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POLYHEDRAL GROUPS AND GRAPH AMALGAMATION PRODUCTS

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A polyhedral group G is defined to be the orientation-preserving subgroup of a discrete reflection group acting on hyperbolic 3-space $\mathbb{H}^3$, and having a fundamental polyhedron of finite volume. A special presentation for G is obtained from the geometry of the polyhedron. This gives G the structure of a graph amalgamation product, and which, in some cases, splits as a free product with amalgamation. The simplest examples of polyhedral groups are the so-called tetrahedral groups. Other examples are given amongst the the groups PGL(2, O_m), where O_m is the ring of algebraic integers in the quadratic imaginary field $Q(\sqrt{-m})$, $m > 0$.

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1. Introduction

We are concerned with the orientation-preserving subgroups of discrete reflection groups which act on hyperbolic 3-space $\mathbb{H}^3$. This paper explores the connection between the combinatorial structure of a fundamental polyhedron for the discrete reflection group, and the way in which this subgroup can be decomposed with respect to free products with amalgamation. Our main result deals with graph amalgamation products (Theorem 3). The simplest groups of this type are tetrahedral groups, and these are studied in Section 4. We then conclude with more complicated, and, perhaps, more characteristic examples amongst the projective general linear groups over discrete rings of algebraic integers. The algebraic structure of some of these Bianchi groups has been studied by Fine [8]. Recently, Fine's work has been extended by Flöge [9], who has also related the presentation to the geometry of a fundamental polyhedron.

A graph amalgamation product is a factor group of the graph product of a graph of groups. The latter construction has been investigated by many authors, e.g. Bass and Serre. Graph amalgamation products appear to be of independent interest. Many of the discrete groups studied by Bianchi [5] have such a structure in an obvious way. We have illustrated the use of this structure in producing hyperbolic

3-manifolds whose fundamental groups are subgroups of polyhedral groups [6]. Frequently the homology of these groups can be readily computed via the Mayer–Vietoris sequence.

2. Polyhedral groups

A *convex polyhedron* in $\mathbb{H}^3$ is a set with non-empty interior, which is the intersection of finitely many closed half-spaces. By a *Coxeter polyhedron* we mean a convex polyhedron of finite volume in $\mathbb{H}^3$, each of whose interior dihedral angles is an integer submultiple of π.

Let P be a Coxeter polyhedron with specified dihedral angles π/m_{ij} where the ith face meets the jth face. Let G^* denote the group generated by the reflections $R_1, \ldots, R_n$ in the faces of P. Then G^* has a presentation

$$\langle R_1, \ldots, R_n ;\, R_1^2 = \cdots = R_n^2 = (R_i R_j)^{m_{ij}} = 1 \rangle. \qquad (*)$$

It is well known that G^* is discrete in its action on $\mathbb{H}^3$. Conversely, if G^* is a discrete group of isometries of $\mathbb{H}^3$ which is generated by a finite number of reflections, then the fixed hyperplanes of the reflections in G^* partition $\mathbb{H}^3$ into congruent polyhedra. Coxeter polyhedra in $\mathbb{H}^3$ are described in the work of Vinberg [19] and Andreev [2, 3]. We also refer to Thurston [18, Chapter 13], where, in addition, he introduces the connection with structures called orbifolds. Given a group G which is discrete in its action on $\mathbb{H}^3$, but not necessarily torsion-free, the quotient space $\mathbb{H}^3/G$ is an orbifold.

We call the subgroup of orientation-preserving isometries in the discrete reflection group the *polyhedral group* associated with a Coxeter polyhedron P. Below, we will obtain a special presentation for a polyhedral group which takes into account the combinatorial structure of P.

P may be described more exactly. A vertex of P will be called an interior or ideal vertex, depending on whether it lies in $\mathbb{H}^3$ or $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$.

$\partial\bar{P}$ (including ideal vertices) is topologically a sphere. The edges and vertices of $\bar{P}$ (including ideal vertices) form a finite graph on $\partial\bar{P}$. This graph partitions the sphere into polygons, the faces of P. In fact, each fact of P is a finite-sided hyperbolic polygon.

Suppose now that P has faces $F_1, \ldots, F_n$. Each edge of P is formed by the intersection of two faces, say F_i and F_j: let $a_{ij} = R_i R_j$ where R_i and R_j are reflections in F_i and F_j, respectively. The isometry a_{ij} is a rotation through an angle $2\pi/m_{ij}$ along this edge, where π/m_{ij} is the interior dihedral angle at this edge. The direction of rotation is from F_j to F_i.

The following proposition gives a presentation for a polyhedral group G which takes into account the combinatorial structure. Clearly, a presentation can be obtained from $(*)$ by using the Reidemeister–Schreier subgroup theorem with any $\{1, R_i\}$ as a set of coset representatives. However, the reader can easily satisfy himself that, in general, it is hard to obtain the presentation in the proposition from the one so obtained. Instead we use a geometric argument based on the Poincaré theorem.

