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Higher Schläfli Formulas and Applications

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Abstract. The classical Schläfli formula relates the variations of the dihedral angles of a smooth family of polyhedra in a space-form to the variation of the enclosed volume. We give higher analogues of this formula: for each p, we prove a simple formula relating the variation of the volumes of the codimension p faces to the variation of the 'curvature' – the volumes of the duals of the links in the convex case – of codimension $p+2$ faces. It is valid also for ideal polyhedra, or for polyhedra with some ideal vertices. This extends results of Suárez-Peiró. The proof is through analoguous smooth formulas. Some applications are described.

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Let M_K^{n+1} be the spherical, Euclidean or hyperbolic space of constant curvature K and dimension $n+1 \geqslant 2$. Consider a smooth one-parameter family, $(P_t)_{t\in[0,1]}$, of polyhedra in M having the same combinatorics. Call V_t the volume bounded by P_t, $\theta_{i,t}$ and $W_{i,t}$ the dihedral angle and the $(n-1)$-volume of the codimension 2 face i of P_t. The classical Schläfli formula (see, for instance, [Mil] or [Vin]) relates the variation of V_t and of the angles $\theta_{i,t}$ in the following way:

CLASSICAL SCHLÄFLI FORMULA

$$nK\frac{\mathrm{d}V_t}{\mathrm{d}t} = \sum_i W_{i,t}\frac{\mathrm{d}\theta_{i,t}}{\mathrm{d}t}. \tag{E_0}$$

Although it is not apparent here (it should be made clear below), this formula is related to the variation of the 'integral mean curvature' $\sum_i W_{i,t}\theta_{i,t}$ of P_t. We will give here analogs of this formula for the higher mean curvatures. To state our generalized formulas, we need to introduce some definitions. Let F be a codimension p face of a convex polyhedron $P \subset M_K^{n+1}$, and let $x \in F$. Consider, in $(T_xF)^{\perp}$, the subset of all the unit vectors that 'point into' P, i.e. unit vectors that are velocity vectors of the geodesics starting at x that go into the interior of P. Since the unit sphere in $(T_xF)^{\perp}$ is isometric to S^{p-1}, we get a convex polyhedron in S^{p-1}, denoted by $L(F, P)$, which is, up to isometry, independent of the choice of x and is called *the link* *of F in P*. The dual of the link of F in P, denoted by $L^*(F, P)$, is defined to be the dual polyhedron of $L(F, P)$ in S^{p-1}. It is the subset of S^{p-1} – viewed as the unit sphere in R^p – which consists of all the unit vectors making an angle $\geqslant \pi/2$ with all the vectors in $L(F, P)$. The $(p-1)$-dimensional volume $K(F)$ of $L^*(F, P)$ will be called the *curvature of P at F*, or simply the curvature of F.

For instance, for codimension 2 faces, we recover the exterior dihedral angle of P at F, which is a polyhedral analog of the mean curvature of P along those faces.

For vertices of a polyhedron P in a three-dimensional space-form, on the other hand, $K(F)$ is the singular curvature at F of the metric induced on P.

We can now write the:

HIGHER SCHLÄFLI FORMULAS. For each $1 \leqslant p \leqslant n-1$,

$$K(n-p)\sum_j \frac{\mathrm{d}W_{j,t}}{\mathrm{d}t} K_{j,t} + p\sum_i W_{i,t}\frac{\mathrm{d}K_{i,t}}{\mathrm{d}t} = 0, \qquad (E_p)$$

where j runs over the faces of codimension p and i runs over those of codimension $p+2$ and W_i and W_j denote the volumes of the faces, and K_i, K_j their curvatures.

Although the curvature at a face has only been defined here for convex polyhedra, it also has a meaning for nonconvex ones (this is defined below) and the higher Schläfli formulas still hold. The curvatures we use coincide – up to a normalization– with the Lipschitz-Killing curvatures used by J. Cheeger, W. Müller and R. Schrader ([C-M-S]) for piecewise flat spaces. However, our curvatures show up in a natural way when dealing with polyhedra embedded in simply connected space forms.

The same formula was previously found by E. Suarez-Peiro [SP] in the special case of simplices in the de Sitter space. She used a different, more combinatorial method in the proof, but her approach could probably be extended to prove the more general results that are described below.

Recently, the first author and I. Rivin [Ri-S-a], [Ri-S-b] gave a 'smooth Schläfli formula' for deformations of compact hypersurfaces in space-forms. A remarkable point is that this formula extends to deformations of hypersurfaces in Einstein manifolds, and even to deformations of the metric (among Einstein metrics of fixed scalar curvature) in an Einstein manifold with boundary. The (polyhedral) Schläfli formula is then obtained as a corollary.

We follow the same idea to establish the higher Schläfli formulas. We first derive higher smooth Schläfli formulas (Theorem 1) and then obtain the polyhedral ones for convex polyhedra (Theorem 2). The proof of our smooth Schläfli formulas is based on the work of R. Reilly [Re]. The (smooth) higher Schläfli formulas do not seem, however, to extend to deformations of hypersurfaces in Einstein manifolds.

We will then show how the application of a classical duality between H^{n+1} and the de Sitter space S_1^{n+1} leads to the same formulas for some space-like polyhedra in S_1^{n+1}, and indicate why the same formulas hold for nonconvex polyhedra.

We also use Schläfli formulas to prove that some quantities defined using the integral mean curvatures of smooth or polyhedral hypersurfaces are topological invariants. Some of those invariants are related to the Gauss–Bonnet integrands, while others, involving the 'dual volume', seem to be new. One of those formulas was written in [SP] in the case of a hyperbolic simplex.

As another application of our higher Schläfli formulas, we show that some extrinsic quantities for polyhedra and for hypersurfaces are invariant under isometric deformations.

As we mentioned above, J. Cheeger, W. Müller and R. Schrader ([C-M-S]), introduced Lipschitz–Killing curvatures for piecewise flat spaces and used them to approximate Riemannian manifolds. A significant role is played by their generalized Regge formulas which give the variational derivative of the total Lipschitz-Killing curvatures. These formulas coincide with our higher Schläfli formulas when the piecewise flat space is a compact polyhedron in a Euclidean space. Conversely, our higher Schläfli formulas provide generalized Regge formulas for piecewise hyperbolic or spherical spaces. This suggests the possibility to approximate Riemannian manifolds by piecewise hyperbolic or spherical spaces.

1. Smooth Schläfli Formulas

Let Σ be a smooth compact oriented boundaryless n-dimensional manifold and consider an immersion $\phi\colon \Sigma \longrightarrow M_K^{n+1}$. Call I the induced metric on M and ∇ its Levi-Civita connection. Let B denote the shape operator of Σ defined, for any $x \in \Sigma$ and $X \in T_x\Sigma$ by $BX = -D_X N$, where N is the oriented unit normal to Σ and D the Levi-Civita connection on M_K^{n+1}. Denote by II the second fundamental form of ϕ defined by $\mathrm{II}(X, Y) = \langle BX, Y\rangle$.

We will need the elementary symmetric functions S_r of the principal curvatures $k_1, k_2, \dots, k_n$ of the immersion ϕ:

$$S_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \dots k_{i_r} \quad (1 \leqslant r \leqslant n).$$

A deformation of the immersion ϕ is a smooth mapping $\Phi\colon\ [0, 1] \times \Sigma \longrightarrow M_K^{n+1}$ such that $\phi_t\colon\ \Sigma \longrightarrow M$, $t \in [0, 1]$, defined by $\phi_t(x) = \Phi(t, x)$, $x \in \Sigma$, is an immersion and $\phi_0 = \phi$.

Let (after R. Reilly [Re]) $T_r, 0 \leqslant r \leqslant n$, be the Newton transformations (or tensors) defined by $T_r = S_r\mathrm{Id} - S_{r-1}B + \cdots (-1)^r B^r$ or, inductively, by $T_0 = \mathrm{Id}$, $T_r = S_r\mathrm{Id} - BT_{r-1}$. These Newton transformations enjoy the following properties (cf. [Re] or [Ro]):

(1) $\mathrm{Trace}(T_r) = (n - r)S_r$,
(2) $\mathrm{Trace}(BT_r) = (r + 1)S_{r+1}$,
(3) $S_r' = \mathrm{Trace}(B' T_{r-1})$, the derivative being taken with respect to the parameter t of the deformation.

We will often implicitly identify, through the metric, quadratic forms and linear morphisms.

THEOREM 1. *Let Σ be a closed oriented hypersurface in a $(n+1)$-dimensional space-form of constant sectional curvature K. For any deformation of Σ, the corresponding variations of the elementary symmetric functions of the principal curvatures satisfy for each $p \in \{1, \dots, n-1\}$:*

$$K(n-p)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma S_{p-1}\mathrm{d}A = \int_\Sigma \{K(n-p)S'_{p-1} - pS'_{p+1}\}\mathrm{d}A + \\ + \int_\Sigma \tfrac{1}{2}\langle I', B((n-p)KT_{p-2} - pT_p)\rangle \mathrm{d}A \qquad (F_p)$$

where, by convention, $T_{-1} = 0$.

