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The Schläfli formula for polyhedra and piecewise smooth hypersurfaces

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Abstract

The classical Schläfli formula relates the variations of the dihedral angles of a smooth family of polyhedra in a space form to the variation of the enclosed volume. We extend here this formula to immersed piecewise smooth hypersurfaces in Einstein manifolds. This leads us to introduce a natural notion of total mean curvature of piecewise smooth hypersurfaces and a consequence of our formula is, for instance, in Ricci-flat manifolds, the invariance of the total mean curvature under bendings. We also give a simple and unified proof of the Schläfli formula for polyhedra in Riemannian and pseudo-Riemannian space forms. Moreover, we show that the formula makes sense even for polyhedra which are not necessarily embedded.

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Introduction

Let M_K^{n+1} be the spherical, Euclidean or hyperbolic space of constant curvature K and dimension $n+1 \geqslant 2$. Consider a smooth one-parameter family, $(P_t)_{t\in[0,1]}$, of polyhedra in M_K^{n+1} bounding compact domains and having the same combinatorics. Write V_t for the volume bounded by P_t, and let $\theta_{i,t}$ and $\mathrm{Vol}(G_{i,t})$ denote respectively the *interior* dihedral angle and the $(n-1)$-volume of the codimension 2 face $G_{i,t}$ of P_t. The classical Schläfli formula relates the variation of V_t and of the angles $\theta_{i,t}$ in the following way:

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The Classical Schläfli formula

$$nK\frac{dV_t}{dt} = \sum_i \mathrm{Vol}(G_{i,t})\frac{d\theta_{i,t}}{dt}.$$

The Schläfli formula is an important tool in the computation of the volume of hyperbolic and spherical polyhedra. Unlike the Euclidean case, there is no simple formula for the volume of a simplex in hyperbolic or spherical case. The formula was first proved in the 1850's by L. Schläfli for spherical simplices of any dimension. Then H. Kneser gave in 1936 a different proof which also applies to the hyperbolic case [11]. The Euclidean version of the formula was rediscovered by Regge in 1961 [14]. A more modern proof is given in [12]. During the recent years the Schläfli formula and its generalizations proved to be useful in several areas. Rivin and Schlenker [17] gave a smooth analogue of the Schläfli formula for deformations of smooth hypersurfaces in Einstein manifolds relating the variation of the volume bounded by a hypersurface and the integral of the variation of the mean curvature. They used their formula to obtain rigidity results for Ricci-flat manifolds with umbilic boundaries. Other modern applications of the Schläfli formula include the study of ideal polyhedra in hyperbolic space [15], of hyperbolic cone-manifolds [7] and convex cores of hyperbolic manifolds [8]. J.-M. Schlenker and the author [21] obtained Schläfli formulas of higher orders, for deformation of polyhedra, relating the variations of the volumes of the codimension p faces to the variations of the *curvature* of codimension $(p+2)$ faces, for $1 \leqslant p \leqslant n-1$. They deduced some topological invariants of polyhedra under deformations. Following Kneser's proof, Suarez-Peiró [23] extended the Schläfli formula to simplices (bounding compact domains) in pseudo-Riemannian space forms of nonzero constant sectional curvature. She used it to generalize to higher dimensions a formula of Santalo relating the volume of a hyperbolic simplex with the measure of the set of hyperplanes intersecting it.

In this paper, we shall give further extensions of the Schläfli formula and derive some consequences. Our first main result gives a unified proof of Schläfli's formula for polyhedra in all simply connected Riemannian and pseudo-Riemannian space forms (Theorem 2). An important feature of the formula is that it applies to oriented polyhedra which are not necessarily embedded and may have self-intersections. The point is that the variation of volume always makes sense for the type of polyhedra we consider. Our proof of Schläfli's formula uses methods of differential geometry, mainly the divergence theorem, and we believe it is not only simpler than the previous ones but can also be used in more general situations (see also Remark 1 after the proof of Theorem 2). Roughly speaking, the idea is to observe that since the deformation is *through polyhedra* then all codimension one faces of the polyhedron remain totally geodesic during the deformation. This provides us with a vector field defined on each of these faces whose divergence has a simple form—for instance the divergence is zero in the flat case. Then applying the divergence theorem to each of these faces and analyzing the boundary terms on codimension 2 faces leads to the Schläfli formula.

As an illustration of our techniques, our second main result gives a Schläfli-type formula for piecewise smooth hypersurfaces in Einstein manifolds (Theorem 4). It extends the formula obtained by Rivin and Schlenker [17] for smooth hypersurfaces. This leads us to introduce a natural notion of total mean curvature for piecewise smooth hypersurfaces. We obtain, as a corollary, the invariance under isometric deformations of a linear combination of the volume bounded by the hypersurface and the total mean curvature (Corollary 6). For results on deformation and rigidity of piecewise smooth surfaces we refer to the survey paper by Ivanova-Karatopraklieva and Sabitov [10].

As a consequence of the formula in the polyhedral case, we also obtain the invariance under flex (i.e., isometric deformations) of the total mean curvature of polyhedra in (pseudo-)Euclidean spaces, and the invariance of a linear combination of the *volume* and the total mean curvature in (pseudo-)Riemannian space forms of nonzero curvature (Corollary 3). In particular, we recover in this way, the invariance of total mean curvature under flex in the Minkowski 3-space proved recently by Alexandrov [3]. The invariance under flex of the total mean curvature of polyhedra in the Euclidean 3-space was first proved by Alexander [2] and rediscovered and extended to smooth hypersurfaces by Almgren and Rivin [4] (see also [17] and [22] for easier proofs and generalizations to higher order mean curvatures). The study of geometric invariants of polyhedra (and of smooth surfaces) under flex is an interesting part in the subject of flexibility/rigidity of these objects. One of the best achievements in this field is the proof by Sabitov of the invariance of the volume of a polyhedron under flex in $\mathbb{R}^3$, solving the Bellows conjecture (cf. [9,18,19]). It is noteworthy that V. Alexandrov has constructed flexible polyhedra with nondegenerate faces in the Minkowski 3-space and observed that the proof of the invariance of the volume carries over to this case [3].

Our article is organized as follows. In Section 1, we recall some basic definitions including a description of the class of polyhedra we consider. We also introduce the notion of the *variation of volume* for these polyhedra. Section 2 gives the proof of the Schläfli formula and its consequence in the polyhedral case. Part of Section 2 is devoted to make precise the adequate notion of dihedral angle for the polyhedra under consideration. Finally, Section 3 is devoted to the case of piecewise smooth hypersurfaces.