Proposition 1. *Let G be a polyhedral group with Coxeter polyhedron P. Then G is generated by the rotations a_{ij}. The defining relations consist in power relations $a_{ij}^{m_{ij}} = 1$, together with a loop relation at each vertex of P. A loop relation at a vertex says that the product of the rotations on consecutive edges crossed by a simple loop around the vertex is 1, where either a_{ij} or $\bar{a}_{ij} = a_{ji}$ is used, as appropriate.*

Proof. The relations $a_{ij}^{m_{ij}} = (R_iR_j)^{m_{ij}} = 1$ follow directly from the relations for the reflection group G^*. Also, at each vertex v of $\bar{P}$ we have the following. Let $F_1, F_2, \ldots, F_m$ be the (consecutive) faces which meet at v (relabel if necessary). Since $a_{12} = R_1R_2$, $a_{23} = R_2R_3$, and so on, we clearly obtain a loop relation $a_{12}a_{23} \cdots a_{m1} = 1$.

To show that these relations are sufficient to define G, Poincaré's Theorem on fundamental polyhedra will be used. We refer to Maskit [15] for a statement and proof of Poincaré's Theorem.

Let $*$ be a point in the interior of P, and let C_i be the cone of the face F_i with respect to $*$ using geodesics. Then $P = \bigcup_i C_i$. We double the polyhedron by defining $DP = P \bigcup_i R_i(C_i)$.

The faces of DP are $\{R_i(F_{ij}), R_j(F_{ij})\}$, where F_{ij} is the cone with respect to $*$ of an edge of P where F_i and F_j intersect. Since

$$a_{ij}(R_j(F_{ij})) = R_iR_j(R_j(F_{ij})) = R_i(F_{ij}),$$

we see that the faces of DP are identified in pairs by the a_{ij}. From the construction of DP we see that the set of ideal vertices of DP is the set of ideal vertices of P. Near an ideal vertex, DP looks like the cartesian product of a euclidean polygon with a ray through the ideal vertex.

Now each edge of DP has a vertex of P for at least one endpoint. An edge of DP which is not an edge of P we call a new edge. Let v be a vertex of P which lies on the faces $F_1, \ldots, F_n$ of P, where the faces are enumerated in order around a simple loop around v. Let E be the geodesic segment from v to $*$. There are n new edges $\{R_i(E)\}$ at v with $a_{i+1,i}(R_i(E)) = R_{i+1}R_i(R_i(E)) = R_{i+1}(E)$.

The edges of DP fall into cycles under the action of G. Each old edge is a cycle by itself, with dihedral angle $2\pi/m_{ij}$. The n new edges at v form a cycle with angle sum 2π. Using the Poincaré Theorem, we obtain as defining relations for G, the relations $a_{ij}^{m_{ij}} = 1$ corresponding to the old edges, and for each vertex of P, a loop relation $a_{12}a_{23} \cdots a_{n1} = 1$. $\square$

3. Graph amalgamation products

Recently, Karrass, Pietrowski and Solitar [11] have introduced a product of groups defined in terms of free products with amalgamation. They have established subgroup theorems for these which are in the same spirit as their theorems on subgroups of amalgamated free products and HNN-extensions [12, 13].

Suppose that a graph is given where the vertices are groups and on the edges are written groups which are isomorphic to subgroups of the adjacent vertices. The *graph amalgamation product* of this graph of groups is the group given with the generators and relations of the vertex groups, together with relations which come from amalgamating the indicated edge groups between adjacent vertex groups.

An alternative description may be given using the Bass–Serre graph product of such a graph of groups. In this graph product a maximal tree is chosen, free products with indicated edge amalgamations are formed in this tree, and then an HNN-extension is formed using the remaining edges. The graph amalgamation product is the factor group obtained by setting the stable letters in the HNN-extension equal to 1.

If the graph of groups is a polygon then we call the graph amalgamation product a *polygonal product.* For a polygonal product, the case of most interest is when the edge groups surrounding a vertex intersect trivially. It is not hard, then, to see that for a polygon with four or more sides, the polygonal product can be split (in general in several ways) as a free product with amalgamation. It can then be shown that the polygonal product contains isomorphic copies of each of the vertices. For a triangular product the situation is more complicated.

In addition, when the vertex groups are finite (and in other special cases) the Karrass, Pietrowski and Solitar subgroup theorem gives an efficient way to find presentations for all the torsion free subgroups of finite index. This was used by Brunner, Frame, Lee and Wielenberg [6] when investigating subgroups of the Picard group. The Picard group is a quadrangular product of finite groups.

Consider now a polyhedral group G as discussed in Proposition 1, and let P be the Coxeter polyhedron for G^*. No vertex of P is equivalent to another vertex of P under the action of G. If v is an interior vertex then a sufficiently small sphere S centered at v can be chosen so that $DP \cap S$ is a fundamental polygon for the action of the stabilizer G_v of v on S. Thus G_v is a discrete 2-dimensional spherical group. Likewise, if v is an ideal vertex then a horosphere S centered at v can be chosen so that $DP \cap S$ is a fundamental polygon for the action of G_v on S, and G_v is a discrete 2-dimensional euclidean group.

Orientation preserving discrete 2-dimensional spherical groups are well known to be the finite triangle groups. The orientation preserving discrete 2-dimensional euclidean groups which occur are the triangle groups $(3, 3, 3)$, $(2, 3, 6)$ and $(2, 4, 4)$, and in addition the 'rectangular pillow' group,

$$R = \langle x, y, z\,;\, x^2 = y^2 = z^2 = (xyz)^2 = 1\rangle.$$

(This is given the notation $p2$ in [7]. A classification of discrete 2-dimensional spherical and euclidean groups is given in [7].)