Proof. We need to know the derivative S'_r, for $r \geqslant 1$. First, we have the following formula (see [Ro], where a similar formula is derived for 'normal' deformations but the proof extends easily to general ones):

$$\langle B'u, v\rangle = -\langle \nabla_u N', v\rangle - \langle D_{Bu}\xi, v\rangle + \langle R(\xi, u)N, v\rangle, \quad u, v \in T\Sigma, \qquad (1.1)$$

where R denotes the curvature tensor of M_K^{n+1} and ξ the deformation vector field, that is, $\xi = \partial\Phi/\partial t$. Note that N' is tangent to Σ since N_t is unitary for each t. Let $\{e_i\}$ be a local orthonormal frame on Σ, using (3) we obtain

$$S'_r = -\sum_{i=1}^{n}\langle \nabla_{T_{r-1}(e_i)}N', e_i\rangle - \sum_{i=1}^{n}\langle D_{BT_{r-1}(e_i)}\xi, e_i\rangle + K\langle \xi, N\rangle\langle T_{r-1}(e_i), e_i\rangle. \qquad (1.2)$$

Now, a nice feature of the Newton tensors T_k is that they are divergence-free (cf. [Re]), this implies that, for any vector field V on Σ (see [Ro] for a direct proof):

$$\mathrm{Trace}(u \to \nabla_{T_k(u)}V) = \mathrm{Trace}(u \to \nabla_u T_k(V)). \qquad (1.3)$$

Together with (1), we thus can rewrite (1.2) in the simpler form

$$S'_r = -\mathrm{div}_\Sigma(T_{r-1}(N')) - \tfrac{1}{2}\langle I', BT_{r-1}\rangle + K(n-r+1)S_{r-1}\langle \xi, N\rangle. \qquad (1.4)$$

Integrating and using the divergence theorem, we get

$$\int_\Sigma S'_r\,\mathrm{d}A = -\int_\Sigma \tfrac{1}{2}\langle I', BT_{r-1}\rangle \mathrm{d}A + K(n-r+1)\int_\Sigma S_{r-1}\langle \xi, N\rangle \mathrm{d}A. \qquad (1.5)$$

Also, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma S_r\,\mathrm{d}A = \int_\Sigma S'_r + \frac{S_r}{2}\langle I', I\rangle \mathrm{d}A.$$

Taking into account the inductive relation $T_r = S_r\mathrm{Id} - BT_{r-1}$, this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma S_r\,\mathrm{d}A = \int_\Sigma \tfrac{1}{2}\langle I', T_r\rangle + K(n-r+1)\int_\Sigma S_{r-1}\langle \xi, N\rangle \mathrm{d}A.$$

Let $\xi^\perp$ be the tangential part of ξ to Σ, using (2) we have

$$\tfrac{1}{2}\langle I', T_r\rangle = \mathrm{div}_\Sigma(T_r(\xi^\perp)) - (r+1)S_{r+1}\langle \xi, N\rangle. \tag{1.6}$$

Therefore:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma S_r\,\mathrm{d}A = -(r+1)\int_\Sigma S_{r+1}\langle \xi, N\rangle \mathrm{d}A + K(n-r+1)\int_\Sigma S_{r-1}\langle \xi, N\rangle \mathrm{d}A. \tag{1.7}$$

Observe that (1.7) also holds for $r = 0$ with the convention $S_{-1} = 0$.

Now, the smooth Schläfli formulas follow by replacing in (1.7) (written for $r = p - 1$) the two integrals in the right-hand side by their expressions obtained from (1.5) (for $r = p - 1$ and $r = p + 1$ respectively). □

Note that the formula of Theorem 1 can be somewhat simplified, since the two leftmost terms can be brought together; as a result, again for each $p \in \{1, \dots, n-1\}$:

$$2\left\{K(n-p)\int_\Sigma S_{p-1}\,\mathrm{d}A' + p\int_\Sigma S'_{p+1}\,\mathrm{d}A\right\} = \int_\Sigma \langle I', B((n-p)KT_{p-2} - pT_p)\rangle \mathrm{d}A,$$

where $\mathrm{d}A'$ is the variation of the volume element of I on Σ.

2. Higher Polyhedral Schläfli Formulas

We now use, as announced, the smooth Schläfli formulas to derive the polyhedral ones for convex polyhedra:

THEOREM 2. *Let P be a convex polyhedron in a $(n+1)$-dimensional Riemannian space form with sectional curvature K; for any deformation of P, and for each $1 \leqslant p \leqslant n-1$, we have*

$$K(n-p)\sum_j \frac{\mathrm{d}W_{j,t}}{\mathrm{d}t}K_{j,t} + p\sum_i W_{i,t}\frac{\mathrm{d}K_{i,t}}{\mathrm{d}t} = 0, \tag{E_p}$$

where j runs over the faces of codimension p and i runs over those of codimension $p+2$; W_i and W_j denote the volumes of the faces, and K_i, K_j their curvatures, i.e. the volumes of the duals of their links.

Proof. Call P_ϵ the set of points at distance ϵ of P on the outside (i.e. on the side of P which is concave). For ϵ small enough, P_ϵ is a C^1 piecewise smooth hypersurface and can be decomposed as $P_\epsilon = \bigcup_{m=1}^{m=n+1} P_{\epsilon,m}$, where $P_{\epsilon,m}$ is the set of points where the normal meets P on a codimension m face. We will first see that the smooth Schläfli formulas apply to P_ϵ, at $t = 0$, provided the deformation preserves its smooth parts. In particular, it applies to P_ϵ, at each t, for the deformation induced by the deformation P_t. We emphasize here that these formulas do not apply to general C^1 and piecewise smooth hypersurfaces. Denote by $P^i_{\epsilon,m}$ the set of points that project on the codimension m face F_i. The smooth parts of P_ϵ are the $P^i_{\epsilon,m}$. Moreover, $P^i_{\epsilon,m}$ is isometric to the product $b(\epsilon)F_i \times c(\epsilon)L^*(F_i)$, where $b(\epsilon) = 1 + \mathrm{o}(1)$ and $c(\epsilon) = \epsilon + \mathrm{o}(\epsilon)$ are functions of ϵ. For instance, in the hyperbolic space H^{n+1}, $b(\epsilon) = \cosh(\epsilon)$ and $c(\epsilon) = \sinh(\epsilon)$.

Let $x \in P^i_{\epsilon,m}$. If $u, v \in T_x P_{\epsilon,m}$ correspond to vectors orthogonal to $T_x F_i$, then, orienting P_ϵ by the inward unit normal,

$$\mathrm{II}_\epsilon(u, v) = \alpha(\epsilon) I_\epsilon(u, v), \tag{2.1}$$

where $\alpha(\epsilon) = (1/\epsilon) + \mathrm{O}(\epsilon)$ is a function of ϵ. For instance, in the hyperbolic space H^{n+1}, $\alpha(\epsilon) = \coth \epsilon$. In case u, v correspond to vectors in $T_x F_i$, then

$$\mathrm{II}_\epsilon(u, v) = \begin{cases} \dfrac{1}{\alpha(\epsilon)} I_\epsilon(u, v), & \text{for } K \neq 0, \\ 0, & \text{for } K = 0. \end{cases} \tag{2.2}$$

For the elementary symmetric functions of the principal curvatures, we have, for each $1 \leqslant r \leqslant n$:

$$\text{On } P_{\epsilon,m}, \quad S_{\epsilon,r} = \begin{cases} \sum_{k=0}^{k=r} \binom{m-1}{r-k} \binom{n-m+1}{k} \alpha(\epsilon)^{r-2k}, & \text{for } K \neq 0, \\ \binom{m-1}{r} \alpha(\epsilon)^r, & \text{for } K = 0, \end{cases} \tag{2.3}$$

with the conventions: $\binom{s}{k} = 0$ for s and $\binom{s}{0} = 1$.

