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VOLUMES OF HYPERBOLIC THREE-MANIFOLDS

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§1. INTRODUCTION

BY "hyperbolic 3-manifold" we will mean an orientable complete hyperbolic 3-manifold M of finite volume. By Mostow rigidity the volume of M is a topological invariant, indeed a homotopy invariant, of the manifold M. There is in fact a purely topological definition of this invariant, due to Gromov. The set of all possible volumes of hyperbolic 3-manifolds is known to be a well-ordered subset of the real numbers and is of considerable interest (for number theoretic aspects see, for instance, [2], [13]) but remarkably little is known about it: the smallest element is not known even approximately, and it is not known whether any element of this set is rational or whether any element is irrational. For more details see Thurston's Notes [7]. In this paper we prove a result which, among other things, gives some metric or analytic information about the set of hyperbolic volumes.

Given a hyperbolic 3-manifold M with h cusps, one can form the manifold

$$M_\kappa = M_{(p_1, q_1, \dots, p_h, q_h)}$$

obtained by doing a (p_i, q_i)-Dehn surgery on the i-th cusp, where (p_i, q_i) is a coprime pair of integers, or the symbol ∞ if the cusp is left unsurgered. This notation is well defined only after choosing a basis m_i, ℓ_i for the homology $H_1(T_i)$, where T_i is a torus cross section of the i-th cusp. Then (p_i, q_i)-Dehn surgery means: cut off the i-th cusp and paste in a solid torus to kill $p_i m_i + q_i \ell_i$.

Thurston [7] showed that M has a hyperbolic structure for all κ near $\infty = (\infty, \dots, \infty)$ and that

$$\lim_{\kappa \to \infty} \mathrm{Vol}(M_\kappa) = \mathrm{Vol}(M), \quad \mathrm{Vol}(M_\kappa) < \mathrm{Vol}(M) \quad (\kappa \neq \infty). \tag{1}$$

Moreover, he describes a result of Jørgensen which shows that the only accumulation points of the set

$$\mathrm{Vol} = \{\mathrm{Vol}(M) : M \text{ is a hyperbolic 3-manifold}\} \subset \mathbb{R}$$

arise in this way (namely, given any constant $C > 0$ there are finitely many hyperbolic 3-manifolds such that any hyperbolic 3-manifold with volume less than C is obtained from one of them by Dehn surgery). Thus to know what Vol looks like, we would like to know how the numbers $\mathrm{Vol}(M_\kappa)$ tend to their limit. To express the answer, we introduce positive definite binary quadratic forms $Q_1, \dots, Q_h$ of determinant one as follows: the torus T_i associated to the i-th cusp has a Euclidean structure well-defined up to similarity and the pair (p_i, q_i) corresponds to a closed geodesic $p_i m_i + q_i \ell_i$ (m_i, ℓ_i the chosen meridian and longitude) on T_i and we define

$$Q_i(p, q) = (\text{length of } p m_i + q \ell_i)^2 / (\text{volume of } T_i).$$

THEOREM 1A. *With the above notations,*

$$\mathrm{Vol}(M_\kappa) = \mathrm{Vol}\, M - \pi^2 \sum_{i=1}^{h} \frac{1}{Q_i(p_i, q_i)} + 0\left(\sum \frac{1}{p_i^4 + q_i^4}\right).$$

A surprising aspect of this result is that the difference of volumes depends to a high order only on the geometry at the cusps and not on the rest of M.

The right-hand side of Theorem 1A can also be expressed in terms of the geometry of M_κ: Let L_i be the length of the short geodesic γ_i on M_κ which is the core of the solid torus added at

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the i-th cusp by the Dehn surgery (with $L_i = 0$ if $(p_i, q_i) = \infty$); then, as we shall see (Proposition 4.3),

$$L_i = 2\pi Q_i(p_i, q_i)^{-1} + 0\left(\sum_j \frac{1}{p_j^4 + q_j^4}\right) \tag{2}$$

and hence Theorem 1A is equivalent to:

THEOREM 1B. $\mathrm{Vol}(M_\kappa) = \mathrm{Vol}(M) - \frac{\pi}{2}\sum_{i=1}^{h} L_i + 0(\sum L_i^2)$.

As a consequence of Theorem 1A, we can determine how fast the limit in (1) is reached and thus determine the metric structure of Vol near its limit points.

COROLLARY. *Let $\{M_\nu\}$ be the set of all hyperbolic 3-manifolds obtained from a given hyperbolic 3-manifold M by doing Dehn surgery on a single cusp of M. Then*

$$\# \{\nu : Vol(M_\nu) < Vol(M) - 1/x\} = 6\pi x + 0(x^{1/2})$$

as $x \to \infty$. If the Riemann hypothesis is true, then the exponent 1/2 can be replaced by 67/148 $+\varepsilon$ for any $\varepsilon > 0$.

