Geometriae Dedicata **55**: 115–123, 1995.

The Generalized Tilt Formula

MAKOTO SAKUMA[1] and JEFFREY R. WEEKS[2]
[1]*Department of Mathematics, Faculty of Science, Osaka University, Toyonaka, Osaka 560, Japan*
e-mail: sakuma@math.wani.osaka-u.ac.jp
[2]*The Geometry Center, 1300 South Second Street, Minneapolis, MN 55454, U.S.A.*
e-mail: weeks@geom.umn.edu

(Received: 21 January 1993; revised version: 15 September 1994)

Abstract. A convex hull construction in Minkowski space defines a canonical cell decomposition for a cusped hyperbolic n-manifold. An algorithm to compute the canonical cell decomposition uses the concept of the 'tilt' of an n-simplex relative to each of its $(n-1)$-dimensional faces. An essential tool for computing tilts is the tilt theorem. The tilt theorem was previously known only in dimensions $n \leq 3$, and the proof was needlessly complicated. Here we offer a new, simplified proof which applies in all dimensions. We also offer a second geometric interpretation of the tilt.

Mathematics Subject Classifications (1991): Primary: 51M10; secondary: 51M09, 57M50.

0. Introduction

A convex hull construction in Minkowski space defines a canonical cell decomposition of a cusped hyperbolic n-manifold ([EP], [W]). An efficient algorithm for computing canonical cell decompositions relies on the so-called tilt formula ([W]). The algorithm allows the computer program SnapPea to quickly determine whether two cusped hyperbolic 3-manifolds are isometric, and to compute the symmetry group of a cusped hyperbolic 3-manifold. In particular, SnapPea can decide whether two hyperbolic knots or links are equivalent, and can compute the symmetry group of a hyperbolic knot or link ([HeW]). Further applications of the canonical cell decomposition appear in [AHW], [HiW], [HMW] and [SW].

Unfortunately, the proof of the tilt formula given in [W, Th 1 and 2] is needlessly complicated, and is restricted to 2- and 3-dimensional manifolds. The present paper generalizes the tilt formula to n dimensions, provides a conceptually clean proof, and offers a second geometric interpretation of the tilt. Section 1 reviews the definition of the tilt. Section 2 states and proves our generalization of the tilt formula. Section 3 offers a second geometric interpretation of the tilt. Section 4 comments on remaining obstacles to computing higher-dimensional canonical decompositions.

1. The Definition of the Tilt

This section briefly reviews the definition of the tilt. For more complete background and motivation, including the use of the tilt in computing the canonical cell decomposition, please see [W].

We will work in the Minkowski space model of hyperbolic n-space. The Minkowski space $\mathbf{E}^{n,1}$ is the real vector space $\mathbf{R}^{n+1}$ with the inner product $\langle \mathbf{x},\ \mathbf{y}\rangle = -x_0y_0 + x_1y_1 + \cdots + x_ny_n$. Hyperbolic n-space is the set $\mathbf{H}^n = \{\mathbf{x} \in \mathbf{E}^{n,1} | \langle \mathbf{x},\ \mathbf{x}\rangle = -1 \text{ and } x_0 > 0\}$. Throughout this paper let T denote an ideal triangulation for a cusped hyperbolic n-manifold M. Choose horospherical cross-sections of the cusps bounding equal volumes. The preimage of the cusp cross-sections in the universal cover $\mathbf{H}^n$ is an infinite set S of horospheres, invariant under the action of the group of covering transformations. In the Minkowski space model, each horosphere is the intersection of $\mathbf{H}^n$ with a hyperplane W whose normal vector is lightlike; we associate to each horosphere the unique vector $\mathbf{v}$ such that $\langle \mathbf{v},\ \mathbf{w}\rangle = -1$ for all $\mathbf{w} \in W$. Let V be the set of points on the light cone corresponding to the horospheres in S. For an n-simplex F of T, let $\tilde{F}$ be a lift of F to the universal cover $\mathbf{H}^n$, and let $\hat{F}$ be the convex hull in $\mathbf{E}^{n,1}$ spanned by the points $\{\mathbf{v}_0,\ \mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ corresponding to the ideal vertices of $\tilde{F}$. The normal vector $\mathbf{p}$ to $\hat{F}$ is defined by the condition $\langle \mathbf{p},\ \mathbf{x}\rangle = -1$ for all $\mathbf{x} \in \hat{F}$. Let E_i be the face of F opposite the ideal vertex corresponding to $\mathbf{v}_i$. Let $\hat{E}_i$ (resp. $\tilde{E}_i$) be the face of $\hat{F}$ (resp. $\tilde{F}$) corresponding to E_i, and let $\mathbf{m}_i$ be the outward pointing unit normal to the hyperplane in Minkowski space containing $\hat{E}_i$ and the origin.