It follows that a vertex v of P has either (i) exactly 3 edges incident (see Fig. 1) with $G_v = \langle a_{11}, a_{12}, a_{13}\,;\, a_{ij}^{m_{ij}} = 1,\ a_{11}a_{12}a_{13} = 1\rangle$, a spherical or euclidean triangle

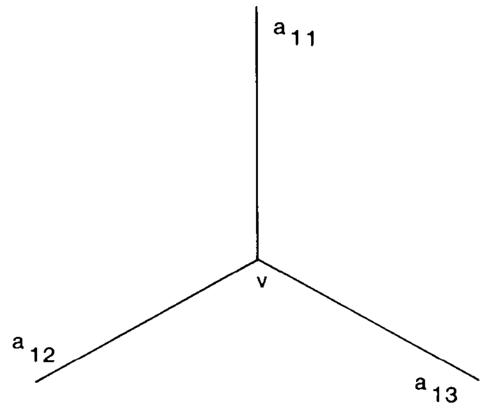

Fig. 1.

group; or (ii) exactly 4 edges incident (see Fig. 2) with $G_v = \langle a_{11}, a_{12}, a_{13}, a_{14};\ a_{ij}^2 = 1,\ a_{11}a_{12}a_{13}a_{14} = 1\rangle$, the rectangular pillow group R.

It is clear now how we may visualize the presentation for G given in Proposition 1 with reference to the Coxeter polyhedron P. At each vertex v is the group G_v which is the stabilizer of v; on edges are rotations a_{ij} of order m_{ij} which are the common subgroups between adjacent vertices used in the amalgamations. This gives G as a graph amalgamation product. We state this as a theorem.

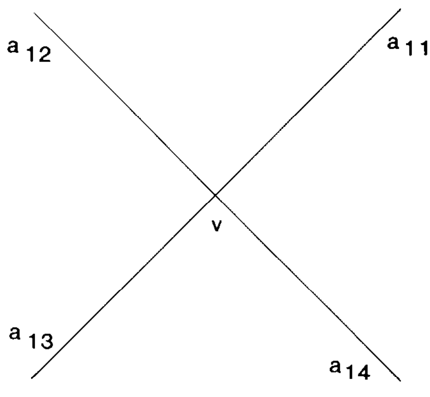

Fig. 2.

Theorem 2. *If G is a polyhedral group, then G is a graph amalgamation product where the graph consists of vertices and edges of a Coxeter polyhedron for G; at each vertex is the stabilizer of that vertex, on the edges are the cyclic groups generated by the rotation on that edge.*

This graph amalgamation product is redundant, in the sense that at least one vertex group may be omitted. In fact, G is a graph amalgamation product of a planar graph. To see this, let v be a vertex of the graph. Then G is generated by all G_w, $v \neq w$, since G_v is generated by its edge groups and these are amalgamated with edge groups in the other G_w. The power relations amongst these generators for G_v are a consequence of defining relations for the G_w, $v \neq w$. Lastly, the loop relation at G_v is also a consequence of the other relations. The boundary of $\bar{P}$ is topologically a sphere. A closed loop on $\partial\bar{P}$ is homotopic to a product of simple loops around single vertices. The product of the rotations on the consecutive edges crossed by the loop (again with appropriate orientation) is a word which represents 1 in G. In particular, the loop relation at any single vertex v is a consequence of the loop relations at the other vertices.

Theorem 3. *Let G be a polyhedral group and P an associated Coxeter polyhedron. Let v be a vertex of P. Then G is a graph amalgamation product of a planar graph of groups: vertex groups are the stabilizers G_w, $w \neq v$, and the edge groups are cyclic groups generated by rotations on the edges of P connecting the vertices w, $w \neq v$.*

Remark 4. The inclusion mapping of the polygonal product represented by a face of P into a graph amalgamation product need not be injective. Likewise, the inclusion mapping of the free product with amalgamation of two adjacent vertex groups along their common edge group need not be injective. However, in many cases these inclusions are injective. The matter can be decided, from the presentations, by building up the graph amalgamation product inductively from the presentations of the polygonal products which represent the faces.

Remark 5. The torsion elements of G are exactly the rotations on an edge of P, together with the conjugates in G of these rotations. If H is a subgroup of G, then G acts transitively by right multiplication on the right cosets of H. A subgroup H of index n corresponds to a homomorphism ϕ of G into the symmetric group S_n, where $\phi(H)$ is a transitive subgroup of S_n and $H = \{g \in G: \phi(g)(1) = 1\}$. The subgroup H is torsion-free if and only if $\phi(g)$ has no fixed point for each torsion element g of G. By the above, the only elements to be checked are the rotations on an edge of P. Furthermore, if H is torsion-free, then the quotient of $\mathbb{H}^3$ by H is a 3-manifold with a hyperbolic structure. In principle, all such hyperbolic manifolds can be produced from the presentation of G as a graph amalgamation product.

4. Tetrahedral groups

The Coxeter polyhedra which are tetrahedra have been classified by Lannér [14], Vinberg [19] and Thurston [18]. These give the simplest illustration of the sort of presentations discussed above; the tetrahedral groups are triangular products. In general, these do not decompose as free products with amalgamation, although often they have subgroups of finite index which do so.