We remark that a similar formula holds trivially for volumes of 2-dimensional hyperbolic orbifolds (= quotients of hyperbolic 2-space by Fuchsian groups, possibly with torsion), with 6π replaced by 2π and an error term $0(1)$. Here Dehn surgery is replaced by the process of filling a cusp by a cone point of varying angle.

The corollary follows from Theorem 1A simply by counting the number of coprime pairs (p, q) with $Q(p, q) < \pi^2 x + 0(1)$. Without the coprimality condition, this would be just the number of lattice points in an ellipse of area $\pi^3 x + 0(1)$ and hence equal to $\pi^3 x + 0(x^{(13/40)+\varepsilon})$ by a result of Wu [12]. (Estimating this error for the quadratic form $Q(p, q) = p^2 + q^2$ is the famous "circle problem" of analytic number theory; the exponent 1/2 is trivial and the best possible exponent is conjectured to be $1/4 + \varepsilon$.) Passing to the coprime points introduces a factor $\zeta(2)^{-1} = 6/\pi^2$ in the leading term and replaces the error term by the one given in the corollary, since Moroz [6] showed that an error $O(x^\alpha)$ for the straight lattice point problem gives (on the Riemann hypothesis) an error $O(x^{(2-\alpha)/(5-4\alpha)+\varepsilon})$ for the coprime problem.

Actually, the M_κ correspond to a discrete subset of an h-complex-parameter family of deformations of the hyperbolic structure on M, and Theorems 1A and 1B remain true for this bigger class if their right-hand sides are suitably interpreted. More precisely, Thurston showed that the deformations of the hyperbolic structure on M (no longer complete) can be holomorphically parametrized by points $\mathbf{u} = (u_1, \ldots, u_h)$ in a neighborhood $\mathscr{D}$ of $O \in \mathbb{C}^h$. The Dehn surgery parameters correspond to a zero-dimensional subset $\{\mathbf{u}_\kappa\} \subset \mathbb{C}^h$, discrete except when some (p_i, q_i) is ∞, such that $M_{\mathbf{u}_\kappa}$ is isometric to the complement of the short geodesics created by Dehn surgery in M_κ. The coordinates $(u_1, \ldots, u_h)$ can be chosen so that the dependence of $\mathbf{u}_\kappa$ on κ is as an odd function of κ of the form

$$u_i = \frac{1}{\alpha_i p_i + \beta_i q_i} + (\text{higher order})$$

for some $\alpha_i, \beta_i \in \mathbb{C}$, $\alpha_i/\beta_i \notin \mathbb{R}$. The higher order terms (at least third order, since $\mathbf{u}_\kappa$ is an odd function of κ) depend on all the (p_j, q_j).

The quantity $\mathrm{Vol}(M_\mathbf{u})$ makes sense and is a real-analytic interpolation of $\mathrm{Vol}(M_\kappa)$. We will explain later how to define a real-analytic function $L_i(\mathbf{u})$ interpolating the L_i in Theorem 1B. This theorem can then be rewritten

$$\mathrm{Vol}(M_\mathbf{u}) = \mathrm{Vol}(M) - \frac{\pi}{2}\sum L_i(\mathbf{u}) + \varepsilon(\mathbf{u}), \tag{3}$$

where $\varepsilon(\mathbf{u}) = 0(\|\mathbf{u}\|^4)$. The function $\varepsilon(\mathbf{u})$ is even in each variable and has a Taylor series expansion $\Sigma c_{\alpha,\beta} u_1^{\alpha_1} \ldots u_h^{\alpha_h} \bar{u}_1^{\beta_1} \ldots \bar{u}_h^{\beta_h}$. Our second main result is that all terms in this expansion vanish except those with all of the α's or all of the β's equal to zero, i.e.:

THEOREM 2. *The function* $\varepsilon(u)$ *defined by* (3) *is the imaginary part of a holomorphic function* $f(\mathbf{u})$ *which is even in each argument* u_i. ($f(\mathbf{u})$ *is unique if we require it to vanish at* $\mathbf{u} = 0$.)