Now, formulas (1.4) and (1.6) apply to all the smooth parts of P_ϵ, i.e. to the $P^i_{\epsilon,m}$. In order to show that the smooth Schläfli formulas apply to P_ϵ, we have to check that the boundary terms that appear after integrating in (1.4) and (1.6) cancel out two by two after summing up. These boundary terms are (using the symmetry of the Newton transformations) of the form $\int_{\partial P^i_{\epsilon,m}} \langle X, T_r(v) \rangle$, for some $1 \leqslant r \leqslant n$, X being a (continuous) vector field on P_ϵ and v the unit exterior conormal to $\partial P^i_{\epsilon,m}$. Now, $P^i_{\epsilon,m}$ and $P^j_{\epsilon,q}$ have a common boundary of nonzero $(n-1)$-measure if and only if $q = m+1$ or (symmetrically) $q = m-1$. Along the common boundary of $P^i_{\epsilon,m}$ and $P^j_{\epsilon,m+1}$, the unit conormal v is tangent to F_i and orthogonal to F_j. A straightforward computation, using (2.1), (2.2) and (2.3), shows that the vectors $T_r(v)$, where v is a vector orthogonal to the common boundary, are equal on the two sides of the common boundary. For instance, in the Euclidean space R^{n+1}, for $P^i_{\epsilon,m}$, we have $T_r(v) = \binom{m-1}{r} \alpha(\epsilon)^r v$, and for $P^j_{\epsilon,m+1}$,

$$T_r(v) = \left[\sum_{k=0}^{k=m} (-1)^k \binom{m}{r-k} \right] \alpha(\epsilon)^r v.$$

The conclusion follows then from the identity

$$\sum_{k=0}^{k=m} (-1)^k \binom{m}{r-k} = \binom{m-1}{r}.$$

So, with obvious notations,

$$\begin{aligned} &K(n-p) \frac{\mathrm{d}}{\mathrm{d}t} \int_{P_\epsilon} S_{\epsilon,p-1} \mathrm{d}A_\epsilon \\ &= \int_{P_\epsilon} \{K(n-p) S'_{\epsilon,p-1} - p S'_{\epsilon,p+1}\} \mathrm{d}A_\epsilon + \int_{P_\epsilon} \tfrac{1}{2} \langle I'_\epsilon, B_\epsilon((n-p)KT_{\epsilon,p-2} - pT_{\epsilon,p}) \rangle \mathrm{d}A_\epsilon. \end{aligned}$$

Using the flow of the unit normal vectors to the P_ϵ, we can identify P_ϵ and $P_{\epsilon'}$ for $\epsilon' \neq \epsilon$, so that we can consider, e.g. I'_ϵ as a 1-parameter family of symmetric 2-tensors on a fixed manifold.

For each $1 \leqslant r \leqslant n$, it follows from (2.3) that

$$\text{On } P_{\epsilon,m}, \quad S_{\epsilon,r} = \begin{cases} \mathrm{O}(\epsilon^{r-2(m-1)}), & \text{if } m < r+1, \\ \dfrac{\binom{m-1}{r}}{\epsilon^r} + \mathrm{O}\left(\dfrac{1}{\epsilon^{r-2}}\right), & \text{if } m \geqslant r+1. \end{cases} \tag{2.4}$$

Therefore

$$\lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}t} \int_{P_\epsilon} S_{\epsilon,r}\, \mathrm{d}A_\epsilon = \sum_i \frac{\mathrm{d}W_{i,t}}{\mathrm{d}t} K_{i,t} + W_{i,t} \frac{\mathrm{d}K_{i,t}}{\mathrm{d}t},$$

where i runs over the codimension $r+1$ faces of P.

We now compute $\lim_{\epsilon\to 0} \int_{P_{\epsilon,m}} \frac{1}{2}\langle I'_\epsilon, B_\epsilon T_{\epsilon,r}\rangle \mathrm{d}A_\epsilon$. Let $x \in P_{\epsilon,m}$ and $u \in T_x P_{\epsilon,m}$. We distinguish four cases:

(1) For $r < m-2$, it follows from (2.1), (2.2) and (2.4) that $B_\epsilon T_{\epsilon,r}(u) = \mathrm{O}\big(1/\epsilon^{m-2}\big)u$. Therefore $\langle I'_\epsilon, B_\epsilon T_{\epsilon,r}\rangle = \mathrm{O}\big(1/\epsilon^{m-2}\big)\langle I'_\epsilon, I_\epsilon\rangle$. Further, using the fact that the volume element on $P_{\epsilon,m}$ is of the form $\epsilon^{m-1} + \mathrm{o}(\epsilon^{m-1})$, we have

$$\int_{P_{\epsilon,m}} \tfrac{1}{2}\langle I'_\epsilon, I_\epsilon\rangle \mathrm{d}A_\epsilon \simeq \epsilon^{m-1} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i W_{i,t} K_{i,t}\right),$$

as $\epsilon \to 0$, i running over the codimension m faces of P. Consequently,

$$\lim_{\epsilon\to 0} \int_{P_{\epsilon,m}} \tfrac{1}{2}\langle I'_\epsilon, B_\epsilon T_{\epsilon,r}\rangle \mathrm{d}A_\epsilon = 0$$

(2) For $r = m-2$: if u corresponds to a vector in $T_x F$, then $B_\epsilon T_{\epsilon,m-2}(u) = \mathrm{O}\big(1/\epsilon^{m-2}\big)u$. While if u corresponds to a vector orthogonal to $T_x F$, then

$$T_{\epsilon,m-2}(u) = \left(\frac{1}{\epsilon^{m-2}} + \mathrm{O}\left(\frac{1}{\epsilon^{m-4}}\right)\right)u$$

and

$$B_\epsilon T_{\epsilon,m-2}(u) = \left(\frac{1}{\epsilon^{m-1}} + \mathrm{O}\left(\frac{1}{\epsilon^{m-3}}\right)\right)u.$$

As before, we deduce, this time, that

$$\lim_{\epsilon\to 0} \int_{P_{\epsilon,m}} \tfrac{1}{2}\langle I'_\epsilon, B_\epsilon T_{\epsilon,m-2}\rangle \mathrm{d}A_\epsilon = \lim_{\epsilon\to 0} \int_{P_{\epsilon,m}} \frac{1}{\epsilon^{m-1}} \frac{1}{2} \langle I'_\epsilon, I_\epsilon\rangle_0 \, \mathrm{d}A_\epsilon,$$

where the term $\frac{1}{2}\langle I'_\epsilon, I_\epsilon\rangle_0$ refers to the metric on the second factors in the canonical isometry between $P_{\epsilon,m}$ and $\bigcup_i b(\epsilon)F_i \times c(\epsilon)\mathrm{L}^*(F_i)$, i running over the codimension m faces of P. Now it is not difficult to check that

$$\int_{P_{\epsilon,m}} \tfrac{1}{2}\langle I'_\epsilon, I_\epsilon\rangle_0 \mathrm{d}A_\epsilon \simeq \epsilon^{m-1}\sum_i W_{i,t}\frac{\mathrm{d}K_{i,t}}{\mathrm{d}t},$$

as $\epsilon \to 0$. Therefore

$$\lim_{\epsilon\to 0}\int_{P_{\epsilon,m}} \tfrac{1}{2}\langle I'_\epsilon, B_\epsilon T_{\epsilon,m-2}\rangle \mathrm{d}A_\epsilon = \sum_i W_{i,t}\frac{\mathrm{d}K_{i,t}}{\mathrm{d}t},$$

with i running over the codimension m faces of P.

(3) For $r = m-1$: again, using (2.1), (2.2) and (2.4), it can be checked that $B_\epsilon T_{\epsilon,m-1}(u) = \mathrm{O}(1/\epsilon^{m-2})u$. Therefore, $\lim_{\epsilon\to 0}\int_{P_{\epsilon,m}}\frac{1}{2}\langle I'_\epsilon, B_\epsilon T_{\epsilon,m-1}\rangle \mathrm{d}A_\epsilon = 0$.

(4) For $r > m-1$: if u corresponds to a vector in T_xF, then, as before, we get: $B_\epsilon T_{\epsilon,r}(u) = \mathrm{O}(\epsilon^{r+1-2(m-1)})u$. Take now another tangent vector v that corresponds to a vector orthogonal to the face F. Taking u, v unitary, we deduce from (2) and (2.4) that $B_\epsilon T_{\epsilon,r}(v) = \mathrm{O}(\epsilon^{r+1-2(m-1)})v$. This implies again that the limit of the integral vanishes.

Furthermore, on the smooth parts of P_ϵ, we have $S'_{\epsilon,k} = 0$ for each k.

The higher Schläfli formulas follow now straightforwardly. □

3. Dualities

The classical 'projective' duality in S^{n+1} is a one-to-one map between points and oriented hyperplanes. It is defined by sending a hyperplane H to the point H^* at distance $\pi/2$ on the geodesic defined by the oriented normal to H (it is independent of where the geodesic starts on H), and by sending a point x to the hyperplane x^* made of all points at distance $\pi/2$ from x, with a natural orientation.

There is also a classical duality (see [Ri-Ho]) between H^{n+1} and the de Sitter space S^{n+1}_1 of dimension $n+1$. It can be defined using the models of H^{n+1} and S^{n+1}_1 as hyperboloids in the Minkowski space R^{n+2}_1. It also sends a point in H^{n+1} to a space-like hyperplane in S^{n+1}_1, and a point in S^{n+1}_1 to an oriented hyperplane in H^{n+1}. Both dualities can be defined in a single 'intrinsic' way, see [Sch98].