DEFINITION 1.1 The tilt t_i of F relative to E_i is the inner product $\langle \mathbf{m}_i,\ \mathbf{p}\rangle$.

2. The Generalized Tilt Formula

The intersection of the cusp cross-section with the ith ideal vertex of F is a Euclidean $(n-1)$-simplex. We call it the ith *vertex cross-section* and measure its size by the radius R_i of its circumscribed sphere (in the Euclidean geometry of the cusp cross-section). The following theorem generalizes Theorems 3.2 and 5.1 of [W].

THEOREM 2.1. *In an ideal triangulation of a cusped hyperbolic n-manifold, the tilt of an ideal n-simplex relative to each of its codimension 1 faces may be computed as*

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} 1 & -\cos\theta_{01} & -\cos\theta_{02} & \ldots & -\cos\theta_{0n} \\ -\cos\theta_{10} & 1 & -\cos\theta_{12} & \ldots & -\cos\theta_{1n} \\ -\cos\theta_{20} & -\cos\theta_{21} & 1 & \ldots & -\cos\theta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\cos\theta_{n0} & -\cos\theta_{n1} & -\cos\theta_{n2} & \ldots & 1 \end{pmatrix} \begin{pmatrix} R_0 \\ R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}.$$

where t_i is the tilt relative to the face opposite vertex i, R_i is the circumradius of vertex cross-section i, and θ_{ij} is the dihedral angle between the faces opposite vertices i and j.

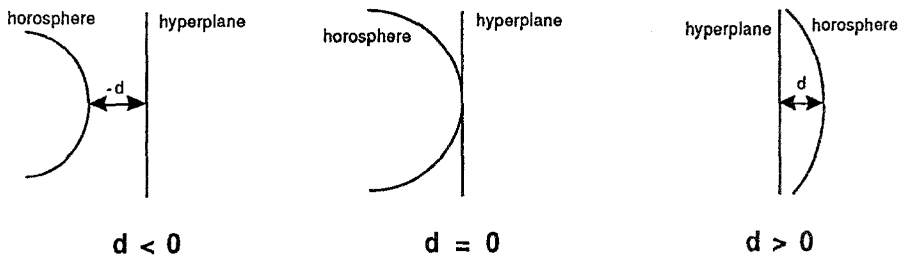

Fig. 2.1. The signed distance from a hyperplane to a horosphere is the distance d by which the horosphere extends past the hyperplane.

DEFINITION 2.2. The *signed distance from a hyperplane to a horosphere* is the distance by which the horosphere extends past the hyperplane. The signed distance may be positive, negative or zero, as shown in Figure 2.1.

Let d_i denote the signed distance from the hyperplane containing E_i to the horosphere containing the cusp cross-section at vertex i.

ORGANIZATIONAL NOTE. We have organized the proof of Theorem 2.1 and its supporting lemmas in a top-down fashion: we begin with the overall plan, and gradually fill in more details. We hope this top-down organization makes the proof easy to read and understand. The actual logical dependence among the lemmas is as follows:

$$\begin{tikzcd}
& \text{Theorem 2.1} & \\
\text{Lemma 2.3} \arrow[ur] \arrow[r] & \text{Lemma 2.5} \arrow[u] & \text{Lemma 2.7} \arrow[ul] \\
\text{Lemma 2.4} \arrow[u] & \text{Lemma 2.6} \arrow[u] &
\end{tikzcd}$$

PROOF OF THEOREM 2.1. Lemma 2.3 shows that the vectors $\{\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_n\}$ form a basis for Minkowski space. Relative to this basis, $\mathbf{m}_k = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, $\mathbf{p} = (R_0, R_1, \ldots, R_n)$ by Lemma 2.5, and the metric is given by the matrix computed in Lemma 2.7. Therefore