Thurston [9] points out that the Chern–Simons invariant of M (which by Meyerhoff [5] can be defined even if M has cusps) can be considered as an imaginary part for the volume. Precisely, one can associate to a hyperbolic manifold M an invariant $I(M) \in \mathbb{C}^*$ whose absolute value is $e^{2/\pi \operatorname{Vol}(M)}$ and whose argument is the Chern–Simons invariant of M. Similarly a geodesic γ on a hyperbolic 3-manifold has a natural invariant $\lambda(\gamma)$ (the ratio of the eigenvalues of the associated element of $PSL_2\,\mathbb{C}$) whose absolute value is $e^{\text{length}(\gamma)}$ and whose argument is the "torsion" of γ. Thus Theorem 2 can be reformulated as

$$|I(M_\kappa)\cdot \prod_{j=1}^{h} \lambda(\gamma_j)| = |I(M)e^{\frac{2}{\pi i}f(\mathbf{u}_\kappa)}|. \tag{4}$$

CONJECTURE†. *Equation* (4) *remains true if the absolute value signs are removed.*

This conjecture is not only very natural in view of Theorem 2 but is also supported by the following:

(a) Meyerhoff's Thesis [5] implies (after resolving a discrepancy in the normalization) that the function

$$u_\kappa \to I(M_\kappa)\cdot \prod_{i=1}^{h} \lambda(\gamma_i)$$

is continuous at $\mathbf{u} = 0$, even though the constituents $I(M_\kappa)$ and $\lambda(\gamma_i)$ are not.

(b) The conjecture is compatible with the conjecture implicit at the bottom of p. 22 of Dupont and Sah [3] related to the extended Hilbert Problem No. 3. Specifically, their conjecture implies that $I(M)e^{\frac{2}{\pi i}f(\mathbf{u}_\kappa)}$ and $I(M_\kappa)\prod_j \lambda(\gamma_j)$ differ by a root of unit for each κ.

The function $f(u)$ of Theorem 2 is closely related with the way the structure of the cusps of M varies as we deform the hyperbolic structure. This is our third main result. To describe it we must be more explicit about the coordinates $u_1, \ldots, u_h$ (see section 4 for more details). For any closed curve γ on M the invariant $\lambda(\gamma)$ described above (ratio of eigenvalues of the holonomy) is well defined up to inversion. Let $\mathbf{u}_i = \pm\log(\lambda(m_i))$ where m_i is the chosen meridian at the i-th cusp. Since $\lambda(m_i) = 1$ for the complete hyperbolic structure on M, we may choose the branch of logarithm so $(u_1, \ldots, u_h) = 0$ for this structure. Then small deformations of the hyperbolic structure on M are parametrized in a $2^h\!:\!1$ way by $\mathbf{u}$ in a neighborhood of $0 \in \mathbb{C}^h$. Using the longitudes ℓ_i instead of the meridians m_i gives different coordinates $v_1, \ldots, v_h$ with $v_i = \pm\log(\lambda(\ell_i))$. There is a natural choice of signs so that the v_i are analytic functions of the chosen coordinates $(u_1, \ldots, u_h)$ (in fact a choice of sign for either u_i or v_i corresponds to a choice of orientation for the line in hyperbolic 3-space fixed by the holonomy of m_i and ℓ_i).

The ratio $\tau_i(\mathbf{u}) = v_i/u_i$ is holomorphic in a neighborhood of $0 \in \mathbb{C}^h$. The hyperbolic structure $M_\mathbf{u}$ is complete at the i-th end of M (i.e. this end is still a true cusp) if and only if $u_i = 0$; in this case, $\tau_i(\mathbf{u})$ is the modulus of the similarity class of euclidean structures on the marked torus T_i associated with this cusp‡. In particular, the numbers $\tau_i^0 = \tau_i(0, \ldots, 0)$ ($i = 1, \ldots, h$) are the moduli of the cusps of the original hyperbolic manifold M. We assume we have chosen m_i and ℓ_i as an oriented basis of homology at the i-th cusp (the torus T_i inherits an orientation from M); then $\tau_i^0 \in \mathbb{H}$ (the upper half plane) for each i.

In terms of the variables u_i and v_i the length $L_i(\mathbf{u})$ introduced above is given by

$$L_i(\mathbf{u}) = -\frac{1}{2\pi}\operatorname{Im}(u_i\bar{v}_i) \tag{6}$$

This is proved in section 4. The following Theorem describes the final ingredient $f(u)$ of the

† Tomayashi Yoshida has announced a proof of this conjecture in "The η-invariant of hyperbolic 3-manifolds", Preprint (Oct. 1983).

‡ That is, T_i is isomorphic to the quotient of $\mathbb{C}$ by the Z-lattice $\mathbb{Z} < 1, \tau_i(\mathbf{u}) >$, with 1 and $\tau_i(\mathbf{u})$ corresponding to m_i and ℓ_i respectively.

volume formulas also in terms of the u_i and v_i. It depends on a surprising symmetry statement (equation (7) below) about how deforming one cusp affects a second one, which is reminiscent of symmetry results of S. Wolpert [11] on the effect of twist deformations for hyperbolic surfaces.