1. Preliminaries

We first recall some basic facts about pseudo-Riemannian manifolds. A basic reference in the subject is [13]. A pseudo-Riemannian (or semi-Riemannian) manifold (M, g) is a differentiable manifold M endowed with a metric tensor g, that is, a symmetric nondegenerate $(0, 2)$ tensor field on M, of constant index, say ν, $0 \leqslant \nu \leqslant n = \dim M$. For instance, if $\nu = 0$, M is a Riemannian manifold, and if $\nu = 1$, M is a Lorentz manifold. The norm of a tangent vector $u \in TM$ is the complex number $|u| = \sqrt{g(u,u)}$ and is a positive real number in case $g(u,u) > 0$, u is then said to be spacelike and we set $\varepsilon(u) = 1$. In the case $g(u,u) < 0$, $|u| = \sqrt{g(u,u)}$ is positive pure imaginary, u is then said to be timelike and we set $\varepsilon(u) = -1$. Finally if $|u| = 0$ and $u \neq 0$, u is said to be a null vector.

The pseudo-Euclidean space $\mathbb{R}^{n+1}_\nu$ is $\mathbb{R}^{n+1}$ endowed with the pseudo-Riemannian metric defined by the bilinear form of index ν:

$$\langle \mathbf{x}, \mathbf{y} \rangle = -\sum_{i=1}^{\nu} x_i y_i + \sum_{i=\nu+1}^{n} x_i y_i,$$

where $\mathbf{x} = (x_1, \dots, x_{n+1})$ and $\mathbf{y} = (y_1, \dots, y_{n+1})$.

The pseudo-Euclidean spaces are flat and, for instance, $\mathbb{R}^{n+1}_1$ is the Minkowski space of dimension $(n+1)$.

The *pseudosphere* of radius $r > 0$ in $\mathbb{R}^{n+1}_\nu$ is the hyperquadric

$$\mathbb{S}^n_\nu(r) = \left\{ \mathbf{x} \in \mathbb{R}^{n+1}_\nu \mid \langle \mathbf{x}, \mathbf{x} \rangle = r^2 \right\}$$

endowed with the metric induced from that of $\mathbb{R}^{n+1}_{\nu}$. It has dimension n, index ν and constant sectional curvature $1/r^2$. A particularily important space among these is the de Sitter space $\mathbb{S}^n_1(1)$ which is a complete, simply connected for $n \geqslant 2$, Lorentzian manifold of constant curvature one.

The *pseudohyperbolic space* of radius $r > 0$ in $\mathbb{R}^{n+1}_{\nu+1}(r)$ is the hyperquadric

$$\mathbb{H}^n_\nu(r) = \left\{\mathbf{x} \in \mathbb{R}^{n+1}_{\nu+1} \mid \langle \mathbf{x}, \mathbf{x}\rangle = -r^2\right\},$$

endowed with the metric induced from that of $\mathbb{R}^{n+1}_{\nu+1}$. It has dimension n, index ν and constant sectional curvature $-1/r^2$.

Connected components of pseudospheres and pseudohyperbolic spaces are, up to isometry, the only complete simply connected pseudo-Riemannian manifolds of constant sectional curvature. The totally geodesic submanifolds of $\mathbb{S}^n_\nu(r)$ and $\mathbb{H}^n_\nu(r)$ are the connected components of their intersection with linear subspaces of $\mathbb{R}^{n+1}$. For this reason polyhedra make sense in these spaces. In the sequel, we shall denote the simply connected pseudo-Riemannian space form of dimension $n+1$, index ν and constant curvature K, by $M^{n+1}_\nu(K)$.

Let (M, g) be a pseudo-Riemannian manifold of dimension n and Levi-Civita connection D. For a vector field X on M its divergence is given by:

$$\operatorname{div} X = \sum_i \varepsilon(e_i)\langle D_{e_i} X, e_i\rangle,$$

where $e_1, \dots, e_n$ is an orthonormal frame. If (M, g) is orientable, there is defined a global volume form and for pseudo-Euclidean spaces $\mathbb{R}^{n+1}_\nu$, it coincides with the Euclidean volume form (cf. [13]).

Polyhedra. We will consider compact oriented polyhedra which may have self intersections. They are defined as follows: Let Σ be an n-dimensional simplicial complex which is homeomorphic to a compact oriented manifold. A polyhedron in $M^{n+1}_\nu(K)$, modeled on Σ is a continuous map $P : \Sigma \to M^{n+1}_\nu(K)$ such that its restriction to each simplex of any dimension k of Σ takes its images in a totally geodesic submanifold of dimension k and is injective and smooth up to the boundary of the simplex. By abuse of language we call $P(\Sigma)$ a polyhedron too. Images of simplices of Σ will be called faces of P and these two objects will often be implicitly identified.

The volume function. In case a polyhedron P in a simply connected pseudo-Riemannian space form $M^{n+1}_\nu(K)$ is embedded and (its image) bounds a uniquely determined compact domain then naturally the volume of P is set to be the volume of that domain. However, in general, a compact embedded polyhedron does not necessarily bound a compact domain because the spaces $M^{n+1}_\nu(K)$ are in general not contractible. This happens, for instance, in the de Sitter space for an important class of convex and compact polyhedra with spacelike faces, namely those which are the duals of convex compact polyhedra in hyperbolic space through the classical duality between the hyperbolic space and the de Sitter space (cf. [16]). Moreover we are considering general polyhedra which are not necessarily embedded. One way to overcome the difficulty in defining the volume in this general situation is to observe that the *variation of volume* always makes sense. We proceed as in [5], where the problem is tackled for smooth hypersurfaces.

By a *smooth deformation* P_t of a polyhedron $P : \Sigma \to M^{n+1}_\nu(K)$ we mean a continuous map $\Psi : [0, t] \times \Sigma \to M^{n+1}_\nu(K)$ such that $\psi_t = \Psi(t, .)$ defines a polyhedron P_t for each t and $\psi_0 = P$. We assume moreover that for each simplex Δ of Σ, the restriction of Ψ to $[0, t] \times \Delta$ is smooth up to the boundary. A face of a polyhedron P is said to be nondegenerate if the metric induced on it is nondegenerate. Consider a smooth deformation P_t of an oriented polyhedron P. The *volume function* (or maybe more adequately the *balance of volume*, cf. [6]) is by definition the function $V:[0,t]\to\mathbb{R}$ given by:

$$V_t = \int\limits_{[0,t]\times P} \Psi^\star\, dM,$$

dM being the volume form on $M_\nu^{n+1}(K)$.