In both cases, a convex polyhedron P is sent to another convex polyhedron P^*, which can be in S^{n+1} if $P \subset S^{n+1}$ and in S^{n+1}_1 (and space-like) if $P \subset H^{n+1}$. A k-face F of P is associated to the 'dual' $(n-k)$-face F^* of P^*, and the metric induced on F^* is exactly the metric on the dual $L^*(F, P)$ of the link of F as defined in the introduction. Conversely, the metric induced on F is the same as the metric on the dual of the link of F^*; if $P \subset H^{n+1}$, then $F^* \subset P^* \subset S^{n+1}_1$, so $L(F^*, P^*) \subset S^n_1$, and $L^*(F^*, P^*) \subset H^n$.

In other terms, if we call $W^*_{i,t}$ (resp. $W^*_{j,t}$) the volume of the p-face i of P^*_t, and $K^*_{i,t}$ its curvature, then $W^*_{i,t} = K_{i^*,t}$, $K^*_{i,t} = W_{i^*,t}$, where i^* is the face of P_t dual of i. Using those transformations in the higher Schläfli formula of Theorem 2 shows that

$$K(n-p)\sum_j \frac{\mathrm{d}K^*_{j,t}}{\mathrm{d}t} W^*_{j,t} + p\sum_i K^*_{i,t}\frac{\mathrm{d}W^*_{i,t}}{\mathrm{d}t} = 0,$$

where j and i run over codimension p and codimension $p+2$ faces of P, respectively. Setting $q = n-p$:

$$Kq\sum_j \frac{\mathrm{d}K_{j,t}}{\mathrm{d}t} W_{j,t} + (n-q)\sum_i K_{i,t}\frac{\mathrm{d}W_{i,t}}{\mathrm{d}t} = 0,$$

where i now goes over faces of codimension q, and j over faces of codimension $q+2$ of P^*.

This is almost the same formula as in Theorem 2. This means that we have nothing new in S^{n+1} (the higher Schläfli formulas are pariwise dual), but, in H^{n+1}, it leads to the higher Schläfli formulas in the de Sitter space.

COROLLARY 3. *Let P be a convex space-like polyhedron in S^{n+1}_1, which is dual to a convex hyperbolic polyhedron; for any deformation $(P_t)_{t\in[0,1]}$ of P among convex space-like polyhedra, and for each $1 \leqslant q \leqslant n-1$, we have:*

$$(n-q)\sum_j \frac{\mathrm{d}W_{j,t}}{\mathrm{d}t} K_{j,t} - q\sum_i W_{i,t}\frac{\mathrm{d}K_{i,t}}{\mathrm{d}t} = 0,$$

where j runs over faces of codimension q of P and i runs over those of codimension $q+2$, W_i and W_j denote the volumes of the faces, and K_i, K_j their curvatures.

It would actually have been possible to prove directly those de Sitter higher Schläfli formula, by proving first their smooth analogues, and then applying the same ϵ-neighborhood argument as above.

The classical Schläfli formula is known to extend well to the de Sitter setting (see [SP]) beyond the duals of convex hyperbolic polyhedra. It would be interesting to know whether the higher Schläfli formulas described above also extend to general polyhedra in the de Sitter space or in other pseudo-Riemannian space-forms.

4. Two Special Cases

The proof of Theorem 1 above also leads to a proof of the smooth Schläfli formula of [Ri-S-a], which is simpler and more straightforward than the one given in [Ri-S-a]. But the result is not exactly the one obtained by replacing p by 0 in the formula of Theorem 1, because some terms disappear for $p=0$, and some coefficients change. The formula ends up as (see [Ri-S-a]):

$$nK\frac{\mathrm{d}V}{\mathrm{d}t} = \int_\Sigma S'_1 + \tfrac{1}{2}\langle I', B\rangle \mathrm{d}A \qquad (F_0)$$

The classical Schläfli formula of the introduction follows as Theorem 2 follows from Theorem 1 above. Note that the dual formulas were proved directly in [Ri-S-a].

The second interesting case happens for $p = 1$ in Theorem 1. $S_0 = 1$, so the integral of S_0' is zero, and $T_{-1} = 0$ too, so that the formula of Theorem 1 becomes

$$(n-1)K\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} + \int_\Sigma S_2' + \tfrac{1}{2}\langle I', BT_1\rangle \mathrm{d}A = 0, \tag{F_1}$$

where $\mathbf{A} = \int_\Sigma \mathrm{d}A$ is the total area of Σ.

Going to the polyhedral case, we find that

$$(n-1)K\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} + \sum_i W_{i,t}\frac{\mathrm{d}K_{i,t}}{\mathrm{d}t} = 0, \tag{E_1}$$

where $\mathbf{A}$ is again the total area and i goes over codimension 3 faces. In other terms, the angle at codimension 1 faces, which doesn't make sense, has to be taken equal to 1.

If $n = 2$ (i.e. in total dimension 3) this formula is similar to the Gauss–Bonnet formula for polyhedra: it shows that the total curvature of a polyhedron, which is the sum of K times the area plus the singular curvatures at the vertices, is constant.

The formula above for the variation of the area can be proved in a rather simple and purely combinatorial way, by using twice the usual Schläfli formula, applying it once to the codimension 1 faces, and once to the duals of the links of the codimension 3 faces. It does not seem easy, however, to generalize this approach to prove Theorem 2 directly from the classical Schläfli formula, because some terms don't add up well when $p \geqslant 2$.

For $p = n - 1$ and $K = 0$, our Schläfli formula again recovers an elementary geometric fact. In this case, the sum over codimension $n - 1$ faces doesn't appear because of the coefficient K, and the only remaining term is the sum over the vertices, which is $n - 1$ times the sum of the variations of the curvatures at the vertices. The curvature at a vertex is just the area of the image of this vertex by the Gauss map (which takes a point p to the set of the oriented normals to the support plane at p) so the sum of the curvatures at all vertices is constant, and equal to the volume of the n-sphere.

5. Ideal Polyhedra

The classical Schläfli formula has no real meaning for ideal polyhedra, since the lengths of the edges are then infinite. It is possible, however (see [Ri]), to show that it still holds under a slightly different formulation.

But both sides of the higher Schläfli formulas (E_p) remain meaningful for ideal polyhedra (or polyhedra having some ideal vertices) because the volumes of the faces of dimension at least two are finite, as are the curvatures of all faces. We can therefore state the:

THEOREM 4. *Let P be a convex hyperbolic polyhedron, which has some ideal vertices. Let $(P_t)_{t\in[0,1]}$ be a deformation of P among finite volume polyhedra, which is smooth* *when considered in a projective model of H^{n+1}, such that P_t has the same combinatorics as P and the same ideal vertices. Then, for any $p \in \{1, \dots, n-1\}$, formula (E_p) holds.*

It should also be possible to prove that the same formulas remain valid when some ideal vertices become nonideal during the deformation (but the polyhedron should remain of finite volume).

Proof. Choose a projective mapping ρ: $H^{n+1} \to B_0^{n+1}(1)$, where $B_0^{n+1}(r)$ is the ball of radius r centered at 0 in R^{n+1}. For each $t \in [0, 1]$, let $Q_t = \rho(P_t)$. Define Q_t^ϵ be the image of Q_t by the homothety of ratio $1 - \epsilon$ and center 0, and P_t^ϵ as $\rho^{-1}(Q_t^\epsilon)$. $(P_t^\epsilon)_{t \in [0,1]}$ is a 1-parameter family of compact polyhedra in H^{n+1}, having the same combinatorics as P, and such that, for fixed t, $P_t^\epsilon \to P_t$, for instance in the sense that $P_t^\epsilon \subset P_t$, while $V(P_t^\epsilon) \to V(P_t)$.

Now choose $p \in \{1, 2, \dots, n-1\}$. Let j be a codimension p face, and i a codimension $p+2$ face of P. For $t \in [0, 1]$ and $\epsilon \in (0, 1)$, call $W_{i,t}^\epsilon$ and $W_{j,t}^\epsilon$ the volumes of i and j in P_t^ϵ, and $K_{i,t}^\epsilon$ and $K_{j,t}^\epsilon$ their curvatures. Then, as $\epsilon \to 0$:

(1) $W_{i,t}^\epsilon \to W_{i,t}$, because the vertices of i for P_t^ϵ converge to the corresponding vertices of i for P_t;
(2) $K_{j,t}^\epsilon \to K_{j,t}$, because the face which is dual to j for $P_{j,t}^\epsilon$ converges to the face dual to j for $P_{j,t}^\epsilon$, which is an 'ordinary' $(p-1)$-dimensional space-like face in S_1^{n+1}.