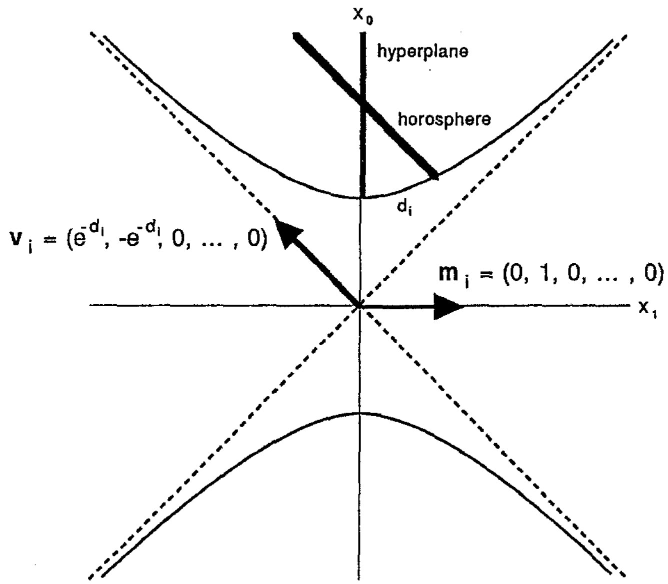

Fig. 2.2. Lemma 2.4 shows that the signed distance from a hyperplane to a horosphere along their common perpendicular may be computed using the inner product of their normal vectors.

$$\begin{aligned} t_k &= \langle \mathbf{m}_k, \mathbf{p} \rangle \\ &= (0, \ldots, 0, 1, 0, \ldots, 0) \\ &\quad \begin{pmatrix} 1 & -\cos\theta_{01} & -\cos\theta_{02} & \cdots & -\cos\theta_{0n} \\ -\cos\theta_{10} & 1 & -\cos\theta_{12} & \cdots & -\cos\theta_{1n} \\ -\cos\theta_{20} & -\cos\theta_{21} & 1 & \cdots & -\cos\theta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\cos\theta_{n0} & -\cos\theta_{n1} & -\cos\theta_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} R_0 \\ R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} \end{aligned}$$

and the theorem follows. □

LEMMA 2.3. *The set* $\{\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_n\}$ *forms a basis for Minkowski space, and is dual to the basis* $\{-\mathrm{e}^{d_0}\, \mathbf{v}_0, -\mathrm{e}^{d_1}\, \mathbf{v}_1, \ldots, -\mathrm{e}^{d_n}\, \mathbf{v}_n\}$.

Proof. For $i \neq j$, $\mathbf{v}_j$ lies in the hyperplane orthogonal to $\mathbf{m}_i$, so $\langle \mathbf{m}_i, \mathbf{v}_j \rangle = 0$. By Lemma 2.4, $\langle \mathbf{m}_i, \mathbf{v}_i \rangle = -e^{-d_i}$. It follows that $\{\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_n\}$ and $\{-\mathrm{e}^{d_0}\mathbf{v}_0, -\mathrm{e}^{d_1}\mathbf{v}_1, \ldots, -\mathrm{e}^{d_n}\mathbf{v}_n\}$ are dual; that is, $\langle \mathbf{m}_i, -\mathrm{e}^{d_j}\mathbf{v}_j \rangle = \delta_{ij}$. This duality implies that each set is linearly independent, and therefore forms a basis for Minkowski space. □

LEMMA 2.4. $\langle \mathbf{m}_i, \mathbf{v}_i \rangle = -\mathrm{e}^{-d_i}$.

Proof. The hyperplane determined by $\mathbf{m}_i$ and the horosphere determined by $\mathbf{v}_i$ have a unique common perpendicular. Choose coordinates so that this common perpendicular intersects the hyperplane at $(1, 0, \ldots, 0)$, and the horosphere at $(\cosh d_i, \sinh d_i, 0, \ldots, 0)$ (see Figure 2.2). In this coordinate system, $\mathbf{m}_i = (0, 1, 0, \ldots, 0)$. The vector $\mathbf{v}_i$ lies somewhere on the ray $(t, -t, 0, \ldots, 0)$. To find t, use the condition that $\langle \mathbf{v}_i, \mathbf{w} \rangle = -1$ for all points on the horosphere: $\langle (t, -t, 0, \ldots, 0), (\cosh d_i, \sinh d_i, 0, \ldots, 0) \rangle = -t\, \mathrm{e}^{d_i} = -1$, so $t = \mathrm{e}^{-d_i}$. Hence $\langle \mathbf{m}_i, \mathbf{v}_i \rangle = -\mathrm{e}^{-d_i}$. □

LEMMA 2.5. $\mathbf{p} = (R_0, R_1, \ldots, R_n)$ *relative to the basis* $\{\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_n\}$.

