THREE-DIMENSIONAL HYPERBOLIC MANIFOLDS WITH GENERAL FUNDAMENTAL POLYHEDRON

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1. As Thurston showed [1], the volumes of three-dimensioanl hyperbolic manifolds, i.e., quotient-spaces H^3/Γ of the Lobachevskii space H^3 by a discrete group Γ of isometries acting without fixed points, form a totally ordered set and the number of manifolds having fixed volume is finite.

For any natural number N in [2] there is constructed a polyhedron of finite volume in H^3 which is fundamental for at least N different 3-manifolds. The proof is based on the consideration of subgroups of finite index of the Picard group PSL (2, Z [i]) and the corresponding noncompact manifolds.

In the present note we give a number of right-angled polyhedra with the analagous property but now for closed manifolds, namely we prove the following

THEOREM. For any natural number N there exists a right-angled polyhedron in H^3 which is fundamental for at least N nonisometric closed orientable hyperbolic three-manifolds.

2. The proof of the theorem is constructive and is based on the method of construction of the series of hyperbolic three-manifolds described in [3].

Let R be a right-angled polyhedron in Lobachevskii space H^3, G be the group generated by reflections in the faces of R, and S the group of its symmetries. We note that the stabilizer in g of each vertex of R is isomorphic to the eight-element group $Z_2 \oplus Z_2 \oplus Z_2$, and we consider an epimorphism $\varphi: G \to Z_2 \oplus Z_2 \oplus Z_2$. Let α, β, γ be three linearly independent elements of $Z_2 \oplus Z_2 \oplus Z_2$, and $\delta = \alpha + \beta + \gamma$.

LEMMA 1 [3]. If the epimorphism φ carries reflections in faces of R into the elements α, β, γ, δ and the images of reflections in any three faces having common vertex are different, then $H^3/\mathrm{Ker}\,\varphi$ is an orientable hyperbolic three-dimensional manifold.

It is obvious that to each epimorphism φ satisfying the hypotheses of the lemma there corresponds a regular coloring of the faces of R in four colors α, β, γ, δ. Conversely, each such coloring defines an epimorphism φ, satisfying the hypotheses of the lemma. We shall determine when colorings of faces correspond to different manifolds.

Definition. A group G generated by reflections is said to be naturally maximal if its extension $G^* = \langle G, S\rangle$ is a maximal discrete group, i.e., is not a proper subgroup of any discrete group of isometries of H^3.

LEMMA 2. Let G be a nonarithmetic naturally maximal group and Γ_1, Γ_2 be the kernels of epimorphisms φ_1 and φ_2 satisfying the hypotheses of Lemma 1. If Γ_1 is isomorphic to Γ_2 then there exists an element $s \in S$ such that $\Gamma_1 = s^{-1}\Gamma_2 s$.

Proof. By virtue of Lemma 1, Γ_1 and Γ_2 are fundamental groups of closed hyperbolic three-manifolds so by the Mostow rigidity theorem there exists an element $h \in \mathrm{Isom}(H^3)$ such that $\Gamma_1 = h^{-1}\Gamma_2 h$. We consider the commensurate of the group G, $C(G) = \{\gamma \in \mathrm{Isom}(H^3)$: $\gamma^{-1}G\gamma$ is commensurate with G$\}$. Since $\Gamma_1 \triangleleft_8 G$ and $\Gamma_2 \triangleleft_8 G$, and hence $\Gamma_1 = h^{-1}\Gamma_2 h \triangleleft_8 h^{-1}Gh$, one has $h \in C(G)$. By hypothesis G is nonarithmetic so by Margulis' theorem [4] C (G) is discrete, but $G^* = \langle G, S\rangle$ is a maximal discrete group and $G^* \subset C(G)$, so $C(G) = G^*$. Consequently, $h = g \cdot s$, where $g \in G$, $s \in S$, but Γ_1, Γ_2 are normal subgroups of G so $\Gamma_1 = s^{-1}\Gamma_2 s$. The lemma is proved.

We note that if the kernels Γ_1 and Γ_2 are conjugate by an element of the group S, then the elements of the group G lying on one coset with respect to Γ_1, lie in one coset with respect to Γ_2 also so faces of R colored in the same color under the epimorphism φ_1 are colored in the same color under the epimorphism φ_2. Thus, N colorings of faces of R which are not equivalent with respect to its symmetries give N nonisomorphic subgroups of index 8 in G and hence N different manifolds.