THEOREM 3.

$$\frac{\partial v_i}{\partial u_j} = \frac{\partial v_j}{\partial u_i} \; (i, j = 1, \ldots, h). \tag{7}$$

Hence there exists a "potential function" $\Phi(u_1, \ldots, u_h)$ *such that* $\Phi(0) = 0$, *and*

$$v_i(\mathbf{u}) = \frac{1}{2}\frac{\partial \Phi}{\partial u_i} \; (i = 1, \ldots, h). \tag{8}$$

$\Phi(u_1, \ldots, u_h)$ *is even in each argument and if* Φ_κ *denotes its degree k part (so* $\Phi_k = 0$ *for k odd) then*

$$f(u) = -\frac{1}{8}\sum_{k=0}^{\infty} (k-2)\Phi_k(u_1, \ldots, u_k). \tag{9}$$

Note that (8) implies

$$\Phi_2(u) = \sum_{i=1}^{h} \tau_i^0 u_i^2 \tag{10}$$

so $f(\mathbf{u})$ together with the moduli τ_i^0 of the cusps determine Φ and hence the dependence of the v_i on the u_i, as well as vice versa.

For example, if there is just one cusp and $v = v_1$ is given by a power series

$$v = \sum_{n=1}^{\infty} c_n u^n \tag{11}$$

in $u = u_1$, then

$$c_1 = \tau^0, \quad c_n = 0 \quad \text{for } n \text{ even}, \tag{12}$$

and by (6), (7), and (8) the volume formulas become

$$\mathrm{Vol}\,(M_\mathbf{u}) = \mathrm{Vol}\,(M) - \frac{1}{8i}\sum_{n=1}^{\infty}\left(\frac{n-1}{n+1}c_n u^{n+1} + c_n u^n \bar{u} - \bar{c}_n u \bar{u}^n - \frac{n-1}{n+1}\bar{c}_n u^{n+1}\right), \tag{13}$$

from which the various ingredients can clearly be recovered (see section 6 for an example and more details in the one cusp case).

The proofs of our results are based on the combinatorics of ideal triangulations of hyperbolic 3-manifolds, which we describe in §2. Our main combinatorial results, Theorem 2.2, is a general property of "triangulations" with no 0-simplicies of open 3-manifolds and should be a useful tool for other purposes.

§2. COMBINATORICS OF IDEAL TRIANGULATIONS

The hyperbolic objects whose volumes are best understood are ideal tetrahedra (geodesic tetrahedra with all vertices at infinity). To study the volume of our manifold M, it would be useful if we could decompose it into such tetrahedra, i.e., tetrahedra whose vertices are at the cusps of M. This is in general not possible. However, Thurston [8] has shown that any hyperbolic 3-manifold is obtainable from an ideally triangulated one by Dehn surgeries on some of the cusps. We therefore will assume that M is the result of $k-h$ Dehn surgeries on a manifold N with k cusps which is triangulated as

$$N = S_1 \cup \ldots \cup S_n, \tag{14}$$

where each S_ν is a tetrahedron with its vertices deleted.

The triangulation in (1) is going to be considered as a purely topological one, i.e. as a combinatorial triangulation of an open 3-manifold N homeomorphic to the interior of a compact manifold with a boundary consisting of k tori. Thus N has k "toral" ends. We are going to put the structure of an ideal hyperbolic tetrahedron on each S_ν, imposing the compatibility conditions required to give N a smooth but not necessarily complete hyperbolic structure. We will then impose further conditions, corresponding to the Dehn surgeries at $k-h$ of the ends of N, which say that these ends can be completed by adding a circle to give a smooth neighborhood of a closed geodesic. The remaining h ends of N will initially be complete, and thus have the structure of true hyperbolic cusps; this corresponds to the initial manifold M of the introduction. The manifolds M_κ are obtained by deforming the hyperbolic structures on the tetrahedra S_ν away from this initial position. To describe the situation more accurately we must introduce some notation.

A general reference for hyperbolic triangulations, compatibility conditions, etc., is Thurston [7, Chapter 4]. The reader may wish to refer to the example in section 6 while reading the following general discussion.