Proof.
$$
\begin{aligned}
\mathbf{p} &= \textstyle\sum \langle \mathbf{p}, -\mathrm{e}^{d_i}\mathbf{v}_i \rangle \mathbf{m}_i && (\{\mathbf{m}_0, \ldots, \mathbf{m}_n\} \text{ and } \{-\mathrm{e}^{d_0}\mathbf{v}_0, \ldots, -\mathrm{e}^{d_n}\mathbf{v}_n\} \\
& && \quad \text{are dual by Lemma 2.3)} \\
&= \textstyle\sum \mathrm{e}^{d_i}\mathbf{m}_i && (\langle \mathbf{p}, \mathbf{v}_i \rangle = -1, \text{ by the definition of } \mathbf{p}) \\
&= \textstyle\sum R_i \mathbf{m}_i && \text{(Lemma 2.6).}
\end{aligned}
$$
□

LEMMA 2.6. $R_i = \mathrm{e}^{d_i}$.

Proof. Position the ideal n-simplex in the upper half space model of $\mathbf{H}^n$ so that vertex i is at infinity, and the hyperplane containing the opposite face is a Euclidean hemisphere of radius one (Figure 2.3). A vertex cross-section tangent to the opposite face has circumradius 1. More generally, a vertex cross-section a signed distance d from the opposite face has circumradius e^d. □

Comment. The techniques of Section 3 below allow one to prove Lemma 2.6 wholly within the Minkowski space model of $\mathbf{H}^n$, without recourse to the upper half space model.

Comment. The appearance of the circumradii R_i in the statement of Theorem 2.1 has no deep significance. One could replace the vector $(R_0, R_1, \ldots, R_n)$ with the equivalent vector $(\mathrm{e}^{d_0}, \mathrm{e}^{d_1}, \ldots, \mathrm{e}^{d_n})$. The reason for using the circumradii is that they are easily computed in the computer program which finds canonical cell decompositions for cusped hyperbolic 3-manifolds.

LEMMA 2.7. *Relative to the basis* $\{\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_n\}$, *the Minkowski space metric is*

$$
\begin{pmatrix}
1 & -\cos\theta_{01} & -\cos\theta_{02} & \cdots & -\cos\theta_{0n} \\
-\cos\theta_{10} & 1 & -\cos\theta_{12} & \cdots & -\cos\theta_{1n} \\
-\cos\theta_{20} & -\cos\theta_{21} & 1 & \cdots & -\cos\theta_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\cos\theta_{n0} & -\cos\theta_{n1} & -\cos\theta_{n2} & \cdots & 1
\end{pmatrix}.
$$

Proof. Because the $\mathbf{m}$'s are unit vectors, $\langle \mathbf{m}_i, \mathbf{m}_i \rangle = 1$. For $i \neq j$, the angle between $\mathbf{m}_i$ and $\mathbf{m}_j$ equals the exterior angle between faces E_i and E_j, hence

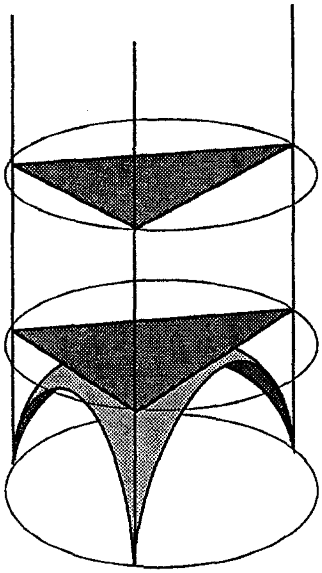

Fig. 2.3. A vertex cross-section a signed distance d from the opposite face has circumradius e^d.

$\langle \mathbf{m}_i, \mathbf{m}_j \rangle$ equals the cosine of the exterior angle, which is the negative of the cosine of the interior angle. □

3. A Geometric Interpretation of the Tilt

Theorem 3.1 interprets the tilt within the intrinsic geometry of the hyperbolic space, without reference to the construction in Minkowski space.