Consider now a tetrahedron 1234 with dihedral angles π/λ_i, π/μ_j (see Fig. 3) (λ_i, μ_i are written on the appropriate opposite edges in the diagram). The vertex groups are the triangle groups $G_1=(\lambda_1, \lambda_2, \mu_3)$, $G_2=(\lambda_3, \lambda_1, \mu_2)$, $G_3=(\mu_1, \lambda_3, \lambda_2)$, $G_4=(\mu_1, \mu_2, \mu_3)$; these are spherical or euclidean depending on whether the vertex is an interior or ideal vertex of $\mathbb{H}^3$. Here $(l, m, n)=\langle x, y;\, x^l=y^m=(xy)^n=1\rangle$.

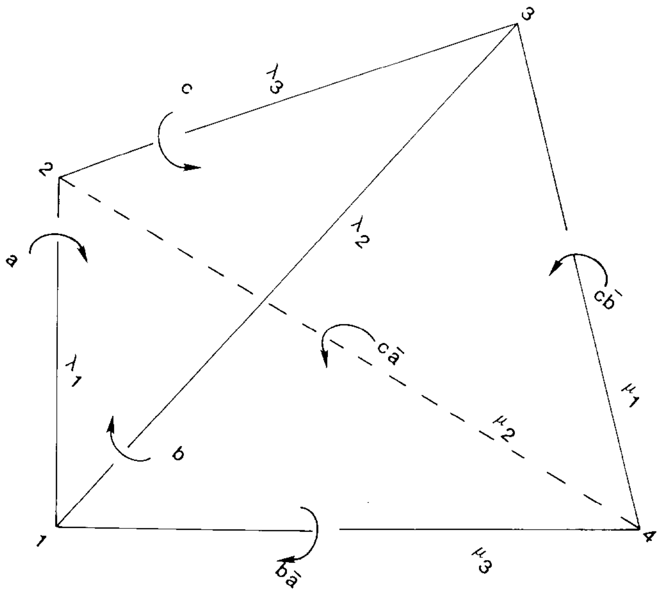

Fig. 3.

If we now choose a, b, c to be rotations of order λ_1, λ_2, λ_3 on the indicated edges, then, by using loop relations, we see that the tetrahedral group $T(\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3)$ is generated by a, b, c. In fact,

$$T=\langle a, b, c;\, a^{\lambda_1}=b^{\lambda_2}=c^{\lambda_3}=(b\bar{c})^{\mu_1}=(c\bar{a})^{\mu_2}=(b\bar{a})^{\mu_3}=1\rangle.$$

With this selection of generators T is a triangular product. It is clear that this presentation favors face 123, and G_4 is omitted. (See Fig. 4.) Omitting any one of G_1, G_2 or G_3 would result in T being written differently as a triangular product. This illustrates Theorem 3.

There are 9 tetrahedra with vertices all interior in $\mathbb{H}^3$, and there are 23 tetrahedra with some ideal vertices.

It is convenient to list the groups as follows (they are listed using Coxeter diagrams in [16]):

$T_1(2,2,3;3,5,2)$, $T_2(2,2,3;2,5,3)$, $T_3(2,2,4;2,3,5)$.

$T_4(2,2,5;2,3,5)$, $T_5(2,3,3;2,3,4)$, $T_6(2,3,4;2,3,4)$,

$T_7(2,3,3;2,3,5)$, $T_8(2,3,4;2,3,5)$, $T_9(2,3,5;2,3,5)$,

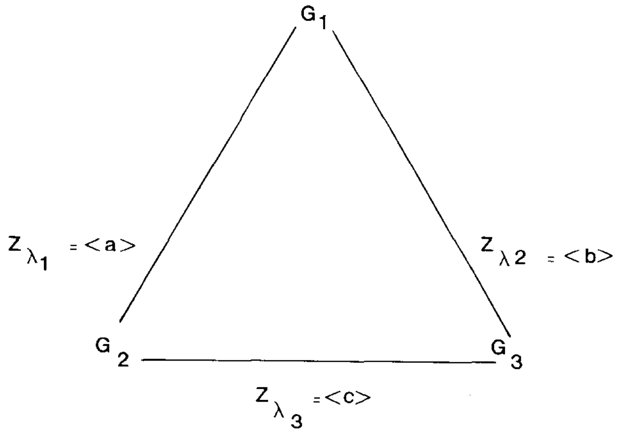

Fig. 4.

and

$$\begin{array}{lll} T^1(3,2,2;6,2,3), & T^2(2,2,3;2,6,3), & T^3(3,2,2;4,2,4), \\ T^4(4,2,2;6,2,3), & T^5(5,2,2;6,2,3), & T^6(6,2,2;6,2,3), \\ T^7(4,2,2;4,2,4), & T^8(2,3,3;2,6,3), & T^9(3,2,4;3,2,6), \\ T^{10}(6,2,3;5,2,3), & T^{11}(3,2,6;3,2,6), & T^{12}(2,3,3;2,4,4), \\ T^{13}(4,2,3;4,2,4), & T^{14}(4,2,4;4,2,4), & T^{15}(3,2,2;2,3,6), \\ T^{16}(3,2,2;2,4,4), & T^{17}(4,2,2;2,4,4), & T^{18}(3,2,3;3,3,3), \\ T^{19}(3,2,2;3,3,3), & T^{20}(4,2,2;3,3,3), & T^{21}(5,2,2;3,3,3), \\ T^{22}(6,2,2;3,3,3), & T^{23}(3,3,3;3,3,3). & \end{array}$$

For example, $T^4(4,2,2;6,2,3)$ has a Coxeter diagram $\bullet\overset{4}{—}\bullet\overset{3}{—}\bullet\overset{6}{—}\bullet$. The group T^4 is then a triangular product of finite groups (see Fig. 5).