We remind the reader that an ideal tetrahedron S is described completely (up to isometry) by a single complex number z in the upper half plane such that the euclidean triangle cut out of any vertex of S by a horosphere section is similar to the triangle in $\mathbb{C}$ with vertices 0, 1 and z. We write $S = S(z)$. The numbers z, $1-(1/z)$ and $1/(1-z)$ give the same tetrahedron; to specify z uniquely, we must pick an edge of S (the dihedral angle at this edge wil be $\arg(z)$). We make such a choice for each S_ν and write

$$S_\nu = S(z_\nu) \qquad (\nu = 1, \ldots, n).$$

Then to each edge of S_ν is associated one of the three numbers

$$z_\nu, \quad 1 - \frac{1}{z_\nu}, \quad \text{and} \quad \frac{1}{1-z_\nu}, \tag{15}$$

the *modulus* of the edge, opposite edges of S_ν having the same modulus (see Fig. 1). The necessary and sufficient condition that (1) gives a (not necessarily complete) hyperbolic manifold is that at each edge e of N the tetrahedra S_ν abutting e "close up" as one goes around e (Fig. 2) and thus that the product of the corresponding moduli of the S_ν at e is $e^{2\pi i}$ (here the

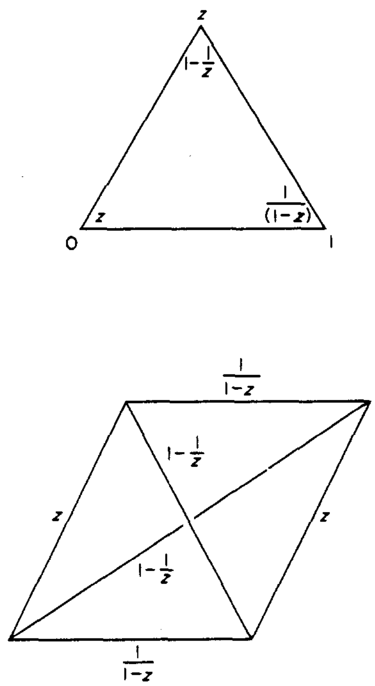

Fig. 1.

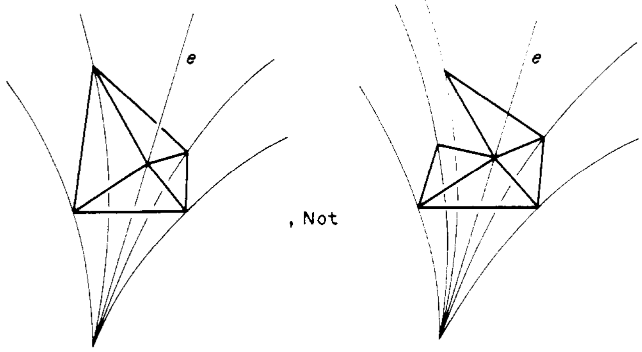

Fig. 2.

product is to be taken in the universal covering $\tilde{\mathbb{C}}^*$ of $\mathbb{C}^*$, that is, the sum of the arguments should be exactly 2π). From (15) we see that these moduli have the form $\pm z_\nu^{r'}(1-z_\nu)^{r''}$ with $(r', r'') \in \{(1,0), (-1,1), (0,-1)\}$, so the gluing condition at edge e gives (computing now only in $\mathbb{C}^*$)

$$\prod_{\nu=1}^{n} z_\nu^{r'_\nu}(1-z_\nu)^{r''_\nu} = \pm 1$$

for some integers r'_ν, r''_ν depending on e (notice that r'_ν and r''_ν are not necessarily still ≤ 1 in absolute value since more than one edge of S_ν may coincide with edge e of N). The fact that N has Euler characteristic zero (N is the interior of a compact 3-manifold with boundary a union of tori) implies by a simple calculation that the number of edges is equal to the number n of tetrahedra. We number the edges by an index j and write the edge relations as

$$\prod_{\nu=1}^{n} z_\nu^{r'_{j\nu}}(1-z_\nu)^{r''_{j\nu}} = \pm 1 \quad (j = 1, \ldots, n) \tag{16}$$

Once we have chosen the numbers z_ν (satisfying the compatibility conditions), N acquires a smooth hyperbolic structure, in general incomplete. We need additional conditions to ensure that the completion of N is M. The torus T associated to any one of the k ends of N has a similarity structure given by its triangulation into the $(0, 1, z)$ triangles cut off by horospheres at the vertices of the S_ν. To each vertex of each triangle of this triangulation is associated a number μ, the modulus of the corresponding edge. If γ is an oriented closed simplicial path, we define $\mu(\gamma)$ to be $(-1)^{|\gamma|}$ times the product of these moduli for the triangle vertices touching γ on the right, where $|\gamma|$ is the number of 1-simplices of γ (see Fig. 3).