Now the following points can be easily checked:

(1) if i is of dimension at least 1 or is a nonideal vertex, then $(L(i, P_t^\epsilon))_{t \in [0,1]}$ is a smooth 1-parameter family of spherical polyhedra, which C^1-converge to $(L(i, P_t))_{t \in [0,1]}$;
(2) if v is an ideal vertex of P, then $(L(v, P_t^\epsilon))_{t \in [0,1]}$ is a 1-parameter family of polyhedra in S^n which converges to a point; but, upon 'renormalization' by a factor $c(\epsilon)$ (that is, multiplication of the metric on S^n by $c(\epsilon)$, where $c(\epsilon) \to \infty$ when $\epsilon \to 0$) it converges to $(L(v, P_t))_{t \in [0,1]}$, which is a 1-parameter family of Euclidean polyhedra (defined as the intersection of P_t with a horosphere centered on v, close enough to v).

Therefore, if p is an edge of the link $L(i, P)$ and if $l_{p,t}$ and $k_{i,p,t}$ are its length and the curvature of $L(i, P)$ at p, respectively, in P_t, and if $l_{p,t}^\epsilon$ and $k_{i,p,t}^\epsilon$ are the same quantities in P_t^ϵ, then as $\epsilon \to 0$: $\mathrm{d}l_{p,t}^\epsilon/\mathrm{d}t \to \mathrm{d}l_{p,t}/\mathrm{d}t$, $k_{i,p,t}^\epsilon \to k_{i,p,t}$. The (classical) dual Schläfli formula applied to $L(i, P_t^\epsilon)$ and to $L(i, P_t)$ therefore shows that $\mathrm{d}K_{i,t}^\epsilon/\mathrm{d}t \to \mathrm{d}K_{i,t}/\mathrm{d}t$. as $\epsilon \to 0$.

If q is a codimension 2 face of j, call $w_{j,q,t}$ and $\theta_{j,q,t}$ its volume and the dihedral angle of j at q respectively for P_t, and let $w_{j,q,t}^\epsilon$ and $\theta_{j,q,t}^\epsilon$ be the volume of q and the dihedral angle of j at q respectively for P_t^ϵ. Then, for the same reason as above $\mathrm{d}\theta_{j,q,t}^\epsilon/\mathrm{d}t \to \mathrm{d}\theta_{j,q,t}/\mathrm{d}t$, while one can check directly that $w_{j,t}^\epsilon \to w_{j,t}$ and the limit is finite since j has codimension $p \leqslant n-1$, and therefore dimension at least 2. The Schläfli formula applied to j therefore shows that $W^{\epsilon}_{j,t}/\mathrm{d}t \to \mathrm{d}W_{j,t}/\mathrm{d}t$.

We have just seen that all the terms involved in (E_p) are converging as $\epsilon \to 0$; the formula (E_p) for $(P_t)_{t\in[0,1]}$ therefore follows from the same formula for the families $(P^{\epsilon}_t)_{t\in[0,1]}$.

6. Nonconvex Polyhedra

The classical Schläfli formula (which is recalled in the introduction) is additive in the sense that, if a convex polyhedron P is decomposed into two convex polyhedra P', P'' with disjoint interiors, then both terms in the formula for P are the sums of the corresponding terms in the formulas for P' and P''.

Define a (nonconvex) polyhedron as the union of a finite number of convex polyhedra $P_1, \dots, P_N$, with disjoint interiors, but such that P_i shares a codimension 1 face with P_{i+1} for $i \in \{1, \dots, n-1\}$. It follows from the remark on additivity above that, if the classical Schläfli formula holds for convex polyhedra, it also holds for nonconvex ones, since they can be cut into convex pieces.

This doesn't work for the higher Schläfli formulas, so some care is necessary to prove them for nonconvex polyhedra. We actually need first to define the curvature of a polyhedron at a face in the nonconvex case.

Consider a convex polyhedron P which can be cut into two convex pieces P', P''. Then $P = P' \cup P''$, and $P' \cap P''$ is a convex polyhedron in a hyperplane Π. Let F be a codimension p face of P, P' and P''. Then:

PROPOSITION 5. *The curvatures* $K_P(F)$, $K_{P'}(F)$, $K_{P''}(F)$, $K_{P'\cap P''}(F)$ *of* F *in* P, P', P'' *and* $P' \cap P''$ *respectively, are such that*

$$K_{P'}(F) + K_{P''}(F) = K_P(F) + \frac{V(S^{p-1})}{V(S^{p-2})} K_{P'\cap P''}(F),$$

where $V(S^k)$ *is the volume of the canonical* k*-sphere.*

Proof. In this setting, $P' \cap P''$ can be considered either as a (degenerate) polyhedron in the whole $(n+1)$-dimensional ambient space M, or as a polyhedron with non-empty interior in the hyperplane of M which contains it. Observe that the link of F in $P' \cap P''$, $L(F, P' \cap P'')$, considered in the first way, and $\bar{L}(F, P' \cap P'')$ its link in $P' \cap P''$ considered in the second way coincide. However, this is not true for the corresponding dual links. Then the curvature of $P' \cap P''$ is obtained by looking at $P' \cap P''$ as a nondegenerate polyhedron in a n-dimensional space, so $K_{P'\cap P''}(F) = V(\bar{L}^*(F, P' \cap P''))$.

Consider the space of directions orthogonal to F at any point of F. It is isometric to S^{p-1}, and contains polyhedra corresponding to $L(F, P)$, $L(F, P')$, $L(F, P'')$ and $L(F, P' \cap P'')$. Moreover, $L(F, P) = L(F, P') \cup L(F, P'')$ and $L(F, P') \cap L(F, P'') =$

$L(F, P' \cap P'')$. Therefore, $L^*(F, P) = L^*(F, P') \cap L^*(F, P'')$, while $L^*(F, P' \cap P'') = L^*(F, P') \cup L^*(F, P'')$.

Consequently,

$$V(L^*(F, P')) + V(L^*(F, P'')) = V(L^*(F, P)) + V(L^*(F, P' \cap P'')).$$

Now

$$V(L^*(F, P' \cap P'')) = \frac{V(S^{p-1})}{V(S^{p-2})} V(\bar{L}^*(F, P' \cap P''))$$

and the formula follows. □

The proof would actually work in the same way for a convex polyhedron cut into more than two convex pieces, with some additional terms corresponding to the intersections of more than two of the pieces.

It is clear from this proposition that, if the higher Schläfli formulas apply to P' and to P'', then they must apply to P. The first term in the formula is

$$(n-p)K \sum_j W_j' K_j(P) = (n-p)K \sum_j W_j' K_j(P') + (n-p)K \sum_j W_j' K_j(P'') - $$
$$- (n-p)K \frac{V(S^{p-1})}{V(S^{p-2})} \sum_j W_j' K_j(P' \cap P''),$$

where the sums are over $(n+1-p)$-dimensional faces of P, P', P'' and $P' \cap P''$ respectively, and $K_j(P)$ indicates the curvature of face j in P, etc. The second term is

$$p \sum_i W_i K_i'(P)$$
$$= p \sum_i W_i K_i'(P') + p \sum_i W_i K_i'(P'') - p \frac{V(S^{p+1})}{V(S^p)} \sum_i W_i K_i'(P' \cap P''),$$

where the sums are now over $(n-1-p)$-dimensional faces of P, P', P'' and $P' \cap P''$ respectively. Adding the two terms and using a higher Schläfli formula for P' and for P'' leaves us with

$$(n-p)K \frac{V(S^{p-1})}{V(S^{p-2})} \sum_j W_j' K_j(P' \cap P'') + p \frac{V(S^{p+1})}{V(S^p)} \sum_i W_i K_i'(P' \cap P''),$$

where j and i run over the faces of $P' \cap P''$ of dimension $n+1-q$ and $n-1-q$ respectively.

Now an elementary computations shows that, for any $k \geqslant 2$:

$$\frac{V(S^{k+1})}{V(S^k)} = \frac{k-1}{k} \frac{V(S^{k-1})}{V(S^{k-2})},$$

so that the remaining term is simply

$$\frac{V(S^{p-1})}{V(S^{p-2})}\left[((n-1)-(p-1))K\sum_j W_j' K_j(P'\cap P'') + (p-1)\sum_i W_i K_i'(P'\cap P'')\right],$$

where the sum on j is over codimension $p-1$ faces of $P'\cap P''$, and the sum on i is over codimension $p+1$ faces. So this vanishes because of a higher Schläfli formula applied to $P'\cap P''$.

