2+B^2-A^2B^2C^2)T^2 - (1+C^2) = 0,$$

$A = \tan\frac{\pi}{2k}$, $B = \tan\frac{\pi}{2l}$, $C = \tan\frac{\pi}{m}$.

Theorem 4.4. *Let* $B(\frac{0}{k}, \frac{0}{l}, \frac{0}{m})$ *be a hyperbolic cone-manifold obtained by spontaneous surgery with cone angles* $\alpha = \frac{2\pi}{k}$, $\beta = \frac{2\pi}{l}$, $\gamma = \frac{2\pi}{m}$ *on three components of the Borromean rings. Then*

$$\mathrm{Vol}(B(\frac{0}{k}, \frac{0}{l}, \frac{0}{m})) = -2\int_0^T \log\left|\frac{(1+A^2)(1+B^2)(1+C^2)t^4}{(1-t^2A^2)(1-t^2B^2)(1-t^2C^2)}\right|\frac{dt}{t^2+1}$$

where T *is a positive root of the equation*

$$A^2B^2C^2T^4 + (1+A^2+B^2+C^2)T^2 - 1 = 0,$$

$A = \tan\frac{\pi}{2k}$, $B = \tan\frac{\pi}{2l}$, $C = \tan\frac{\pi}{2m}$.

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