Locating the tetrahedron explicitly in $\mathbb{H}^3$, the rotations which generate T^4 can be written down as matrices:

$$a=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad b=\begin{pmatrix}\eta & 1\\ 1 & 0\end{pmatrix},\quad c=\begin{pmatrix}0 & 1\\ \omega & 0\end{pmatrix},$$

where $\omega=\frac{1}{2}(-1+\mathrm{i}\sqrt{3})$ and $\eta=\mathrm{i}\sqrt{2}$.

Various subgroup relationships exist amongst the tetrahedral groups. Some of these can be seen directly geometrically by combining copies of a given tetrahedron along congruent faces, or algebraically, using presentations.

We note that torsion-free subgroups of minimal index in T_4 and T_2 have been computed in [16] and [4]. Al Jubouri [1] has studied torsion-free subgroups of finite index in T_1 and T_3. In the next section we mention some results on torsion-free subgroups of finite index in T^1 and T^{19}.

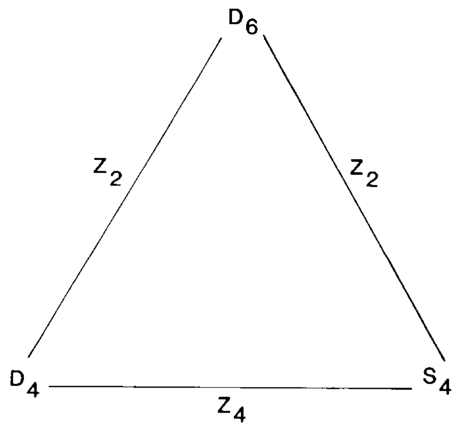

Fig. 5.

5. The Bianchi and related groups

Let O_m denote the ring of integers in the imaginary quadratic number field $Q(\sqrt{-m})$, $m > 0$. By PGL(2, R) we mean the projective general linear group of 2×2 matrices over a ring R; it is the quotient of GL(2, R) by its centre. The Bianchi groups include many of the PSL(2, O_m) and PGL(2, O_m). These are discrete groups which act discontinuously on $\mathbb{H}^3$; PGL(2, O_m) contains PSL(2, O_m) as a subgroup of index 2. Fundamental polyhedra for small values of m have been worked out by Bianchi [5], and Swan [17] has used these to obtain presentations for the groups. The results of Fine [8] and Flöge [9] on the algebraic structure of the PSL(2, O_m) provide an interesting comparison with our results on the PGL(2, O_m) discussed below. The reader is also referred to Hatcher [10] for a discussion of the PGL(2, O_m), especially in connection with the link groups which they may contain.

Some of the PGL(2, O_m) are polyhedral groups and others exhibit related structures. A Coxeter polyhedron for PGL can be deduced from those given in [5, 17]. However, the situation is a bit complicated. Swan describes one-fourth of a fundamental polyhedron for PSL, taking advantage of the symmetry of the region. Bianchi describes a polyhedron for PΓL, where PΓL is PGL with the generator $z \to \bar{z}$ adjoined. In addition, when m is 7, 11, 15 or 19, Bianchi's polyhedron is actually the double of a fundamental polyhedron for PΓL. The element

$$\begin{pmatrix} -1 & w \\ 0 & 1 \end{pmatrix},$$

where

$$w = \tfrac{1}{2}(1 + \mathrm{i}\sqrt{m}),$$

rotates this region onto itself.

These complications are resolved as follows. When m is 1, 2, 3, 5 or 6, Bianchi's polyhedron had dihedral angles which are integer submultiples of π. The group generated by rotations of the appropriate order on the edges is a graph amalgamation product by Theorem 3. The group PGL contains each of these rotations, hence this graph amalgamation product is PGL. (It is a subgroup of PGL of index 1, since the volume of a fundamental polyhedron is correct.) When m is 7, 11, 15 or 19, the dihedral angles are again integer submultiples of π, but the group PGL contains a rotation (the one above) in a line of symmetry of a face of a Coxeter polyhedron. So PGL itself is an extension by an automorphism of order two of a graph amalgamation product. The actual generators can be discovered by patient calculation from Swan's generators, using the geometry as a guide. It is necessary to check the direction of rotation for each generator so as to obtain the correct matrix for the presentation which is read from the Coxeter polyhedron. Some detailed examples are given below.

The cases where $m=1$ or 3 are a bit different from the general case as there are units in O_m other than ± 1.

$\mathrm{PGL}(2, O_1) = T^3(3,2,2;4,2,4)$. Hence $\mathrm{PGL}(2, O_1)$ is a triangular product with vertices $\langle a, b;\, a^2=b^2=(ab)^4=1\rangle = D_4$, $\langle b, c;\, b^2=c^3=(bc)^2=1\rangle = D_3$, and $\langle c, a;\, c^3=a^2=(ca)^4=1\rangle = S_4$.