Now we can prove Theorem 2 for nonconvex polyhedra. First, we have to define the curvature of a nonconvex polyhedron P at a face F. P can be decomposed into a finite number of convex polyhedra $P_1, \dots, P_N$, i.e. $P = \bigcup_i P_i$ and the P_i have pairwise disjoint interiors. Then we can apply the formula of Proposition 5 (actually the analoguous formula for more than two polyhedra if necessary) and call $K_P(F)$ the result. $K_P(F)$ is independent of the decomposition $P = \bigcup_i P_i$ of P into convex polyhedra P_i; if $P = \bigcup_j P_j'$ is another decomposition into convex pieces, taking the finer decomposition $P = \bigcup_{i,j} P_i \cap P_j'$ and applying Proposition 5 shows that the values of $K_P(F)$ obtained by $\bigcup_i P_i$, $\bigcup_j P_j'$ and $\bigcup_{i,j} P_i \cap P_j'$ are identical.

With this definition, it is not too difficult to prove that the higher Schläfli formulas apply to deformations of a nonconvex polyhedron P. Start by choosing a decomposition of P into convex pieces $(P_i)_{1\leqslant i\leqslant N}$, and use the corresponding higher Schläfli formula for the P_i. Then do as in the proof above for $P = P'\bigcup P''$ to get rid of the terms involving the intersection of two or more of the P_i, and the result follows.

LIPSCHITZ-KILLING CURVATURES: An alternate way to define the curvature of a non-convex polyhedron P at a face F could be to use the Lipschitz–Killing curvature defined in [C-M-S]: consider, as before, a decomposition of P into a finite number of convex polyhedra $P_1, \dots, P_N$, i.e. $P = \cup_i i$, the P_i having pairwise disjoint interiors. Let p be the codimension of F. The Lipschitz–Killing curvature of P at F is set to be:

$$R_P(F) = \sum_{\substack{G\supseteq F\\ i=\dim G\leqslant n+1}} (-1)^{i-(n+1-p)}\frac{V(S^{p-1})}{V(S^{p-1-(n+1-i)})}K_G(F),$$

where the sum is over all faces G, in the decomposition, containing F, including F itself, counted *without multiplicity* and $K_F(F)=1$. This is – up to a normalizing factor – the Lipschitz–Killing curvature of order p of the given decomposition of P at F as defined in [C-M-S]. It turns out that the curvature obtained this way coincides with ours (this, by the way, also shows $R_P(F)$ doesn't depend on the choice of the decomposition). This will be a consequence of the following fact:

PROPOSITION 6. *Let P be a convex polyhedron in H^{n+1}, R^{n+1} or S^{n+1}, and let F be a codimension p face of P. Then the curvature of F in P satisfies*

$$K_P(F) = \sum_{\substack{G \supseteq F \\ i=\dim G \leqslant n+1}} (-1)^{i-(n+1-p)} \frac{V(S^{p-1})}{V(S^{p-1-(n+1-i)})} K_G(F)$$

the sum running over all faces G of P containing F, including F itself and P.

Proof. $L(F, P)$ and $L^*(F, P)$ are dual polyhedra in S^{p-1}. The faces of $L(F, P)$ are the $L(F, G)$ for faces G of P containing F, except for F and P. Hence, the faces of $L^*(F, P)$ are their 'dual' faces $L(F, G)^*$ as discussed in Section 3. Call $V(F, G)$ the volume of the link of $L(F, G)^*$ in $L^*(F, P)$. We distinguish two cases:

(1) In case $p = 2j + 1$ is odd, by the Poincaré formula (cf. [Vin]), applied to the polyhedron $L^*(F, P)$ in the even dimensional sphere S^{2j}, we have

$$V(L^*(F, P)) = \frac{1}{2}\left[V(S^{2j}) + \sum_{\substack{G \supseteq F \\ i=\dim G \leqslant n}} (-1)^{i-(n-2j)} \frac{V(S^{2j})}{V(S^{2j-(n+1-i)})} V(F, G) \right],$$

where the sum is taken over all faces G of P containing strictly F, except for P itself. Now it can be checked that $L(L(F, G)^*, L^*(F, P))$ is isometric to $\bar{L}^*(F, G)$, where G is considered as a (nondegenerate) polyhedron in a suitable sphere (see Section 5) . Taking into account that $K_F(F) = 1$, we can therefore write

$$K_P(F) = \frac{1}{2} \sum_{\substack{G \supseteq F \\ i=\dim G \leqslant n}} (-1)^{i-(n-2j)} \frac{V(S^{2j})}{V(S^{2j-(n+1-i)})} K_G(F),$$

the sum being taken over faces G containing F including F itself and excluding P. This formula is the same as the one stated in the proposition since for $p = 2j + 1$ and $G = P$: $(-1)^{i-(n+1-p)} = -1$.

(2) In case $p = 2j$ is even, we have the following formula for the polyhedron $L^*(F, P)$ in the odd-dimensional sphere S^{2j-1} (cf. [Vin]):

$$\sum_{\substack{G \supseteq F \\ i=\dim G \leqslant n}} (-1)^{i-(n+1-2j)} \frac{V(S^{2j-1})}{V(S^{2j-1-(n+1-i)})} V(F, G) = 0,$$

where the sum is over faces G containing F except for P. Again this coincides with the formula stated in the proposition. □

Consider now a decomposition of a polyhedron P (convex or nonconvex) into convex pieces $(P_i)_{1 \leqslant i \leqslant N}$. Replacing in the formula giving $K_P(F)$ in Proposition 5 (or the analoguous formula for more than two pieces if necessary) each curvature of F in an intersection of some of the P_i by its value given by Proposition 6, leaves us with the sum defining $R_P(F)$.

To obtain a complete picture concerning nonconvex polyhedra, we briefly describe how the duals of such polyhedra can be defined. Here $P \subset M_K^{n+1}$ is a closed orientable polyhedron, but it does not have to be convex or even embedded. To simplify things a little, we suppose that two p-faces have an intersection of dimension at most $p-1$.

Suppose first that $n = 1$ and $K = 1$, so that P is an oriented polygon in S^2. At each vertex v of P, there is an incoming edge e_i and an outgoing edge e_o. Consider the set E_v of oriented lines at v between e_i and e_o, i.e. E_v correspond to the segment of length $L < \pi$ between the points corresponding to e_i and to e_o in the unit tangent bundle of S^2 at v. Call v^* the set of points in S^2 which are duals to the oriented lines in E_v. v^* is a geodesic segment of S^2, and the segments corresponding to successive vertices of P have a vertex in common, so that they all add up to a polygon $P^* \subset S^2$ which we call the dual of P.

Now if $K = 0$ or $K = -1$, the same construction works and leads to a dual $P^* \subset S^1$ if $K = 0$, and $P^* \subset S_1^2$ if $K = -1$.

For $n \geqslant 2$, the dual of $P \subset M_K^{n+1}$ can be defined recursively on n. Suppose it is already defined in S^k for $k \leqslant n$; then, for $P \in M_K^{n+1}$, one can define the dual v' of a vertex v of P as the dual of the link of P at v. v' can be considered as a polyhedron in the hyperplane v^* dual to v. Moreover, if v_1 and v_2 are the endpoints of an edge e, then v_1' and v_2' share a codimension 1 face (which is the dual of e). Therefore, the polyhedra duals of the vertices of P again add up as a polyhedron P^* in the space dual to M_K^{n+1}.

This notion of duality shares many properties of the classical duality for convex polyhedra; in particular, P^* is orientable and one can define its 'volume' as for convex polyhedra (for instance, for $P \subset H^{n+1}$, $P^* \subset S_1^{n+1}$ does not bound a compact domain, so its volume has to be defined with respect to a fixed hyperplane, as in the convex case). There is therefore a notion of dual volume also for nonconvex polyhedra.

7. Topological Invariants

We show in this section how the higher Schläfli formulas given above indicate simply that some quantities defined for polyhedral or smooth hypersurfaces in non-flat space-forms are topological invariants. Some of the resulting invariants are consequences of the Gauss–Bonnet theorem, while others are not.

A similar approach was used by E. Suárez-Peiró in [SP] to prove a Gauss–Bonnet type formula for simplices with Riemannian faces in the de Sitter space, as well as a formula on the dual volume of hyperbolic simplex.

Let P be a polyhedron in M_K^{n+1}, we suppose here that $K \neq 0$. For each $k \in \{1, \dots, n-1\}$, call $\mathbf{H}_k$ the kth integral mean curvature of P, i.e.

$$\mathbf{H}_k = \sum_i W_i K_i, \tag{7.1}$$

where i runs over codimension $k+1$ faces of P, K_i is the curvature of face i and W_i is its $(n-k)$-volume. It is natural to define also $\mathbf{H}_0$ as the area $A(P)$ of P, and $\mathbf{H}_n$ as its dual area $A^*(P) = A(P^*)$, i.e. the area of the dual polyhedron. This makes sense because, for $k = 0$, only the n-volume of codimension 1 faces of P appear in (7.1).