Let $\xi_t(p) = \frac{\partial\Psi}{\partial t}(t,p),\ \ p\in\Sigma$, be the deformation vector field of Ψ. Then, we have the following formula for the variation of the volume similar to one known in the smooth case (cf. [5]):

Lemma 1. *Let P be a compact oriented polyhedron in $M_\nu^{n+1}(K)$ with nondegenerate n-dimensional faces and P_t a smooth deformation of P through polyhedra satisfying the same nondegeneracy condition. The variation of the volume is then given by*:

$$\frac{dV_t}{dt} = \sum_j \varepsilon\big(N_j(t)\big) \int\limits_{F_{j,t}} \big\langle \xi_t, N_j(t)\big\rangle\, dF_{j,t},$$

where j runs over the n-dimensional faces $F_{j,t}$ of P_t, $dF_{j,t}$ being the volume form on $F_{j,t}$ and $N_j(t)$ the unit vector field normal to $F_{j,t}$ compatible with the orientation on P.

Remarks.

(1) As usual, the unit field N normal to the face F of P is said to be compatible with the orientation on P if for a positive basis $\{e_1,\dots,e_n\}$ of T_xF, $x\in F$, the basis $\{e_1,\dots,e_n,N\}$ is positive for the orientation of the ambiant space.
(2) Note that by continuity $\varepsilon(N_j(t)) = \varepsilon(N_j)$ for all t through the deformation.

Proof. It is enough to prove the formula for $t=0$. Since the faces of dimension less than n have measure zero in P, we have $V_t = \sum_j \int_{[0,t]\times F_j} \Psi^\star\, dM$. Let $x\in F_j$ be a fixed point and $e_1,\dots,e_n,e_{n+1}=N_j$ a positive orthonormal frame around x. Then, $\Psi^\star\, dM = f(t,x)\,dt\wedge dF_j$, where

$$f(t,x) = \Psi^\star\, dM\left(\frac{\partial}{\partial t}, e_1,\dots,e_n\right) = dM\left(\frac{\partial\Psi}{\partial t}, d\psi_t(e_1),\dots,d\psi_t(e_n)\right)$$

and so:

$$f(0,x) = \varepsilon\big(N_j(0)\big)\big\langle \xi_0, N_j(0)\big\rangle.$$

Therefore:

$$\left.\frac{dV_t}{dt}\right|_{t=0} = \sum_j \left.\frac{d}{dt}\right|_{t=0}\Bigg\{\int\limits_{[0,t]\times F_j} f(t,x)\,dF_j\Bigg\} = \sum_j \int\limits_{F_j} f(0,x)\,dF_j$$

$$= \sum_j \varepsilon\big(N_j(0)\big)\int\limits_{F_j}\big\langle \xi_0, N_j(0)\big\rangle\, dF_j. \qquad \square$$

Remark. It is not difficult to check that the formula is the same if we consider the volume bounded by the polyhedra P_t in case they bound compact domains and orient them by the exterior normal. Also any reasonable definition of the volume, for instance the generalized volume of a polyhedron in the Euclidean space or in the Minkowski space (cf. [3]), leads to the same formula for the variation of the volume.

2. The Schläfli formula for polyhedra

We will treat in a unified way the Riemannian and pseudo-Riemannian cases although the Riemannian case is simpler. We first fix some notations. Let P be an oriented polyhedron in $M_\nu^{n+1}(K)$ and let F_1, F_2 be two adjacent faces of dimension n sharing a face G of dimension $n-1$. Let N_i be the unit normal to F_i, compatible with the orientation on P, and ν_i the unit outward conormal to G in F_i, $i=1,2$. Call Π_i, for $i=1,2$, the totally geodesic hypersurface of $M_\nu^{n+1}(K)$ containing F_i. Then, using the models for $M_\nu^{n+1}(K)$ described before, it can be checked that, for each $x, y \in G$, there is an orientation preserving isometry of $M_\nu^{n+1}(K)$ keeping Π_1 and Π_2 globally invariant, sending x to y and sending $N_i(x)$ to $N_i(y)$ and $\nu_i(x)$ to $\nu_i(y)$, $i=1,2$. Therefore the geometry of the polyhedron P at $x \in G$ does not depend on the choice of $x \in G$. This is the case, in particular, for the 2-plane orthogonal—in $T_x M_\nu^{n+1}(K)$—to $T_x G$, $x \in G$. So for simplicity, for instance, we will say G is orthogonally definite (respectively Lorentzian) if the scalar product induced on the 2-plane orthogonal to $T_x G$, $x \in G$, is definite (respectively Lorentzian).

We now make precise the notion of dihedral angle.

The signed dihedral angle. In order to be able to define dihedral angles we will restrict ourselves to oriented polyhedra P in $M_\nu^{n+1}(K)$ satisfying the following two conditions:

(C1) *the metric induced on each face of dimension n or $n-1$ of P is nondegenerate,*
(C2) *for each face G of dimension $n-1$ of P which is orthogonally definite, the unit normals N_1 and N_2—compatible with the orientation on P—to the two n-dimensional faces F_1 and F_2 sharing G as a common face satisfy $N_1 + N_2 \neq 0$.*

Condition (C1)—which was considered in [23] and is superfluous in the Riemannian case—is rather natural and allows us to define the angle between the normals to adjacent n-dimensional faces. Condition (C2) means that, for a face G which is orthogonally definite, the intersection of the two adjacent faces at G is reduced to G. For a face G violating that condition, the normals to F_1 and F_2 satisfy: $N_1 = -N_2$ and in this case the dihedral angle cannot be defined in a coherent way as it will be clear below. We will attach to each face of dimension $n-1$ of P a signed dihedral angle. It is a real number which depends on the chosen orientation on P and represents in some sense the *mean curvature* of the polyhedron along that face. Note that when one considers compact embedded polyhedra in simply connected space forms it is enough to use the usual interior dihedral angles to state the Schläfli formula. However, if one wants to consider the total mean curvature, then the interior dihedral angle is no longer the appropriate one. Moreover we are considering general polyhedra which do not necessarily bound. The notion of dihedral angle is therefore a little more subtle. We first need to recall the notion of angle between two vectors in the Minkowski plane $\mathbb{R}^2_1$. Such angles were used by Alexandrov [3], Schlenker [20] and Suarez-Peiró [23]. In the definition of Alexandrov and that of Schlenker, the angle is a complex number, whereas Suarez-Peiró uses real numbers. Although not defined in the same way, the notions of angle introduced by these authors are basically equivalent up to some conventions. Indeed, it is, for instance, straightforward to check that the non-oriented angle used by V. Alexandrov (see the definition below) is equal to the modulus of the real part of the angle used by J.-M. Schlenker and is equal to the modulus of the angle used by E. Suarez-Peiró. To avoid distinguishing several cases, we will rather follow the exposition of V. Alexandrov. The oriented angle is a complex multivalued function, which assigns to two non-null nonzero vectors $u, v \in \mathbb{R}^2_1$ a number of the form $\angle uv = \theta_0 - ik\frac{\pi}{2}, k \in \mathbb{Z}, \theta_0 \in \mathbb{R}$, which satisfies the relation: $\langle u, v\rangle = |u||v| \cosh \angle uv$. It has the following properties:

(i) it is additive: if u, v, w are three non-null nonzero vectors and $\angle uv = \theta_1 + ik_1\frac{\pi}{2}$ and $\angle vw = \theta_2 + ik_2\frac{\pi}{2}$, then there exists $n \in \mathbb{Z}$ such that: $\angle uw = (\theta_1 + \theta_2) + i\frac{(k_1+k_2)\pi}{2} + 2\pi in$.
(ii) if the ordered pair of non-null and nonzero vectors u, v is positively oriented and satisfy $\langle u, v\rangle = 0$ (i.e., they are orthogonal) then $\angle uv = -i\frac{\pi}{2} + 2\pi in,\ n \in \mathbb{Z}$.

We refer to [3] for more details. We just mention that an alternate way to introduce the oriented angle in the Minkowski plane is as follows: first note that it is enough to define it for normalized vectors, that is, vectors of norm 1 or i. Consider then an ordered pair of such vectors u and v. If they are of the same type then there exists a unique orientation preserving linear isometry taking u to v. This isometry is a hyperbolic rotation of some real angle θ_0. If u and v are not of the same type then there is a unique orientation preserving anti-isometry taking u to v, and again such an anti-isometry is determined by some real angle θ_0. Now requiring the property $\langle u, v\rangle = |u||v| \cosh \angle uv$ leads naturally to define the angle as a (multivalued) complex function of the form $\angle uv = \theta_0 - ik\frac{\pi}{2}, k \in \mathbb{Z}$, depending on the position of u and v on the different branches of the hyperbolas of normalized vectors. It is not very difficult to check that this definition agrees with the one given by Alexandrov [3] and, for instance, property (i) then follows from the fact that the set of linear and anti-linear isometries is a group under composition.

We shall also need the notion of non-oriented angle between two vectors u, v in a 2-plane Π endowed with a definite or Lorentzian scalar product (cf. [3]). Fix some orientation on Π. Consider first the case where Π is endowed with a definite scalar product and order the vectors u, v so that the oriented angle (with respect to Π) between them is of the form $\phi + 2k\pi,\ 0 \leqslant \phi \leqslant \pi,\ k \in \mathbb{Z}$. Then ϕ is the non-oriented angle between u and v. Consider now the case where Π carries a Lorentz scalar product and order the two vectors u, v so that the real part θ_0 of the oriented angle between u, v is positive. Then, $\phi := \theta_0$ is called the non-oriented angle between u and v. Note that, in both cases, ϕ is independent of the choice of orientation on the plane Π.

Consider now an *oriented polyhedron* P in $M^{n+1}_\nu(K)$ satisfying conditions (C1) and (C2) and G an $(n-1)$-dimensional face of P which is the common face of two n-dimensional faces F_1 and F_2. By assumption (C1) the 2-plane Π_x orthogonal to T_xG, $x \in G$, is nondegenerate. Denote by ϕ the non-oriented angle (in Π_x) between the normals N_1 and N_2 to F_1 and F_2—recall that this is independent of the choice of $x \in G$. Fix an orientation on Π_x. In case $\phi = 0$, we set, by definition, the *signed dihedral angle* θ to be zero. Now consider the case $\phi \neq 0$. Interchanging indices if necessary, we can assume that $\angle N_1N_2 = \phi + 2k\pi, k \in \mathbb{Z}, 0 \leqslant \phi \leqslant \pi$, in case Π_x is definite and $\phi = \Re\angle N_1N_2$ in case Π_x is Lorentzian. We say that G is of positive type if the ordered basis $\{\nu_1, N_1\}$ is positively oriented and of negative type in the opposite case. Observe that because our polyhedron is oriented, the two ordered bases $\{\nu_1, N_1\}$ and $\{\nu_2, N_2\}$ have opposite orientations. Note also that in the definite case condition (C2) is necessary, otherwise we would have $N_1 = -N_2$ and $\phi = \pi$, and there is no canonical way to distinguish between N_1 and N_2. So, in that case, one cannot distinguish between the positive and the negative type and therefore between the two possible values for θ, namely π and $-\pi$. In case $\phi = 0$, the problem is solved because the positive and the negative type both lead to the same value for θ, namely zero. Again, the type of G depends neither on the point $x \in G$ nor on the orientation on Π_x. Now, the *signed dihedral angle* θ at the face G is, by definition, $\theta = \phi$ if G is of positive type and $\theta = -\phi$ if G is of negative type. For instance, for an embedded polyhedron P in a space form, oriented by the exterior normal, then our signed dihedral angles are given by $\theta_i - \pi$, where θ_i denote the interior dihedral angles.

The general Schläfli formula reads then as follows:

Theorem 2. *Let P_t be a smooth deformation of a compact oriented polyhedron P in $M_\nu^{n+1}(K)$, satisfying conditions* (C1) *and* (C2)*, through polyhedra satisfying the same conditions. Let $\theta_{i,t}$ be the signed dihedral angle at the $(n-1)$-dimensional face $G_{i,t}$ of P_t. Then the signed dihedral angles vary in a differentiable way and their variations are related to the variation of volume by the following formula*:

$$nK\frac{dV_t}{dt} = \sum_i \frac{d\theta_{i,t}}{dt}\,\mathrm{Vol}(G_{i,t}).$$

Proof. Fix some value t_0 of the parameter t. To simplify notations we will sometimes drop the reference to the parameter t for $t = t_0$ and we will identify (metrically) Σ—the abstract simplicial complex parameterizing P—with its image $\psi_{t_0}(\Sigma)$. The basic observation is that the deformation is through *polyhedra*, so each face F, of dimension n, remains totally geodesic through the deformation. Call N_t the unit normal to F_t compatible with the orientation on P. For each $u \in TF$ and each t, we have: $D_{d\psi_t(u)}N_t = 0$, where D denotes the Levi-Civita connection on $M_\nu^{n+1}(K)$. Therefore: $D_{\frac{\partial\Psi}{\partial t}}D_{d\psi_t(u)}N_t = 0$. This can be rewritten as follows:

$$D_{d\psi_t(u)}D_{\frac{\partial\Psi}{\partial t}}N_t + R\left(d\Psi\left(\frac{\partial}{\partial t}\right), d\psi_t(u)\right)N_t = 0,$$

where R is the curvature tensor of $M_\nu^{n+1}(K)$. Taking the value at $t = t_0$, we get:

$$D_u N' + R(\xi, u)N = 0,$$

where as before: $\xi(x) = \frac{\partial\Psi}{\partial t}(t_0, x),\ x \in P$, is the deformation vector field of Ψ at $t = t_0$ and $N' = \frac{D}{dt}N(t_0)$. Note that since N_t is unitary for each t, N' is tangent to F. It follows that

$$\mathrm{div}_F(N') = nK\langle \xi, N\rangle.$$

We now apply the divergence theorem. First, we apply Stokes' theorem, which is valid for manifolds with piecewise smooth boundary (see for instance [1]) and then we can apply the divergence theorem in pseudo-Riemannian manifolds since the set where the metric degenerates has measure zero (cf. [24]). We get:

$$\sum_i \varepsilon(\nu_i)\int_{G_i}\langle N', \nu_i\rangle\, dG_i = nK\int_F \langle \xi, N\rangle\, dF,$$

the sum being taken over $(n-1)$-dimensional faces G_i of F and ν_i being the unit outward conormal to G_i in F. Now, multiply both members by $\varepsilon(N)$ and then sum over n-dimensional faces F_j of P. Note that in the left-hand side of the previous equation, each face G_i appears twice since it belongs to two

n-dimensional faces $F_{i,1}$ and $F_{i,2}$. With obvious notations, we end with:

$$\sum_i \int_{G_i} \left\{\varepsilon(N_{i,1})\varepsilon(\nu_{i,1})\langle N'_{i,1}, \nu_{i,1}\rangle + \varepsilon(N_{i,2})\varepsilon(\nu_{i,2})\langle N'_{i,2}, \nu_{i,2}\rangle\right\} dG_i = nK \sum_j \varepsilon(N_j) \int_{F_j} \langle \xi, N_j\rangle \, dF_j$$

where i runs over $(n-1)$-dimensional faces and j runs over n-dimensional faces.

Taking into account Lemma 1, to complete the proof we have to show that:

$$\varepsilon(N_{i,1})\varepsilon(\nu_{i,1})\langle N'_{i,1}, \nu_{i,1}\rangle + \varepsilon(N_{i,2})\varepsilon(\nu_{i,2})\langle N'_{i,2}, \nu_{i,2}\rangle = \frac{d\theta_{i,t}}{dt}. \tag{2.1}$$

To simplify notations, we drop the index i and consider a general $(n-1)$-dimensional face G which is the intersection of two n-dimensional faces F_1 and F_2. Since $N_k(t)$ is unitary for each t, N'_k is tangent to F_k. So, we may write:

$$N'_k = \varepsilon(\nu_k)\langle N'_k, \nu_k\rangle \nu_k + u_k, \quad \text{where } u_k \in TG,\ k = 1, 2.$$

We now distinguish two cases:

(i) G is orthogonally definite: fix an orientation on the 2-plane Π_x orthogonal to T_xG (for any fixed $x \in G$). Let ϕ_t be the non-oriented angle between $N_1(t)$ and $N_2(t)$. Assume first $\phi_{t_0} \neq 0$. Recall that by assumption (C2), we also have $\phi_{t_0} \neq \pi$. Interchanging indices if necessary, we can assume ϕ_t coincides with (some determination of) the oriented angle between $N_1(t)$ and $N_2(t)$: $\angle N_1(t)N_2(t) = \phi_t$ for t lying in an *open* interval containing t_0 by continuity. So, ϕ_t is differentiable on that interval since $\phi_t = \arccos\langle N_1(t), N_2(t)\rangle$ and $\langle N_1(t), N_2(t)\rangle \neq \pm 1$ by assumption. Taking the derivatives in the equation: $\langle N_1(t), N_2(t)\rangle = \cos\phi_t$, at $t = t_0$, we get: $-\frac{d\phi_t}{dt}\sin\phi_{t_0} = \langle N'_1, N_2\rangle + \langle N_1, N'_2\rangle$. But $\langle N'_1, N_2\rangle = \langle N'_1, \nu_1\rangle\langle \nu_1, N_2\rangle$. Now $\langle \nu_1, N_2\rangle = \cos\angle \nu_1 N_2 = \cos(\angle \nu_1 N_1 + \angle N_1 N_2) = \mp \sin\phi_{t_0}$ according to G being of positive or negative type respectively. Therefore,

$$\frac{d\phi_t}{dt}\sin\phi_{t_0} = \pm\left\{\langle N'_1, \nu_1\rangle + \langle N'_2, \nu_2\rangle\right\}\sin\phi_{t_0}$$

according to G being of positive or negative type respectively. Since we assumed $\phi_{t_0} \neq 0, \pi$, we can conclude that at $t = t_0$:

$$\pm\frac{d\phi_t}{dt} = \langle N'_1, \nu_1\rangle + \langle N'_2, \nu_2\rangle \tag{2.2}$$

according to G being of positive or negative type respectively. Now, if G is of positive (respectively negative) type at $t = t_0$, then the same is true for t in a neighborhood of t_0 and so $\theta_t = \phi_t$ (respectively $\theta_t = -\phi_t$) on this interval and (2.1) follows.

Now in case $\phi_{t_0} = 0$, we can assume without loss of generality that the basis $\{\nu_1, N_1\}$ is, for $t = t_0$, positively oriented. This is also true by continuity for t in a neighborhood of t_0. Consider the determination $\alpha(t)$ of the oriented angle $\angle N_1(t)N_2(t)$, for t close to t_0, such that $\alpha(t_0) = 0$. Then it can be checked directly that $\theta(t) = \alpha(t)$. This shows differentiability of $\theta(t)$ at $t = t_0$. Now if ϕ_t is not identically zero near t_0 then (2.1) is satisfied at t_0 by the previous case and continuity in t. If ϕ_t is identically zero near t_0, then near t_0, $N_1(t) = N_2(t)$ and $\nu_1(t) = -\nu_2(t)$ and again (2.1) is trivially satisfied.