LEMMA 2.1. *The number $\mu(\gamma) \in \mathbb{C}^*$ depends only on the homotopy class of γ and defines a homomorphism $\pi_1(T) = H_1(T) \to \mathbb{C}^*$.*

Proof. In fact, the similarity structure on T defines a holonomy homomorphism $\pi_1(T) \to \mathrm{Sim}(\mathbb{E}^2) = \mathrm{Aff}(\mathbb{C})$ and $\mu(\gamma)$ is just the derivative of the holonomy of the element of $\pi_1(T)$ represented by γ (see [7, §4]), from which the lemma follows. However, there is a direct combinatorial proof: any deformation of γ within its homotopy class can be obtained by successive elementary steps of the type illustrated in Fig. 4, and since the product of moduli

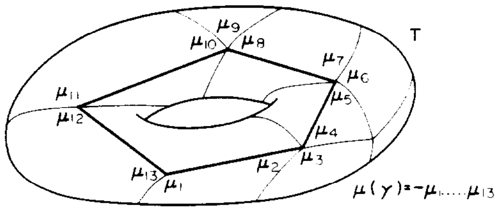

Fig. 3.

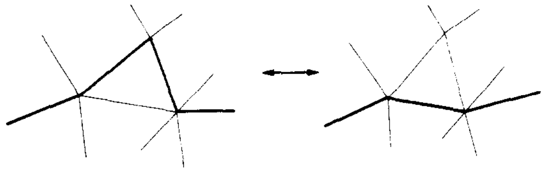

Fig. 4.

around a vertex is $+1$ (the consistency condition at an edge) and the product of moduli of the vertices of a triangle is -1 (cf. (15)), the lemma follows. ∎

Notice that μ is actually obtained from a homomorphism $\tilde{\mu}: H_1(T) \to \tilde{\mathbb{C}}^*$, obtained by considering the moduli in $\tilde{\mathbb{C}}^*$ and replacing $(-1)^{|\gamma|}$ by $e^{-\pi i}|\gamma|$. The condition that the end of N corresponding to the torus T is complete, i.e. a true hyperbolic cusp, is that $\tilde{\mu}$ be trivial, since this exactly says that the similarity structure on T is euclidean. On the other hand, at the $k-h$ Dehn surgered ends, one primitive element $[\gamma] \in H_1(T)$ has been killed by Dehn surgery and the $\tilde{\mu}$ of this γ must be $e^{\pm 2\pi i}$. In particular $\mu(\gamma) = 1$, and since $\mu(\gamma)$ is a product of edge moduli, each of which is one of the numbers (15), this relation takes the form

$$\prod_{\nu=1}^{n} z_\nu^{r'_\nu}(1-z_\nu)^{r''_\nu} = \pm 1.$$

Since this relation has exactly the same form as the consistency relations at edges (16), we use the same letter j to index surgered ends as we used for edges (with j now ranging from $n+1$ to $n+k-h$) and write the complete set of necessary conditions in the uniform form

$$\prod_{\nu=1}^{n} z_\nu^{r'_{j\nu}}(1-z_\nu)^{r''_{j\nu}} = \pm 1 \quad (j = 1, \ldots, n+k-h). \tag{17}$$

We note that these are only necessary conditions and not sufficient, since we are considering them in $\mathbb{C}^*$ rather than $\tilde{\mathbb{C}}^*$. However, if we start at the value $\mathbf{z}^0 = (z_1^0, \ldots, z_n^0)$ corresponding to the hyperbolic structure on M and deform $\mathbf{z} = (z_1, \ldots, z_n)$ preserving conditions (17), then, of course, the corresponding $\tilde{\mathbb{C}}^*$ conditions are preserved, so we do get a deformation of the hyperbolic structure on M.

We write $N(\mathbf{z})$ for N with the hyperbolic structure determined by $\mathbf{z} = (z_1, \ldots, z_\nu)$ satisfying the compatibility conditions. Thus M is the completion of $N(\mathbf{z}^0)$ and the M_κ and $M_\mathbf{u}$ of the Introduction are completions of suitable $N(\mathbf{z})$.

The h unsurgered cusps of $N(\mathbf{z}^0)$, corresponding to the cusps of M, will be indexed by the letter i, $1 \le i \le h$. Corresponding to the basis ℓ_i, m_i of $H_1(T_i)$ fixed in section 1, we have for this manifold additional relations $\mu(\ell_i) = 1$, $\mu(m_i) = 1$, $(i = 1, \ldots, h)$ as explained above, and again these can be written in the form

$$\prod_{\nu=1}^{n} (z_\nu^0)^{l'_{i\nu}}(1-z_\nu^0)^{l''_{i\nu}} = \pm 1 \quad (i = 1, \ldots, h) \tag{18}$$

$$\prod_{\nu=1}^{n} (z_\nu^0)^{m'_{i\nu}}(1-z_\nu^0)^{m''_{i\nu}} = \pm 1 \quad (i = 1, \ldots, h) \tag{19}$$

for some integers $l'_{i\nu}, l''_{i\nu}, m'_{i\nu}, m''_{i\nu}$ (which are well defined, however, only after choosing closed simplicial curves on the T_i representing the homology classes ℓ_i, m_i).