Explicit generators are

$$a=\begin{pmatrix}0 & \mathrm{i}\\ 1 & 0\end{pmatrix},\qquad b=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad c=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}.$$

In fact, $\mathrm{PGL}(2, O_1)$ contains rotations along all the geodesics on the unit hemisphere whose projection into the unit disc are shown in Fig. 8. An integer n associated to a geodesic indicates a dihedral angle of π/n between the hemisphere and a plane orthogonal to $\mathbb{C}$. It is easy to see that $T^{16}=\langle a, f, c\rangle$ has index 2 and the subgroup $T^{12}=\langle a, g, c\rangle$ has index 4. (The tetrahedra have one face and three vertices on the hemisphere, three faces on planes orthogonal to $\mathbb{C}$, and one vertex at ∞.) The Picard group $H=\mathrm{PSL}(2, O_1)$ is generated by b, c, d, e and H', its derived group, is generated by c, d, h, g. The matrices here are

$$d=\begin{pmatrix}0 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix},\qquad e=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad f=\begin{pmatrix}0 & \mathrm{i}\\ -1 & 0\end{pmatrix},\qquad g=bdb,\qquad h=ece.$$

The Picard group H is the quadrangular product in Fig. 6, and H' is the quadrangular product in Fig. 7.

An easy calculation using the Mayer–Vietoris sequence shows that $H_2(H,\mathbb{Z})\cong\mathbb{Z}_6$ and $H_2(H',\mathbb{Z})\cong\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_6$.

$\mathrm{PGL}(2, O_3) = T^1(3,2,2;6,2,3)$. Hence $\mathrm{PGL}(2, O_3)$ is a triangular product with vertices $\langle a, b;\, a^2=b^2=(ab)^6=1\rangle = D_6$, $\langle b, c;\, b^2=c^3=(bc)^2=1\rangle = D_3$, and $\langle c, a;\, c^3=a^2=(ca)^3=1\rangle = A_4$.

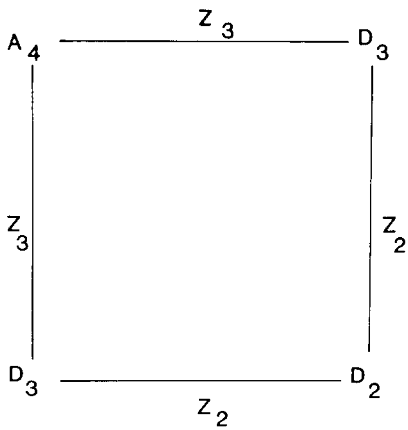

Fig. 6.

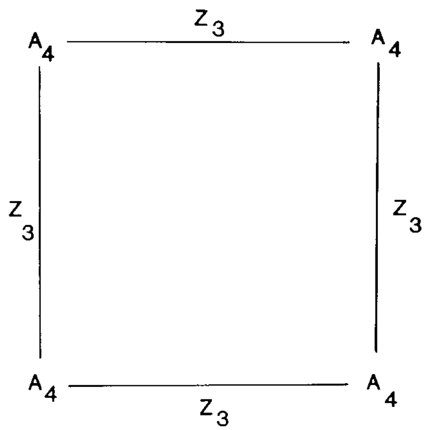

Fig. 7.

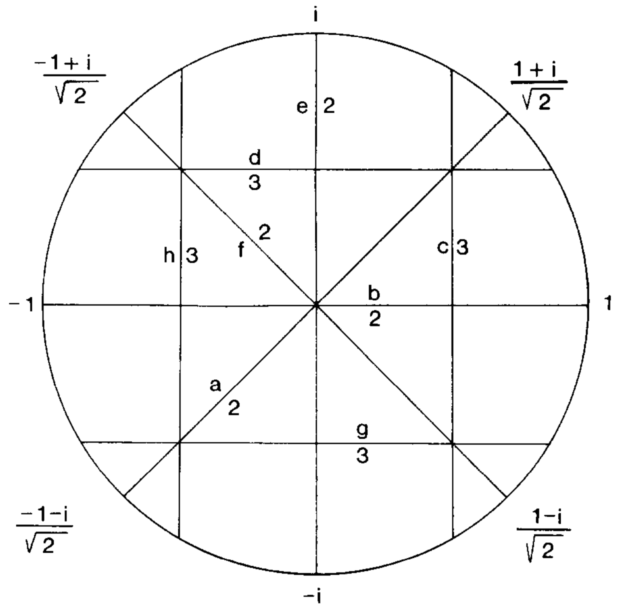

Fig. 8.

Explicit generators are

$$a=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad b=\begin{pmatrix}0 & 1\\ \omega & 0\end{pmatrix},\qquad c=\begin{pmatrix}0 & \omega\\ -\omega^2 & -1\end{pmatrix},$$

where $\omega=\frac{1}{2}(-1+\mathrm{i}\sqrt{3})$.

Again, $\mathrm{PGL}(2, O_3)$ contains rotations along the geodesics whose projections into the unit disk are shown in Fig. 9. Visible subgroups include $T^2=\langle b, g, h\rangle$, $T^6=\langle a, b, e\rangle$, $T^{11}=\langle e, f, g\rangle$, $T^{15}=\langle a, h, i\rangle$, $T^{18}=\langle a, c, i\rangle$, $\mathrm{PSL}(2, O_3)=T^{19}=\langle a, c, d\rangle$, $T^{22}=\langle b, e, f\rangle$ and $T^{23}=\langle c, i, j\rangle$.