THEOREM 3.1. *With notation as in Section 1, assume the cusp cross-sections at the vertices of an ideal n-simplex $\tilde{F}$ are equidistant from a point $Q \in \mathbf{H}^n$. Let F', E_i' and Q' be the orthogonal projections of $\tilde{F}$, $\tilde{E}_i$ and Q onto the cusp cross-section at some vertex $j \neq i$. Let δ be the signed distance (measured in the Euclidean cusp cross-section) from E_i' to Q', where the sign of δ is negative if Q' and F' are on the same side of E_i', and positive if they are on opposite sides. Then δ equals the tilt t_i of F relative to E_i.*

NOTE. The distance δ is independent of j.

The proof of Theorem 3.1 uses the following two lemmas. As in Sections 1 and 2, we represent horospheres by vectors on the light cone.

LEMMA 3.2. *If δ is the hyperbolic distance from a point $\mathbf{u} \in \mathbf{H}^n$ to the horosphere determined by the vector $\mathbf{v}$, then $\langle \mathbf{u}, \mathbf{v} \rangle = -e^{\delta}$.*

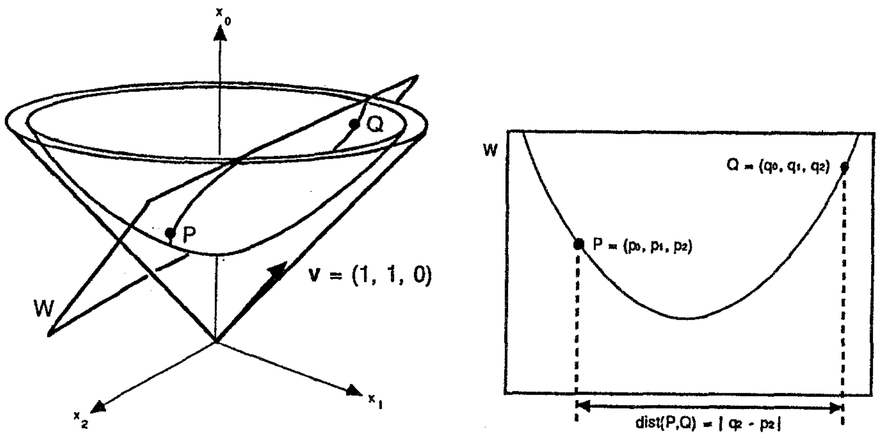

Fig. 3.1. The plane W containing the horocycle inherits a degenerate metric from Minkowski space. The distance between the points P and Q, measured along the horocycle, is simply the difference of their third coordinates.

Proof. Choose coordinates so that $\mathbf{u} = (1,\ 0, \ldots,\ 0)$ and the closest point on the horosphere is at $\mathbf{w} = (\cosh\ \delta,\ \sinh\ \delta,\ 0, \ldots,\ 0)$. The vector $\mathbf{v}$ must lie somewhere on the ray $(t,\ t,\ 0, \ldots,\ 0)$; the condition $\langle \mathbf{v},\ \mathbf{w} \rangle = -1$ implies $\mathbf{v} = (\mathrm{e}^{\delta},\ \mathrm{e}^{\delta},\ 0, \ldots,\ 0)$. Therefore $\langle \mathbf{u},\ \mathbf{v} \rangle = -\mathrm{e}^{\delta}$. □

We now address the question of how to measure distances along a horosphere. A simple 2-dimensional example motivates the general method. Consider the horocycle corresponding to the vector $\mathbf{v} = (1,\ 1,\ 0)$ (Figure 3.1). The plane W containing the horocycle inherits a degenerate metric from the ambient Minkowski space metric, so to measure the distance *along the horocycle* between two points P and Q, we need only subtract their third coordinates! That is, the distance along the horocycle from $P = (p_0,\ p_1,\ p_2)$ to $Q = (q_0,\ q_1,\ q_2)$ is $|q_2 - p_2|$.

The same principle applies in higher dimensions. A horosphere is defined by an n-dimensional hyperplane in $(n,\ 1)$-dimensional Minkowski space. The hyperplane inherits a degenerate metric. When we mod out by the equivalence relation $\{\mathbf{u} \sim \mathbf{v} \text{ iff } |\mathbf{u} - \mathbf{v}| = 0\}$, we project down to an $(n - 1)$-dimensional Euclidean space. Distances in this Euclidean quotient space equal distances along the horosphere.