(ii) G is orthogonally Lorentzian: Let ϕ_t be the non-oriented angle between $N_1(t)$ and $N_2(t)$. Assume first that $\phi_{t_0} \neq 0$. Fixing some orientation on Π_x and interchanging indices if necessary, we can assume $\phi_t = \Re\beta_t$, where β_t is some determination of $\angle N_1(t)N_2(t)$ for t in a neighborhood of t_0.

Take the derivative with respect to t in the relation:

$$\big\langle N_1(t), N_2(t)\big\rangle = \big|N_1(t)\big|\big|N_2(t)\big| \cosh\beta_t,$$

taking into account that $|N_i(t)| = |N_i|$ for all t by the nondegeneracy condition and continuity and that $\frac{d\beta_t}{dt} = \frac{d\phi_t}{dt}$. We obtain at $t = t_0$:

$$\langle N_1', N_2\rangle + \langle N_1, N_2'\rangle = |N_1||N_2|\frac{d\phi_t}{dt}\sinh\beta_{t_0}. \tag{2.3}$$

But

$$\begin{cases} \langle N_1', N_2\rangle = \varepsilon(\nu_1)\langle N_1', \nu_1\rangle\langle N_2, \nu_1\rangle, \\ \langle N_2', N_1\rangle = \varepsilon(\nu_2)\langle N_2', \nu_2\rangle\langle N_1, \nu_2\rangle. \end{cases} \tag{2.4}$$

Now, $\langle N_2, \nu_1\rangle = |N_2||\nu_1| \cosh\angle\nu_1 N_2$, and $\cosh\angle\nu_1 N_2 = \cosh(\angle\nu_1 N_1 + \angle N_1 N_2) = \cosh(\mp i\frac{\pi}{2} + \beta_{t_0}) = \mp i|N_1||N_2|\sinh\beta_{t_0}$, according to G being of positive or negative type respectively. Moreover, we have $|\nu_1| = -i\varepsilon(\nu_1)|N_1|$. We end with:

$$\begin{cases} \langle N_2, \nu_1\rangle = \mp\varepsilon(\nu_1)|N_1||N_2|\sinh\beta_{t_0}, \\ \langle N_1, \nu_2\rangle = \mp\varepsilon(\nu_2)|N_1||N_2|\sinh\beta_{t_0} \end{cases} \tag{2.5}$$

according to G being of positive or negative type respectively.

From (2.3), (2.4) and (2.5), we get:

$$\big\{\langle N_1', \nu_1\rangle + \langle N_2', \nu_2\rangle\big\}\sinh\beta_{t_0} = \mp\frac{d\phi_t}{dt}\sinh\beta_{t_0}.$$

Now, since we assumed $\phi_{t_0} \neq 0$, we have $\beta_{t_0} \neq ik\pi,\ k \in \mathbb{Z}$, this implies that

$$\langle N_1', \nu_1\rangle + \langle N_2', \nu_2\rangle = \mp\frac{d\phi_t}{dt}. \tag{2.6}$$

If G is of positive (respectively negative) type then the same is true for t in an interval containing t_0 and $\theta_t = \phi_t$ (respectively $\theta_t = -\phi_t$) on that interval. Since $\{\nu_k, N_k\}$ is an orthonormal basis of the orthogonal of G, which is Lorentz, we have: $\varepsilon(\nu_k)\varepsilon(N_k) = -1$, $k = 1, 2$ and (2.1) follows.

Consider now the case $\phi_{t_0} = 0$, we can assume without loss of generality that the basis $\{\nu_1, N_1\}$ is, for $t = t_0$, positively oriented for a fixed orientation on Π_x. This is then true for t close to t_0 by continuity. Consider the determination $\alpha(t)$ of the oriented angle $\angle N_1(t)N_2(t)$, for t close to t_0, satisfying $\alpha(t_0) = 0$ in case $N_1(t_0) = N_2(t_0)$ and $\alpha(t_0) = -i\pi$ in case $N_1(t_0) = -N_2(t_0)$. Then it is directly checked that $\theta(t) = \Re\alpha(t)$, for t close to t_0. This shows differentiability of $\theta(t)$ at t_0.

If ϕ_t is not identically zero near t_0 then (2.1) is satisfied at t_0 by the previous case and continuity in t.

If ϕ_t vanishes identically near t_0, then either $\alpha(t)$ is identically zero and then $N_1(t) = N_2(t)$ and $\nu_1(t) = -\nu_2(t)$ for t near t_o, or $\alpha(t)$ is identically $-i\pi$ and hence $N_1(t) = -N_2(t)$ and $\nu_1(t) = \nu_2(t)$ for t near t_0. In both cases Eq. (2.1) is trivially satisfied (the both members vanish). □

Remarks.

(1) The Euclidean version of the Schläfli formula was rediscovered by T. Regge in a celebrated paper [14]. His proof, which is quite hard to follow, uses the divergence theorem. It should be pointed out that, in this case, his argument is similar to our ours, although presented differently. Indeed, using the

divergence theorem, T. Regge establishes first the Minkowski formula: in the Euclidean case, for a face F of dimension n, one has $\sum \mathrm{Vol}(G_i)\nu_i = 0$, where the sum is taken over the $(n-1)$-faces G_i of F (cf. Eq. (6) in [14]). Then he derives the same equation as our Eq. (2.1) and then plugging this into the sum $\sum_i \frac{d\theta_{i,t}}{dt} \mathrm{Vol}(G_{i,t})$, he observes that summing instead over the n-faces, one obtains for each such face F a term of the form $\langle N', \sum \mathrm{Vol}(G_i)\nu_i\rangle$, and this concludes the proof. It should also be noticed that the same type of calculation, using the Minkowski formula, was done in [2] and [3] to prove the invariance of the total mean curvature in the Euclidean and Minkowski spaces respectively.

(2) As we mentioned in the introduction, J.-M. Schlenker and the author [21] have found higher order Schläfli-type formulae for embedded polyhedra in space forms. The proof is through analoguous smooth formulae. It would be interesting to extend the idea we used here to find a different proof of those formulae. This would apply to polyhedra that are not necessarily embedded as well.

The total mean curvature of a compact oriented polyhedron P in $M_\nu^{n+1}(K)$ is, using the previous notations, by definition the quantity:

$$\mathbf{H} = \frac{1}{n} \sum_i \theta_i \,\mathrm{Vol}(G_i).$$

Recall that a deformation of a polyhedron of dimension n is called a flex if it does not change the metric on each of its n-dimensional faces, that is, it changes only the dihedral angles. A direct consequence of Theorem 2 is the following corollary which extends in the pseudo-Riemannian case the result obtained by Alexandrov [3] for polyhedra in the Minkowski 3-space, $\mathbb{R}^3_1$.