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3. We construct polyhedra of [3] satisfying the hypotheses of Lemma 2 and give epimorphisms explicitly defining nonisomorphic fundamental groups.

Let ABCA'B'C' be a triangular prism. We draw an edge DE with vertices D and E lying on BB' and CC'. By Andreev's theorem [5] for any integer $n \geq 5$ there exists in the Lobachevskii space H^3 a bounded convex hexahedron ABCA'B'C'DE with dihedral angles π/n, $\pi/4$, and $\pi/4$ for the edges AA', BD, and EC' respectively and with right angles for the remaining edges. Let $\Delta(n)$ be the group generated by reflections in the faces of the hexahedron. Its Coxeter scheme contains as a subscheme the scheme of a triangular group with signature (2, 4, n). By [6, p. 91], if the group $\Delta(n)$ is arithmetic, then (2, 4, n) must be arithmetic also, but as shown in [7], this is only possible if n = 5, 6, 7, 8, 10, 12, 18. Consequently, for all other values of n the group $\Delta(n)$ is nonarithmetic.

By the action of the dihedral group D_n, the stabilizer of the edge AA' in $\Delta(n)$, from 2n copies of our hexahedron we get a right-angled (2n + 2)-hedron R(n) whose lateral surface consists of two regular right-angled n-gons and 2n right-angled pentagons. In particular, R(5) is a right-angled dodecahedron. We denote by S(n) the group of symmetries of R(n) and by G(n) the group generated by reflections in the faces of R(n) which has the following representation: (Generators)

$$g_1,\ g_2,\ \ldots,\ g_{2n+2},$$

(relations)

$$g_{2n+2}^2 = 1,\ g_i^2 = 1,\ g_i g_{i+1} = g_{i+1} g_i\ (i = 1,\ \ldots,\ 2n-1);$$
$$g_i g_{2n+1} = g_{2n+1} g_i,\ g_{n+i} g_{2n+2} = g_{2n+2} g_{n+i}\ (i = 1,\ \ldots,\ n);$$
$$g_i g_{n+i} = g_{n+i} g_i,\ g_i g_{n+i+1} = g_{n+i+1} g_i\ (i = 1,\ \ldots,\ n-1);$$
$$g_1 g_n = g_n g_1,\ g_{n+1} g_{2n} = g_{2n} g_{n+1},\ g_n g_{2n} = g_{2n} g_n.$$

Following [3] we call closed orientable hyperbolic manifolds $H^3/\mathrm{Ker}\ \varphi$, where $\varphi\colon G \to Z_2 \oplus Z_2 \oplus Z_2$ is an epimorphism satisfying the hypotheses of Lemma 1 manifolds of Löbell type.

Since G(n) is a subgroup of finite index of $\Delta(n)$ is naturally, they are simultaneously nonarithmetic. In addition it is proved in [8] that for $n \geq 6$ the group G(n) is naturally maximal. Consequently, udner the restrictions indicated the group G(n) satisfies the hypothesis of Lemma 2.

4. Let n = 6k, k > 3, and we give epimorphisms with nonisomorphic kernels. Let $\varphi\colon G(n) \to Z_2 \oplus Z_2 \oplus Z_2$ Let $(i_1, i_2, \ldots, i_k)$ be different collections of the numbers 1, 2, 3, arranged in nondecreasing order. There are (2k + 1) such collections altogether. We define an epimorphism $\varphi_{i_1 \ldots i_k} = (f_{i_1}, \ldots, f_{i_k})\colon G(6k) \to Z_2{}^3$ by the following rule: if $j = 6p + q$, $1 \leq q \leq 6$, then we set

$$\varphi_{i_1 \ldots i_k}(g_j) = f_{i_{p+1}}(g_q),$$
$$\varphi_{i_1 \ldots i_k}(g_{n+j}) = f_{i_{p+1}}(g_{6+q})\ (j = 1,\ \ldots,\ n),$$
$$\varphi_{i_1 \ldots i_k}(g_{2n+1}) = \alpha,\ \varphi_{i_1 \ldots i_k}(g_{2n+2}) = \beta,$$