LEMMA 3.3. *Let* $\mathbf{u} \in \mathbf{H}^n$ *be a point, let* $E \subset \mathbf{H}^n$ *be a hyperplane, and let* H *be a horosphere centered at an ideal point of* E *and containing* $\mathbf{u}$. *Then the signed distance from* E *to* $\mathbf{u}$, *measured along the horosphere* H, *is* $\langle \mathbf{m},\ \mathbf{u} \rangle$, *where* $\mathbf{m}$ *is a unit normal to the hyperplane* Π *in Minkowski space which contains* E. *The signed distance will be positive or negative according to whether* $\mathbf{u}$ *and* $\mathbf{m}$ *lie on the same side of* Π.

Proof. Let $\mathbf{v}$ be the vector corresponding to the horosphere H. Let D be the 3-plane of Minkowski space spanned by $\mathbf{u}$, $\mathbf{m}$, and $\mathbf{v}$ (Figure 3.2), and let $D^{\perp}$

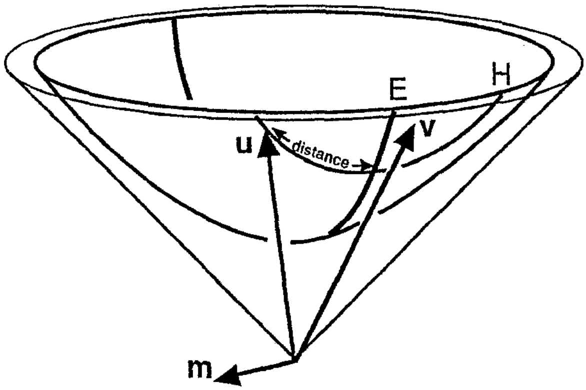

Fig. 3.2. The signed distance from the hyperplane E to the point $\mathbf{u}$, measured along the horosphere H, is $\langle \mathbf{m}, \mathbf{u} \rangle$.

be its orthogonal complement (not shown). The discussion preceding this lemma says that to measure distances along H, we simply project orthogonally onto the $(n-1)$-plane spanned by $D^{\perp}$ and $\mathbf{m}$, and measure using its Euclidean metric. It follows that the distance from E to $\mathbf{u}$ along H is just the length of the orthogonal projection of $\mathbf{u}$ onto $\mathbf{m}$, which is $\langle \mathbf{m}, \mathbf{u} \rangle$. □

PROOF OF THEOREM 3.1. Let $\mathbf{q}$ and $\mathbf{q}'$ be the points Q and Q' of $\mathbf{H}^n$, thought of as unit vectors in Minkowski space. We first show that q is a multiple of $\mathbf{p}$. Because Q is equidistant from the vertex cross-sections of F, Lemma 3.2 implies that $\langle \mathbf{q}, \mathbf{v}_i \rangle$ is constant for all i. This implies that $\mathbf{q}$ is orthogonal to $\hat{F}$, hence $\mathbf{q} = \alpha \mathbf{p}$ for some $\alpha \in \mathbf{R}$.

By Lemma 3.3 we may compute the distance δ from E_i' to Q' as the inner product $\langle \mathbf{m}_i, \mathbf{q}' \rangle$. The vector $\mathbf{q}'$ lies in the 2-plane spanned by the vectors $\mathbf{q}$ and $\mathbf{v}_j$. Therefore any vector which is orthogonal to $\mathbf{q}$ and $\mathbf{v}_j$ must be orthogonal to $\mathbf{q}'$ as well. The vector $\mathbf{m}_i + s\mathbf{v}_j$ is orthogonal to $\mathbf{v}_j$ for any choice of $s \in \mathbf{R}$, and is orthogonal to $\mathbf{q}$ iff

$$\begin{aligned} s &= -\frac{\langle \mathbf{m}_i, \mathbf{q} \rangle}{\langle \mathbf{v}_i, \mathbf{q} \rangle} \\ &= -\frac{\langle \mathbf{m}_i, \mathbf{p} \rangle}{\langle \mathbf{v}_i, \mathbf{p} \rangle} \quad \text{(because } \mathbf{q} = \alpha\mathbf{p}) \\ &= \langle \mathbf{m}_i, \mathbf{p} \rangle \quad \text{(because } \langle \mathbf{v}_i, \mathbf{p} \rangle = -1 \text{ by the definition of } \mathbf{p}). \end{aligned}$$