It would also make sense to define $\mathbf{H}_{-1}$ as the volume $V(P)$ of P, and $\mathbf{H}_{n+1}$ as the dual volume $V^*(P) = V(P^*)$ of P, i.e. the volume of the dual polyhedron.

The dualities we consider here are the one in $S^{n+1}(K)$ and the one between $H^{n+1}(-K)$ and de Sitter space $S_1^{n+1}(K)$ of constant curvature $K > 0$. These dualities follow from those in Section 3 by a change of scale.

Remember that, for $k \in N$, $k!!$ is the product of all integers between 1 and k with the same parity as k. By a deformation (in the ambiant space) of a polyhedron we mean a general deformation, that is, a deformation through polyhedra which does not change the combinatorial structure. In Section 8 we derive consequences of our generalized Schläfli formulae in the particular case of isometric deformations. We can now state the main result of this section:

THEOREM 7. *The following quantities are invariant under deformations of P:*

(1) *for $n = 2m$ even:*

$$C_A(P) = \sum_{q=0}^{m} K^{m-q} \frac{(2q-1)!!(2m-2q-1)!!}{(2m-3)!!} \mathbf{H}_{2q}$$

and if P is convex

$$C_V(P) = n(K^m V + \epsilon |K|^m V^*) + \sum_{q=1}^{m} \frac{K^{m-q}}{\binom{m-1}{q-1}} \mathbf{H}_{2q-1},$$

where ϵ = sign of K.

(2) *for $n = 2m-1$ odd:*

$$C_V(P) = nV + \sum_{q=0}^{m-1} \frac{2^q q!(2m-2q-3)!!}{K^{q+1}(2m-3)!!} \mathbf{H}_{2q+1},$$

with $(-1)!! = 1$, and if P is convex:

$$C_{V^*}(P) = n\epsilon |K|^{n/2} V^* + \sum_{q=1}^{m} 2^{q-1} K^{q-1} \frac{(q-1)!(2m-2q-1)!!}{(2m-3)!!} \mathbf{H}_{2m-2q}.$$

Note that the assumption that P is convex in some of the cases above is not crucial; if P is not convex, however, one should use the notion of dual volume of a non-convex polyhedron, which we have only sketched at the end of Section 6.

Proof. The constancy of $C_A(P)$ for $n = 2m$ follows from the linear combination:

$$\sum_{q=1}^{m} K^{m-q} \frac{(2q-3)!!(2m-2q-1)!!}{(2m-3)!!}(E_{2q-1}).$$

The second formula is obtained by the following linear combination of the usual, dual, and higher Schläfli formulas:

$$\epsilon |K|^m (E_n) - K^m (E_0) + \sum_{q=1}^{m-1} K^{m-q} \frac{(m-q-1)!(q-1)!}{2\ (m-1)!}(E_{2q}).$$

To prove that $C_V(P)$ is constant for $n = 2m - 1$, consider the equation:

$$-(E_0) + \sum_{q=1}^{m-1} \frac{2^{q-1}(q-1)!(2m-2q-3)!!}{K^q (2m-3)!!}(E_{2q}).$$

Finally, $C_{V^*}(P)$, which is dual of $C_V(P)$, is shown to be constant by using the dual equation. □

Using the same proofs, but with the smooth higher Schläfli instead of the polyhedral ones, leads to analog smooth results:

THEOREM 8. *Let Σ be a convex smooth hypersurface in M_K^{n+1}, suppose that $K \neq 0$. Let $\mathbf{H}_r = \int_\Sigma S_r$, for $r = 0, \ldots n$, be the integral mean curvatures of Σ computed with respect to the exterior unit normal. Then the following quantities are invariant under deformations of Σ:*

(1) *for $n = 2m$ even:*

$$C_V(\Sigma) = -n(K^m V + \epsilon |K|^m V^*) + \sum_{q=1}^{m} \frac{K^{m-q}}{\binom{m-1}{q-1}} \mathbf{H}_{2q-1}$$

and

$$C_A(\Sigma) = \sum_{q=0}^{m} K^{m-q} \frac{(2q-1)!!(2m-2q-1)!!}{(2m-3)!!} \mathbf{H}_{2q},$$

with ϵ = sign of K.

(2) *for $n = 2m - 1$ odd:*

$$C_V(\Sigma) = nV - \sum_{q=0}^{m-1} \frac{2^q q!(2m-2q-3)!!}{K^{q+1}(2m-3)!!} \mathbf{H}_{2q+1}$$

and

$$C_{V^*}(\Sigma) = n\epsilon|K|^{n/2}V^* - \sum_{q=1}^{m} 2^{q-1}K^{q-1}\frac{(q-1)!(2m-2q-1)!!}{(2m-3)!!}\mathbf{H}_{2m-2q}.$$

For $n = 2m$, the dimension of M is odd. C_A is then related to the Gauss–Bonnet theorem applied to the hypersurface. For $n = 2m - 1$, the dimension of M is even, and C_V is related to the Gauss–Bonnet theorem 'with boundary' applied to the interior of the domain bounded by the hypersurface.

For $n = 2$, the only nontrivial quantity is C_V, which is simply

$$C_V(P) = 2K(V + V^*) + \mathbf{H}_1.$$

The fact that this quantity is constant was already remarked by I. Rivin.

Once those quantities are known to be constant, it is easy to find their values (for each immersion into M given up to deformation) by choosing a special hypersurface and computing the value of the invariant.

The quantities C_A for n even and C_V for n odd, which do not include V^*, are also constant for hypersurfaces in constant curvature space-forms (e.g. hyperbolic manifolds).

8. Bending Invariants

We examine here the special case of isometric deformations of polyhedra and hypersurfaces. A deformation of a hypersurface is said to be isometric, or a bending, if it leaves invariant the metric induced on the hypersurface. There are no known examples of smooth closed hypersurfaces (in fact, not even C^2 ones) admitting nontrivial smooth isometric deformations (i.e. not through rigid motions of the ambient space). Consider the rth integral mean curvatures $\mathbf{H}_r = \int_\Sigma S_r$, for $r = 0, \dots n$. For r even, this depends only on the induced metric on the hypersurface since this is even the case for S_r as it follows from Gauss equation. However this not true for r odd as it can be checked on the standard example (and its obvious generalizations to higher dimensions) of a toplogical sphere which admits two isometric and noncongruent embeddings in $\mathbb{R}^3$ (see [Spi], p. 307). The following result may be viewed as a kind of rigidity in a weak sense.

THEOREM 9. *Let Σ be a smooth closed and oriented hypersurface in M_K^{n+1}. Then, for the choice of the unit normal compatible with the orientation of Σ, the following quantities are invariant under isometric deformations of Σ: $nKV - \mathbf{H}_1$ and $\mathbf{H}_r$ for $r \geqslant 2$; V being the* (*oriented*) *volume enclosed by Σ.*

This result was first proved for $r = 1$ by F. Amgren and I. Rivin ([Al-Ri]) by noticing firstly that the analoguous statement for polyhedra is true (see below) and then extending it to smooth hypersurfaces using geometric measure theory methods. Direct proofs using differential-geometric tools were then given independently in [Ri-S-a] and [So]. In [Ri-S-a] the stronger result that the S_r, for $r \geqslant 2$, are pointwise invariant under isometric deformations, is proved.

Remark. We do not need to assume Σ is embedded. Indeed the enclosed volume may be defined in the immersed case (see [Ra]) and the result still holds.

Proof. The result is a direct consequence of smooth Schläfli formulas, (F_0) for $r = 1$ and (F_{r-1}) for $r \geqslant 2$. □

An open question in this setting is to decide whether the volume enclosed by a hypersurface in R^{n+1} is invariant under isometric deformations. The analoguous statement for closed polyhedra in R^3 was proved recently by I. Sabitov (see e.g. [CSW] and the references given there). The following corollary (see also [Ri-Sch] and [So]) suggests that this should be true also for smooth hypersurfaces. Call Σ_ϵ the parallel hypersurface to Σ at (algebraic) distance ϵ, that is, the hypersurface obtained by going at distance ϵ along the normal at each point (it is indeed a regular hypersurface for ϵ small enough). Call V^ϵ the (oriented) volume enclosed between Σ and Σ_ϵ. Note that, unlike the volume of a tubular neighborhood of Σ, V^ϵ does depend on the extrinsic geometry of Σ (this is clear from the formula below).