We write L', L'', M', M'', R', R'' for the integer matrices $(l'_{i\nu})$, $(l''_{i\nu})$, $\ldots$, $(r'_{j\nu})$, $(r''_{j\nu})$, and combine them into a single $(n+k+h) \times 2n$ matrix U as follows

$$U = \begin{array}{cc} 1 \le \nu \le n & 1 \le \nu \le n \\ \left(\begin{array}{c} L' \\ M' \\ R' \end{array}\right. & \left.\begin{array}{c} L'' \\ M'' \\ R'' \end{array}\right) \end{array} \begin{array}{l} \\ \}1 \le i \le h \\ \}1 \le i \le h \\ \}1 \le j \le n+k-h \end{array} \tag{20}$$

where for convenience we recall our notational conventions:

$v \to$ tetrahedra; $' \to z_v$, $'' \to 1 - z_v$, $(1 \le v \le n)$

$i \to$ unsurged cusps $(1 \le i \le h)$

$j \to$ edges $(1 \le j \le n)$ and surgered cusps $(n < j \le n + k - h)$.

We also write L for the $h \times 2n$ matrix (L', L'') and similarly $M = (M', M'')$, $R = (R', R'')$, so

$$U = \begin{pmatrix} L \\ M \\ R \end{pmatrix}.$$

For any natural number m, denote by J_{2m} the symplectic matrix

$$J_{2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

The fundamental fact about the combinatorics of our triangulation is the following result.

THEOREM 2.2. *Let $U \in M_{n+h+k,\,2n}(\mathbb{Z})$ be the matrix defined in* (20). *Then*

$$U J_{2n} U^t = 2\begin{pmatrix} J_{2h} & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \}\, 2h \\ \}\, n+k-h. \end{matrix} \tag{21}$$

The proof, which is rather long, is postponed to §3.

For any matrix A with $2n$ columns we shall denote by $[A] \subset \mathbb{R}^{2n}$ the row-space of A, that is, the subspace of $\mathbb{R}^{2n}$ generated by the rows of A. On $\mathbb{R}^{2n}$ we have the symplectic form

$$< x, y > \; = \tfrac{1}{2} x J_{2n} y^t$$

Let C denote the matrix $C = \binom{L}{M}$, so $U = \binom{C}{R}$. The content of Theorem 2.2 is that $[U]$ is orthogonal (w.r.t. $<,>$) to $[R]$ and that the rows of C form a symplectic basis of $[C]$; in particular C has maximal rank, namely rank $C = 2h$.

PROPOSITION 2.3. *Rank $R = n - h$ and rank $U = n + h$. Moreover, if $\perp$ denotes orthogonal complement with respect to $<,>$ then $[U]^\perp = [R]$.*

Proof. The representation $[U] = [C] + [R]$ is an orthogonal sum, so $\dim [U] = \dim [C] + \dim [R] = 2h + \dim [R]$. On the other hand, since $[R] \subset [U]^\perp$, we have $\dim [R] \le \dim [U]^\perp = 2n - \dim [U] = 2n - 2h - \dim [R]$, so $\dim [R] \le n - h$ and equality holds if and only if $[R] = [U]^\perp$. Thus it suffices to show $\dim [R] \ge n - h$.

Since $\dim [R] \le n - h$, at most $n - h$ of the consistency relations (17) are independent, so these relations determine a subvariety $V \subset \mathbb{C}^n$ of dimension at least h. We claim V has dimension exactly h at the point $z^0 = (z_1^0, \ldots, z_n^0)$. (This re-proves a result of Thurston [7, §5].) To show this, it suffices to show that the subvariety W of V defined by the h additional relations

$$\prod_{v=1}^{n} z_v^{l'_{iv}} (1 - z_v)^{l''_{iv}} = \pm 1 \quad (i = 1, \ldots, h)$$

(obtained by replacing z_v^0 by z_v in (18)) has dimension 0 at $\mathbf{z}^0$. These h relations specify that the holonomies of the longitudes ℓ_i are parabolic. Since the holonomies of ℓ_i and m_i commute, m_i then also has parabolic holonomy for each i, so the parameter $\mathbf{z} \in W$ corresponds to the complete hyperbolic structure on M, which is unique by Mostow rigidity. Thus, if W had positive dimension, the ideal triangulation of M could be deformed. But this is impossible, since there are only countably many ideal triangulations of M (since there are only countably many geodesics in M which are asymptotic to either a cusp or a closed geodesic in each direction).