We conclude that $\mathbf{m}_i + \langle \mathbf{m}_i, \mathbf{p} \rangle \mathbf{v}_j$ is orthogonal to $\mathbf{q}'$. Therefore $\langle \mathbf{m}_i + \langle \mathbf{m}_i, \mathbf{p} \rangle \mathbf{v}_j, \mathbf{q}' \rangle = 0$, hence $\langle \mathbf{m}_i, \mathbf{q}' \rangle = -\langle \mathbf{m}_i, \mathbf{p} \rangle \langle \mathbf{v}_j, \mathbf{q}' \rangle$. But since $\mathbf{q}'$ lies on the horosphere determined by $\mathbf{v}_j$, $\langle \mathbf{v}_j, \mathbf{q}' \rangle = -1$. Therefore $\delta = \langle \mathbf{m}_i, \mathbf{q}' \rangle = \langle \mathbf{m}_i, \mathbf{p} \rangle = t_i$. □

NOTE. The geometric intuition behind the above proof is that the vectors $\{\mathbf{m}_i + s\mathbf{v}_j | s \in \mathbf{R}\}$ are the unit normals to a family of hyperplanes in $\mathbf{H}^n$ which are orthogonal to the jth vertex cross-section, and parallel to E_i. By solving for s in the above proof, we in effect found the hyperplane passing through $\mathbf{q}$. Note that we never explicitly found the point $\mathbf{q}'$, but knowing which hyperplane it is on determines its projection onto $\mathbf{m}_i$, which by Lemma 3.3 is all that matters.

4. Higher-dimensional Canonical Decompositions

The canonical decomposition algorithm of [W] has not been proved to terminate for dimension $n \leq 3$ (although in practice it always does). In addition, for $n \geq 4$ we have been unable to prove that two adjacent lifts of a single n-simplex must meet at a convex dihedral angle on the hull (Corollaries 1 and 2 of [W] prove this for $n \leq 3$). If two adjacent lifts of a single n-simplex were to meet at a concave dihedral angle, we would not be able to carry out the local retriangulation necessary to remove the concavity. To circumvent these difficulties, we conclude with a challenge.

CHALLENGE. Find an algorithm which computes the canonical cell decomposition of any cusped hyperbolic n-manifold in a finite number of steps.

Acknowledgements

This joint work was done while the first-named author was staying at the University of Toronto and Ruhr University. He would like to express his hearty thanks to Prof. K. Murasugi and Prof. H. Zieschang for their hospitality and encouragement. The second-named author would like to thank the Geometry Center (NSF Science and Technology Research Center for computation and visualization of Geometric Structures) for its support.

References

[AHW] Adams, C., Hildebrand, M. and Weeks, J.: Hyperbolic invariants of knots and links, *Trans. Amer. math. Soc.* **326** (1991), 1–56.

[EP] Epstein, D. B. A. and Penner, R. C.: Euclidean decompositions of noncompact hyperbolic manifolds, *J. Differential Geom.* **27** (1988), 67–80.

[HeW] Henry, S. and Weeks, J.: Symmetry groups of hyperbolic knots and links, *J. Knot Theory Ram.* **1** (1992), 185–201.

[HiW] Hildebrand, M. and Weeks, J.: A computer generated census of cusped hyperbolic 3-manifolds, in E. Kaltofen and S. Watt (eds), *Computers and Mathematics*, Springer-Verlag, 1989, pp. 53–59.

[HMW] Hodgson, C., Meyerhoff, R. and Weeks, J.: Surgeries on the Whitehead link yield geometrically similar manifolds, in B. Apanasov, W. Neumann, A. Reid and L. Siebenmann (eds), *Topology 90, Proc. Research Semester in Low Dimensional Topology at Ohio State University*, de Gruyter Verlag, Berlin, 1992, pp. 195–206.

[SW] Sakuma, M. and Weeks, J.: Examples of canonical decompositions of hyperbolic link complements, *Japanese Journal of Mathematics* **21** (1995).

[W] Weeks, J.: Convex hulls and isometries of cusped hyperbolic 3-manifolds, *Topology Appl.* **52** (1993), 127–149.