Corollary 3. *Let P_t be a flex of a compact oriented polyhedron P in $M_\nu^{n+1}(K)$ through polyhedra satisfying conditions* (C1) *and* (C2). *Then*

$$K \frac{dV_t}{dt} - \frac{d\mathbf{H}_t}{dt} = 0.$$

Remark. If the polyhedron P bounds a compact domain, the corollary asserts the invariance under flex of the quantity $KV - \mathbf{H}$, where V is the volume of the domain bounded by P and $\mathbf{H}$ the mean curvature with respect to the exterior normal.

3. The Schläfli formula for piecewise smooth hypersurfaces

Let $(M, \langle,\rangle)$ be an orientable Einstein manifold of dimension $n+1$, $n \geqslant 2$, and let Σ be a simplicial complex of dimension n which is homeomorphic to a closed oriented differentiable manifold. Consider a mapping $\psi : \Sigma \to M$ such that its restriction to each simplex of dimension n or $n-1$ of Σ is an immersion which is smooth up to the boundary. We will call such a data an immersed piecewise smooth, closed and oriented hypersurface in M. In all what follows we endow the simplices of Σ of dimension n and $n-1$ with the metric induced by ψ. Consider a simplex G_i of Σ of dimension $n-1$ and let $F_{i,1}$ and $F_{i,2}$ be the two faces of Σ, of dimension n, sharing G_i as a common face. Let $N_{i,1}$ and $N_{i,2}$ be the unit normals to $\psi(F_{i,1})$ and $\psi(F_{i,2})$ respectively, and which are compatible with the orientation on Σ and denote by $\nu_{i,1}$ and $\nu_{i,2}$ the exterior unit conormals to G_i in $F_{i,1}$ and $F_{i,2}$ respectively. At each point x of G_i, call Π_x the 2-plane in T_xM orthogonal to $\psi_*(T_xG_j)$ and $\phi_i(x)$ the non-oriented angle (in Π_x)

between $N_{i,1}$ and $N_{i,2}$. Fix some orientation on Π_x. Reindexing if necessary, we can assume that $\phi_i(x)$ coincides with a well chosen determination of the oriented angle, that is, $\phi_i(x) = \angle N_{i,1}N_{i,2}$. Suppose $\phi_i(x) \neq 0, \pi$, we say that x is of positive type if the ordered basis $\{\nu_{i,1}, N_{i,1}\}$ is positively oriented and of negative type in the opposite case. The *signed dihedral angle* $\theta_i(x)$ at x is, by definition, $\theta_i(x) = \phi_i(x)$ if x is of positive type and $\theta_i(x) = -\phi_i(x)$ if x is of negative type. The type of x and hence the signed dihedral angle do not depend on the choice of orientation on Π_x. In case $\phi_i(x) = 0$, we set $\theta_i(x) = 0$. If $\phi_i(x) = \pi$, the signed dihedral angle is not defined and x is said to be *cusp point*. The signed dihedral angle cannot be defined in a coherent way in the previous case as there is no canonical way to distinguish between the two normals and therefore between the two possible values π and $-\pi$.

Consider now a smooth deformation of such a hypersurface which preserves the decomposition into smooth parts. This means we are given a mapping $\Psi : [0,1] \times \Sigma \to M$, such that for each simplex F of dimension n of Σ, the restriction of Ψ to $[0,1] \times F$ is smooth up to the boundary and such that, for each t, $\psi_t := \Psi(t,.) : \Sigma \to M$ defines a piecewise smooth immersion as above and $\psi_0 = \psi$.

Denote by N_t the unit field normal to the smooth parts of $\psi_t(\Sigma)$ which is compatible with the orientation on Σ. Let also V_t be the volume function defined in the same way as in Section 2. In case the mappings ψ_t are embeddings and $\psi_t(\Sigma)$ bound and are oriented by the exterior normal then V_t can be taken to be the enclosed volume. Denote by I_t the first fundamental form of ψ_t and by II_t and H_t its second fundamental form and mean curvature respectively, with respect to the normal N_t (strictly speaking, these objects are defined only on the smooth parts of Σ). We implicitly identify, through the metric, quadratic forms and linear morphisms. Our generalized Schläfli formula, which extends the one obtained for smooth hypersurfaces by Rivin and Schlenker [17], reads as follows:

Theorem 4. *Let $\psi : \Sigma \to M$ be a compact oriented piecewise smooth immersed hypersurface without cusp points in an orientable Einstein $(n+1)$-manifold M with scalar curvature S. Consider a smooth deformation of ψ, through piecewise smooth immersions without cusp points, and preserving its decomposition into smooth parts. Then the signed dihedral angle functions $\theta_{i,t}$ are differentiable in t and their variations and the variations of the volume, the mean curvature and metric on Σ are related by the formula*:

$$\frac{S}{n+1}\frac{dV_t}{dt} = \int_{\Sigma} \left\{ nH_t' + \frac{1}{2}\langle I_t', II_t\rangle \right\} dA_t + \sum_i \int_{G_i} \frac{d\theta_{i,t}}{dt}(x)\, dx_t,$$

where i runs over simplices of dimension $n-1$ of Σ.

To simplify notations we assume $t = 0$ and drop the reference to the parameter t at $t = 0$. We also identify (metrically) Σ with $\psi_0(\Sigma)$. To prove the theorem we need to use the following formula:

$$nH' = -\mathrm{div}_\Sigma(N') - \frac{1}{2}\langle I', II\rangle + \frac{S}{n+1}\langle \xi, N\rangle \tag{3.1}$$

where $\xi(x) = \frac{\partial \Psi}{\partial t}(0,x)$, $x \in \Sigma$, is the deformation vector field of Ψ at $t = 0$. Formula (3.1) is a consequence of the known formula (3.2) below. We include the proof for reader's convenience. Denote by B_t the shape operator associated to ψ_t with respect to the normal N_t and let R be the curvature tensor of M.