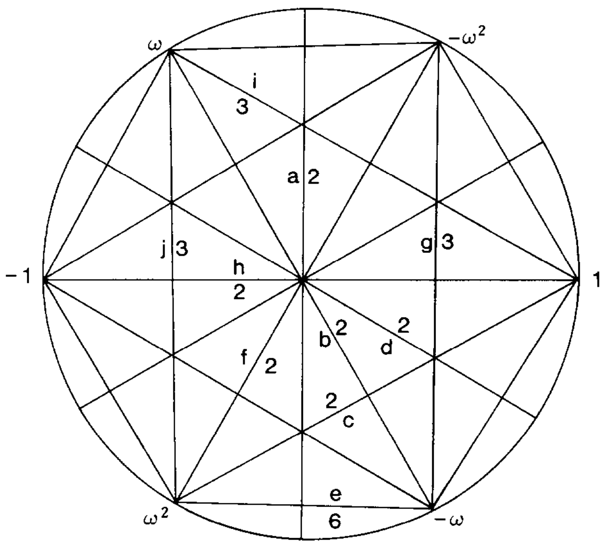

Fig. 9.

Using the same techniques as [6], we have calculated that PSL(2, O_3) contains exactly two non-isomorphic torsion-free subgroups of index 12. One is the figure-eight knot group. The other has a presentation $\langle x, y; \bar{y}x\bar{y}^3x\bar{y} = x^2\rangle$; it is the group of the manifold obtained by 5-surgery on one component of the Whitehead link. PGL(2, O_3) has no torsion-free subgroups of index 12.

For the PGL(2, O_m), $m \neq 1$, 3, we give three representative examples: $m = 2$, 6 and 7.

$m = 2$. PGL(2, O_2) has a fundamental polyhedron which is a pentahedron. It is the quadrangular product in Fig. 10.

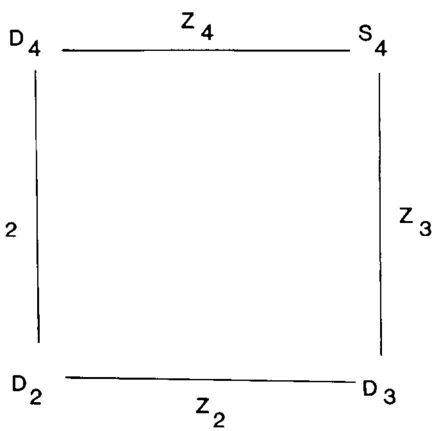

Fig. 10.

$m=6$. PGL(2, O_6) has a complicated fundamental polyhedron. Let $\omega = \mathrm{i}\sqrt{6}$. The group has generators

$$a=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\quad b=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad c=\begin{pmatrix} 1 & -1\\ 1 & 0\end{pmatrix},$$

$$d=\begin{pmatrix} 1+\omega & 2-\omega\\ 2 & -1-\omega\end{pmatrix},\quad f=\begin{pmatrix} -1-\omega & -3+\omega\\ -2 & 1+\omega\end{pmatrix},\quad g=\begin{pmatrix} -5 & 2\omega\\ 2\omega & 5\end{pmatrix}.$$

The generators are rotations along geodesics lying on the three hemispheres corresponding to

$$|z|=1,\qquad |z-\tfrac{1}{2}(1+\omega)|=\tfrac{1}{2},\qquad \left|z-\frac{5}{2\omega}\right|=\frac{1}{2\sqrt{6}}.$$

Eliminating the ideal vertices at ∞ leaves seven vertices. One of these, $\frac{1}{2}\omega$, is an ideal vertex. The projection into $\mathbb{C}$ is given in Fig. 11. Bianchi [5, p. 316] also has a sketch of the polyhedron. PGL(2, O_6) is the graph amalgamation product in Fig. 12.

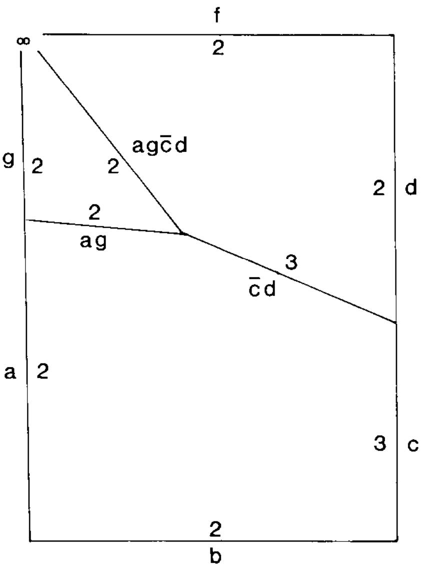

Fig. 11.

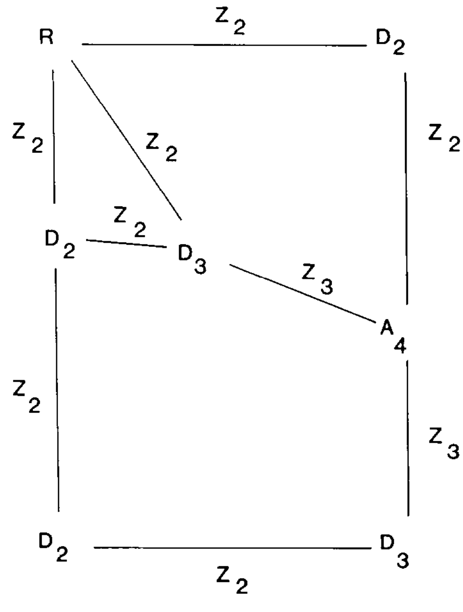

Fig. 12.