COROLLARY 10. *The volume V^ϵ (ϵ small enough) is invariant under isometric deformations of Σ.*

Proof. It is known and not too difficult to check that V^ϵ is given by

$$V^\epsilon = \int_0^\epsilon \int_\Sigma \prod_{i=1}^{i=n} (1 - sk_i)\, \mathrm{d}A \mathrm{d}s = \sum_{r=1}^{r=n} (-1)^r \frac{\epsilon^{r+1}}{r+1} \int_\Sigma S_r \, \mathrm{d}A$$

so the result follows directly from Theorem 8. □

Before stating the polyhedral version of Theorem 9, we make precise some definitions. An isometric deformation, or a bending, of a polyhedron P is a deformation such that each codimension 1 face of P remains congruent (through rigid motions) to itself under the deformation. Cauchy [Cau] proved in 1813 that two compact convex polyhedra in R^3, constructed from pairwise congruent faces assembled in the same order, were in fact congruent themselves; a nice exposition of this theorem, as well as some extensions, can be found in [Sto]. Cauchy's theorem in higher dimensions is an easy consequence (cf [Be] or [Vin]). It follows that such polyhedra are rigid (i.e. admit no nontrivial isometric deformations).

Moreover Cauchy's argument (and, of course, the rigidity consequence) extends to (compact) convex polyhedra in all Riemannian space forms in all dimensions and also to convex polyhedra in H^n with finite volume having some ideal vertices; the proof of the hyperbolic case can be found in [Ri-Ho], which also contains a sketch of the proof in the spherical case. Moreover, the infinitesimal rigidity problems for convex polyhedra in the various (Riemannian or Lorentzian) space-forms are essentially equivalent, since a remarkable construction of Pogorelov takes the problem from one space-form to another. Details and some applications of this to rigidity questions can be found, e.g. in [Sch98], [Sch00] or [Sch01].

A quantity defined for a polyhedron is said to be intrinsic if it depends only on the metric induced on its n-dimensional skeleton. In analogy with the smooth case, the curvature of P at faces of odd codimension is intrinsic (cf. [C-M-S]), in particular the r-th integral mean curvature, $\mathbf{H}_r$, as defined by (7.1), is intrinsic for r even. There exist polyhedra in R^3 admitting nontrivial isometric deformations (cf. [Co]). We can now state the following theorem:

THEOREM 11. *Let P be a polyhedron in M_K^{n+1}. Then $nKV + \mathbf{H}_1$ and $\mathbf{H}_r$, $2 \leqslant r \leqslant n$ are invariant under bendings.*

Proof. (i) The volumes of the faces of P of codimension $\geqslant 1$ are invariant under bendings. For $r = 1$, the result is thus a direct consequence of classical Schläfli formula (E_0) (note that our definition of the curvature at codimension 2 faces involves the external dihedral angles). This was the starting point of the work of F. Almgren and I. Rivin [Al-Ri]. For $r \geqslant 2$, the statement is a consequence of the higher Schläfli formula (E_{r-1}). □

9. Piecewise Space-forms and Generalized Regge Formulas

Let M_K^{n+1} be, as before, the spherical, Euclidean or hyperbolic space of constant curvature K and dimension $n+1$. A piecewise K-space form X^{n+1} of dimension $n+1$ is a simplicial complex which is a triangulation of a compact (n + 1)-dimensional manifold, with or without boundary, endowed with a distance such that:

(i) each q-dimensional face of X is isometric to some q-simplex σ^q of M_K^{n+1}
(ii) X is a path metric space, that is, the distance between each pair of its points equals the infimum of the lengths of curves joining the points.

More details can be found in [C-M-S] (or [La]). The piecewise flat spaces defined in [C-M-S] are more general but only those we consider here are really used by these authors. The Lipschitz–Killing curvature of order p of X is defined on the $(n+1-p)$-skeleton: for a codimension p face σ^{n+1-p}, let

$$R_X(\sigma^{n+1-p}) = \sum_{\sigma^i \supseteq \sigma^{n+1-p}} (-1)^{i-(n+1-p)} \frac{V(S^{p-1})}{V(S^{p-1-(n+1-i)})} K_{\sigma^i}(\sigma^{n+1-p}).$$

The total Lipschitz–Killing curvature of order p of X is then defined as follows:

$$R^p = \sum_{\sigma^{n+1-p}} R_X(\sigma^{n+1-p})|\sigma^{n+1-p}|$$

where $|\sigma^{n+1-p}|$ denotes the volume of σ^{n+1-p}. Now, we can also define the curvatures $K_X(\sigma^{n+1-p})$ in the same way we defined them in Section 6 for polyhedra embedded in

M_K^{n+1}. The same arguments as in Section 6 show these curvatures are the same as Lipschitz–Killing ones and that the higher Schläli formulas apply to piecewise space forms. We therefore have :

PROPOSITION 12 (Generalized Regge formulas). *Let X_t^{n+1} be a 1-parameter family of piecewise K-space forms. Then the variation of the total Lipschitz–Killing curvature of order $p \in \{3, \dots, n+1\}$ is given by:*

$$(R^p)' = \sum_{\sigma^{n+1-p}} R_X(\sigma^{n+1-p})|\sigma^{n+1-p}|' - K\frac{(n+2-p)}{(p-2)} \times \\ \times \sum_{\sigma^{n+3-p}} R_X(\sigma^{n+3-p})|\sigma^{n+3-p}|'.$$

And for $p = 2$:

$$(R^p)' = \sum_{\sigma^{n-1}} R_X(\sigma^{n-1})|\sigma^{n-1}|' - nK\sum_{\sigma^{n+1}} |\sigma^{n+1}|'.$$

Remark. In [C-M-S], the Lipschitz–Killing curvatures are defined under the following normalization

$$\bar{R}_X(\sigma^{n+1-p}) = \frac{1}{V(S^{p-1})} R_X(\sigma^{n+1-p})$$

and, similarly, $\bar{R}^p = (1)/(V(S^{p-1}))R^p$. The generalized Regge formulas then read, for $p \in \{3, \dots, n+1\}$:

$$(\bar{R}^p)' = \sum_{\sigma^{n+1-p}} \bar{R}_X(\sigma^{n+1-p})|\sigma^{n+1-p}|' - \\ - K\frac{(n+2-p)}{(p-2)}\frac{V(S^{p-3})}{V(S^{p-1})} \sum_{\sigma^{n+3-p}} \bar{R}_X(\sigma^{n+3-p})|\sigma^{n+3-p}|'$$

and, for $p = 2$,

$$(\bar{R}^2)' = \sum_{\sigma^{n-1}} \bar{R}_X(\sigma^{n-1})|\sigma^{n-1}|' - K\frac{n}{2\pi}\sum_{\sigma^{n+1}} |\sigma^{n+1}|'.$$

10. In Hyperbolic Manifolds

All the results of the previous sections also hold when the ambient simply connected space-form is replaced by some (well chosen) manifold with constant curvature. We will illustrate this here by considering the special case of convex (smooth or polyhedral) hypersurfaces in a convex cocompact 'quasi-Fuschsian' manifold M.

That is, we consider a hyperbolic complete manifold M homeomorphic to $N \times (0, 1)$, where N is a compact hyperbolic n-manifold. One can obtain 'Fuchsian' examples by considering a group Γ acting on H^n so that the quotient is a compact hyperbolic manifold N, and then considering the natural action of Γ on H^{n+1} (given by the action of Γ on a totally geodesic hyperplane in H^{n+1}). More 'quasi-Fuchsian' examples can be obtained by deforming the action of Γ.

Let $(\Sigma_t)_{t\in[0,1]}$ be a 1 parameter family of convex, compact hypersurfaces in M. Each Σ_t is homeomorphic to N. It is then straightforward to check that the smooth higher Schläfli formulas (F_p) of Theorem 1 extend to this setting (the proof is the same as in Section 1). Using the same limiting argument as in Section 2 also leads to the higher polyhedral Schläfli formulas of Theorem 2.

We also want to define the dual volume of Σ_t in such a way that the dual smooth Schläfli formula holds for deformations of Σ_t.

One way to do this is to lift Σ_t to the universal cover H^{n+1} of M, thus obtaining the universal cover $\tilde{\Sigma}_t$ of Σ_t as a convex complete surface, invariant under the action of Γ.$\tilde{\Sigma}_t$ has a dual hypersurface $\tilde{\Sigma}_t^*$, which is a convex, space-like hypersurface in S_1^{n+1} which is invariant under the action of Γ on S_1^{n+1} induced from the action of Γ on H^{n+1}. One could then check that the $\tilde{\Sigma}_t^*$ and any hemisphere S_0 in S_1^{n+1} bound a domain Ω such that Ω/Γ has finite volume; the dual volume of Σ_t would then be the volume of Ω/Γ.

A simpler approach, however, is to define directly $V^*(\Sigma_t)$ as the volume of the set of hyperplanes intersecting Σ_t. This volume exists because Σ_t is compact. Note that this is possible because $\tilde{\Sigma}$ is globally convex, therefore each hyperplane in H^{n+1} intersects it with a connected intersection.

It is then quite easy to prove that the dual Schläfli formula holds for the deformation (Σ_t). As a consequence, we also recover in this setting the results of Sections 7 and 8 on topological and bending invariants of hypersurfaces.

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