Proposition 5.

$$\langle B'u, v\rangle = -\langle \nabla_u N', v\rangle - \langle D_{Bu}\xi, v\rangle + \big\langle R(\xi, u)N, v\big\rangle, \quad u, v \in T\Sigma. \tag{3.2}$$

Proof. Call g_t the metric induced on (the smooth parts of) Σ by ψ_t. For all $u, v \in T\Sigma$, we have by definition of B_t:

$$g_t(B_t u, v) = -\big\langle D_{d\psi_t(u)} N_t, d\psi_t(v)\big\rangle.$$

Taking the derivative at $t = 0$, we get:

$$\begin{aligned}\left(\frac{d}{dt} g_t\right)\bigg|_{t=0}(Bu, v) + g_0(B'u, v) &= -\langle D_{\frac{\partial\Psi}{\partial t}} D_{d\psi_t(u)} N_t, v\rangle - \big\langle D_u N, D_{\frac{\partial\Psi}{\partial t}} d\psi_t(u)|_{t=0}\big\rangle\\ &= -\langle D_u D_{\frac{\partial\Psi}{\partial t}} N_t, v\rangle + \big\langle R(\xi, u)N, v\big\rangle - \langle D_u N, D_v\xi\rangle.\end{aligned}$$

Now, $(\frac{d}{dt} g_t)|_{t=0}(Bu, v) = \langle D_{Bu}\xi, v\rangle + \langle Bu, D_v\xi\rangle$. Recollecting we obtain formula (3.2). □

Proof of Theorem 4. Let F be a simplex of dimension n of Σ. Integrating (3.1) on F and using the divergence theorem, which is valid in our case (cf. [1]), we obtain:

$$\frac{S}{n+1}\int_F \langle \xi, N\rangle\, dA = \int_F \left\{nH' + \frac{1}{2}\langle I', II\rangle\right\} dA + \sum_i \int_{G_i} \langle N', \nu_i\rangle\, dx \tag{3.3}$$

where i runs over simplices G_i of dimension $n-1$ lying on the boundary of F and ν_i is the unit outward conormal to G_i in F. Take now the sum over n-dimensional faces of Σ. Note that in the right-hand side of the previous equation, each simplex G_i of dimension $n-1$ appears twice since it belongs to two n-dimensional faces $F_{i,1}$ and $F_{i,2}$. With obvious notations, we get:

$$\frac{S}{n+1}\int_\Sigma \langle \xi, N\rangle\, dA = \int_\Sigma \left\{nH' + \frac{1}{2}\langle I', II\rangle\right\} dA + \sum_i \int_{G_i} \left\{\langle N'_{i,1}, \nu_{i,1}\rangle + \langle N'_{i,2}, \nu_{i,2}\rangle\right\} dx,$$

where j runs over n-dimensional simplices and i runs over $(n-1)$-dimensional ones.

Now, proceeding in the same way as in the proof of Theorem 2, case (i), we conclude that, for each point x in G_i such that $\theta_i(x) \neq 0$, the function $\theta_{i,t}(x)$ is differentiable in t at $t = 0$ and its derivative satisfies:

$$\langle N'_{i,1}, \nu_{i,1}\rangle + \langle N'_{i,2}, \nu_{i,2}\rangle = \frac{d\theta_{i,t}}{dt}(x). \tag{3.4}$$

In case $\theta_i(x) = 0$, fixing an orientation on the 2-plane orthogonal to $\psi_*(T_x G_i)$, we can assume without loss of generality that the basis $\{\nu_1, N_1\}$ is positively oriented. This is also true by continuity for t in a neighborhood of 0. Consider the determination $\alpha_{i,t}(x)$ of the oriented angle $\angle N_{i,1}(t)N_{i,2}(t)$, for t close to 0, such that $\alpha_{i,0}(x) = 0$. Then it can be checked directly that $\theta_{i,t}(x) = \alpha_{i,t}(x)$. This shows differentiability of $\theta_{i,t}(x)$ at $t = 0$. Now if $\phi_{i,t}(x)$ is not identically zero near 0, then using the previous case and continuity in t we see that (3.4) is satisfied at 0. If $\phi_{i,t}(x)$ is identically zero near 0, then near 0, $N_{i,1}(t) = N_{i,2}(t)$ and $\nu_{i,1}(t) = -\nu_{i,2}(t)$ and again Eq. (3.4) is trivially satisfied (the both members vanish).

Finally we have:

$$\frac{S}{n+1}\int_{\Sigma}\langle\xi,N\rangle\,dA=\int_{\Sigma}\left\{nH'+\frac{1}{2}\langle I',II\rangle\right\}dA+\sum_i\int_{G_i}\frac{d\theta_{i,t}}{dt}(x)\,dx,$$

and the result now follows because $\frac{dV_t}{dt}=\int_{\Sigma}\langle\xi,N\rangle\,dA$. □

Remark. The same proof shows that the result is true under some weaker hypotheses. Assume, for instance, that the set of cusp points of the immersion ψ has measure zero in $\bigcup_i G_i$, then the signed dihedral angle function $\theta_{i,t}(x)$ is differentiable at $t=0$ for almost every $x\in\bigcup_i G_i$ and the formula is valid. Also one can weaken the regularity hypotheses. For instance one can replace everywhere the smoothness condition by a C^2 one. One can even assume less regularity on ψ on the boundary of the simplices of dimension n by requiring differentiability only on the interior of the simplices of dimension $n-1$. But then one has to put adequate hypotheses which guarantee the convergence of the integrals involved in the formulae. We do not go into the details here in order to keep the basic ideas clear.

The previous considerations and the notion of the (total) mean curvature of polyhedra lead us naturally to define the *total mean curvature of an oriented piecewise smooth hypersurface* without cusp points as follows:

$$\mathbf{H}=\int_{\Sigma}H\,dA+\frac{1}{n}\sum_i\int_{G_i}\theta_i(x)\,dx.$$

By a bending of $\psi:\Sigma\to M$ we mean a deformation ψ_t of ψ preserving the decomposition into smooth parts and such that the metrics induced on the simplices of dimension n and $n-1$ of Σ remain the same for each t. From the previous theorem, we deduce:

Corollary 6. *Let Σ be a piecewise smooth, compact orientable embedded hypersurface without cusp points in an orientable Einstein $(n+1)$-manifold M with scalar curvature S. Assume Σ bounds a compact domain of volume V. Then the quantity $\frac{S}{n(n+1)}V-\mathbf{H}$, where $\mathbf{H}$ is the total mean curvature of Σ with respect to the exterior normal to Σ, is invariant under bendings through piecewise smooth immersions without cusp points.*

Remark. It is clear that if ψ has no cusp points then this is true for small t for any deformation ψ_t of ψ. Also, the corollary can be stated in a more general form using the notion of variation of volume as before, it is neither necessary to assume that the hypersurface bounds a compact domain nor that it is embedded.

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