where for i = 1, 2, 3,

$$f_i(g_{12}) = f_1(g_8) = f_1(g_{10}) = f_2(g_9) = f_3(g_8) = f_3(g_{10}) = \alpha,$$
$$f_i(g_1) = f_1(g_3) = f_1(g_5) = f_2(g_4) = f_3(g_3) = f_3(g_5) = \beta,$$
$$f_i(g_7) = f_1(g_2) = f_1(g_4) = f_2(g_2) = f_2(g_5) = f_2(g_{10}) = f_3(g_9) = f_3(g_{11}) = \gamma,$$
$$f_i(g_6) = f_1(g_9) = f_1(g_{11}) = f_2(g_3) = f_2(g_8) = f_2(g_{11}) = f_3(g_2) = f_3(g_4) = \delta.$$

One verifies directly that the epimorphisms $\varphi_{i_1 \ldots i_k}$ satisfy the hypothesis of Lemma 1. We note that the number of pentagonal faces on the lower layer of R(n) colored in the color γ, is equal to the sum $i_1 + i_2 + \ldots + i_k$, so all (2k + 1) colorings of faces of R(n) induced by epimorphisms $\varphi_{i_1 \ldots i_k}$ are not equivalent with respect to symmetries of the polyhedron 4(n). By Lemma 2 the kernels $\Gamma_1, \ldots, \Gamma_{2k+1}$ are nonisomorphic groups.

Each of the groups indicated $\Gamma_1, \ldots, \Gamma_{2k+1}$ is a fundamental group of a Löbell type manifold obtained from eight copies of the (2n + 2)-hedron R(n). Up to conjugacy in Isom (H^3) we can assume that the right-angled polyhedron $\tilde{R}(n)$, obtained by gluing eight copies of R(n) is fundamental for all Γ_i. Thus, from the polyhedron $\tilde{R}(6k)$ we can glue at least N different Löbell type manifolds. The theorem is proved.

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APPROXIMATION OF FUNCTIONS OF CLASS C(ε) BY SEQUENCES OF DE LA VALLÉE–POUSSIN SUMS

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Let C be the space of continuous 2π-periodic functions f with uniform norm $\|f\|$, $s_n(f)$ be the Fourier sum of order n, $\sigma_{np}(f) = \frac{1}{p+1}\sum_{\nu=n-p}^{n} s_\nu(F)$, $0 \le p \le n$, $n = 0, 1, \ldots$ be the de la Vallée-Poussin sum (clearly, $\sigma_{n0}(f) = s_n(f)$, and $\sigma_{nm}(f) = \sigma_n(f)$ is the Fejér sum), $E_n(F)$ is the best uniform approximation of the function f by trigonometric polynomials of degree no higher than n, $C(\varepsilon) = \{f \in C\colon\ E_n(f) \le \varepsilon_n,\ n = 0, 1, \ldots\}$, where $\varepsilon = \{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of real numbers, $\varepsilon_n \downarrow 0$ as $n \to \infty$, $e(C(\varepsilon), \sigma_{np}) = \sup\{\|f - \sigma_{np}(f)\|: f \in C(\varepsilon)\}$. Everywhere in what follows the relation $\alpha \ll \beta$ ($\alpha \asymp \beta$) denotes the existence of an absolute $c > 0$, for which $\alpha \le c\beta$ ($c^{-1}\beta \le \alpha \le c\beta$).

Two-sided bounds of the magnitude $e(C(\varepsilon, \sigma_{np})$ were obtained in [1-4]. This was first done for Fejér sums $\sigma_n(f)$ by Stechkin [1], then – for Fourier sums by Oskolkov [2]. Finally, for arbitrary de la Vallée–Poussin sums, Stechkin [3] and Damen [4] showed independently that

$$e(C(\varepsilon), \sigma_{np}) \asymp \sum_{\nu=0}^{n} \frac{\varepsilon_{n-p+\nu}}{p+\nu+1}. \quad (1)$$

For $p \ge n/2$ the right part of (1) can be replaced by

$$\frac{1}{p+1}\sum_{\nu=n-p}^{n} \varepsilon_\nu.$$

In this case in [3] (and for p = n – also in [1]) to obtain lower bounds, there is used the function $f(x) = \sum_{n=1}^{\infty} (\varepsilon_{n-1} - \varepsilon_n)\cos nx$, which belongs to the class C(ε) and, in this case,

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