$m=7$. PGL(2, O_7) is generated by the stabilizers of the vertices of a Coxeter polyhedron, together with a rotation in a line of symmetry of a face. This rotation conjugates one vertex group to another. The result is not a graph amalgamation product in the usual way. Rather, it can be described as an amalgamated product of two copies of $D_2 *_{Z_2} D_3$ with the amalgamated subgroup being $\mathrm{PSL}(2, Z) \cong Z_2 * Z_3$.

Let $\omega = \frac{1}{2}(1+i\sqrt{7})$. PGL(2, O_7) has generators

$$a=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\qquad b=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad c=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix},\qquad d=\begin{pmatrix}-1 & \omega\\ 0 & 1\end{pmatrix},$$
$$g=\begin{pmatrix}-1 & \omega\\ -1+\omega & 1\end{pmatrix}.$$

The generators a, b, c and g are rotations along geodesics lying on the two hemispheres corresponding to $|z|=1$ and $|z-\frac{1}{2}\omega|=1/\sqrt{2}$. The hemisphere $|z-\omega|=1$ also contains the intersection of these two hemispheres. The generator d is a rotation along the geodesic from $\frac{1}{2}\omega$ to ∞. A projection into $\mathbb{C}$ of a Coxeter polyhedron and its image under the rotation d is given in Fig. 13. The group is generated by the vertex groups

$$D_2=\langle a, b\,;\, a^2=b^2=(ab)^2=1\rangle,\qquad D_3=\langle b, c\,;\, b^2=c^3=(bc)^2=1\rangle,$$
$$D_3=\langle c, g\,;\, c^3=g^2=(cg)^2=1\rangle,\qquad D_2=\langle g, d\,;\, g^2=d^2=(dg)^2=1\rangle,$$

and there is an extra relation $a=d(cg)d$. Here, $\langle a, b, c\rangle \cong D_2 *_{Z_2} D_3 \cong \langle c, g, d\rangle$. Then, PGL(2, O_7) is the free product with amalgamation of these two groups amalgamating $\langle c, a=d(cg)d\rangle \cong$ PSL(2, Z).

There is a subgroup of index 2 obtained by using $\{1, d\}$ as coset representatives. The generators are $a_0=a$, $b_0=b$, $c_0=c$, g, $a_1=dad$, $b_1=dbd$, and $c_1=dcd$. A projection into $\mathbb{C}$ of the fundamental polyhedron is given in Fig. 14. The group is the graph amalgamation product in Fig. 15. The generator d is an automorphism of

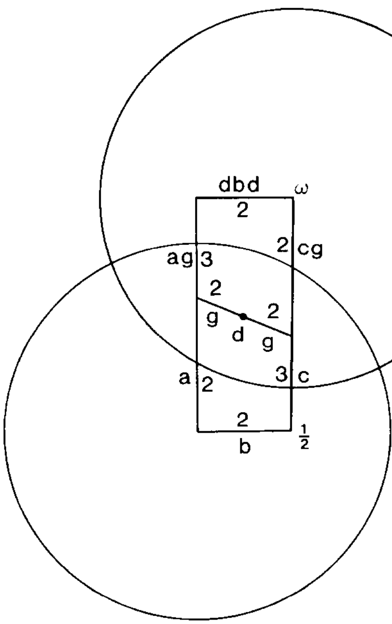

Fig. 13.

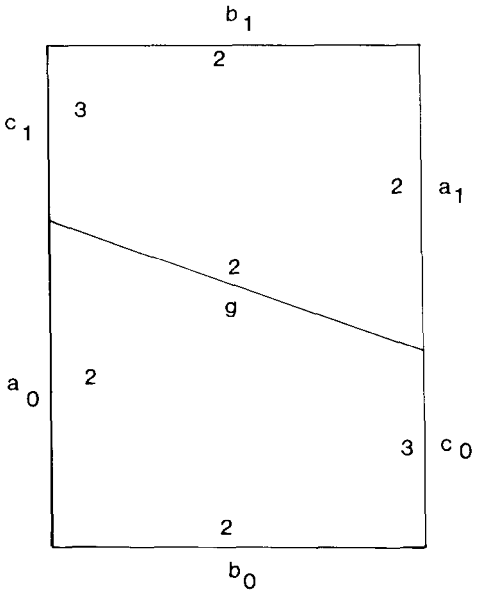

Fig. 14.

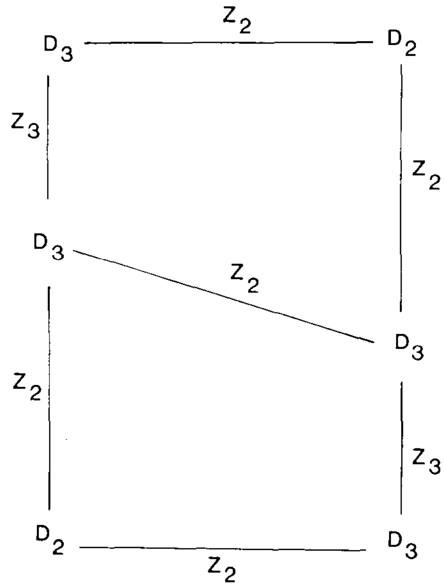

Fig. 15.

this subgroup. Similarly, for $m = 11$, 15 and 19, there is a subgroup of index 2 which is a graph amalgamation product and extending by the automorphism

$$d = \begin{pmatrix} -1 & \omega \\ 0 & 1 \end{pmatrix}$$

gives $\mathrm{PGL}(2, O_m)$.

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