Since V has dimension h at $\mathbf{z}^0$, we can pick a nearby point $\mathbf{z} = (z_1, \ldots, z_n)$ where V has dimension h and is nonsingular (in fact, as we shall see in §4, $\mathbf{z} = \mathbf{z}^0$ is such a point). We rewrite the relations defining V as

$$\sum_{\nu=1}^{n} (r'_{j\nu} \log z_\nu + r''_{j\nu} \log(1-z_\nu)) = \text{const. } (= \log(\pm 1)) \quad (j = 1, \ldots, n+k-h).$$

The Jacobian of this system of equations at $\mathbf{z}$ is RZ, where

$$Z = \begin{pmatrix} \frac{1}{z_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \frac{1}{z_n} \\ \frac{-1}{1-z_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{-1}{1-z_n} \end{pmatrix}.$$

Thus rank $(RZ) = n-h$, so rank $R \geq n-h$, as desired.

The following corollary will be the fundamental tool for proving the volume formulae.

Corollary 2.4. *Suppose $x, y \in \mathbb{R}^{2n}$ satisfy $Rx^t = Ry^t = 0$. Then $xJ_{2n}y^t = \frac{1}{2}xC^tJ_{2h}Cy^t$.*

Proof. Since $0 = Rx^t = RJ_{2n}(xJ_{2n})^t$ the vector xJ_{2n} is in $[R]^\perp$, so by Proposition 2.3 we have $xJ_{2n} \in [U]$, that is, $xJ_{2n} = zU$ for some $z \in \mathbb{R}^{n+k+h}$. Also $UJ_{2n}C^t = 2\binom{J_{2h}}{0}$ by Theorem 2.2, so $xC^t = -xJ_{2n}^2C^t = -2z\binom{J_{2h}}{0}$. Thus $\frac{1}{2}xC^tJ_{2h}Cy^t = -z\binom{J_{2h}}{0}J_{2h}Cy^t = z\binom{C}{0}y^t = z\binom{C}{R}y^t = zUy^t = xJ_{2n}y^t$, as desired.

We close this section with a digression, namely to describe how the k relations between the rows of R, which must exist by Proposition 2.3, can be described explicitly in terms of the combinatorics. In fact these k relations occur among the first n rows of R, that is, the rows which correspond to edges of the triangulation. We denote the submatrix consisting of these rows by R_0. We define a matrix $X = (x_{ij})_{1 \leq i \leq k, 1 \leq j \leq n}$, where, for the purpose of this discussion, the index i indexes *all* ends of N rather than only the unsurgered ones, by letting $x_{ij} \in \{0, 1, 2\}$ be the number of ends of the j-th 1-simplex which terminate at the i-th end of N.

Proposition 2.5. *$XR_0 = 0$ and X has (maximal) rank k.*

Proof. To see that $XR_0 = 0$ we must show

$$\sum_{j=1}^{n} x_{ij}r'_{j\nu} = \sum_{j=1}^{n} x_{ij}r''_{j\nu} = 0 \tag{22}$$

for each torus T_i and tetrahedron S_ν. Each nonzero product $x_{ij}r'_{j\nu}$ in (22) corresponds to an edge of S_ν which ends at the i-th cusp. Hence (22) is trivially true if none of the four vertices of S_ν is at this cusp. If S_ν does have a vertex at the i-th cusp, then there will be three edges j_1, j_2, j_3 contributing to (22) (see Fig. 5). They will each contribute 1 to x_{ij} and will contribute $(1,0)$, $(-1,1)$, $(0,-1)$, respectively, to $(r'_{j\nu}, r''_{j\nu})$ by (15). Since $(1,0)+(-1,1)+(0,-1) = (0,0)$, (22) follows.

To show that X has maximal rank, we show that XX^t is positive definite, i.e. that the quadratic form

$$\xi = (\xi_1, \ldots, \xi_n) \mapsto \sum_{i=1}^{k} \sum_{i'=1}^{k} \sum_{j=1}^{n} x_{ij}x_{i'j}\xi_i\xi_{i'} \tag{23}$$

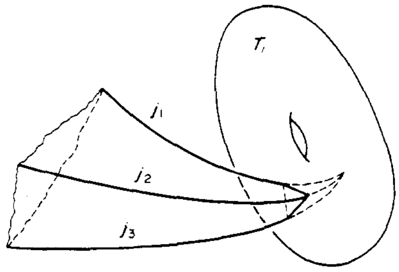

Fig. 5.

is positive for $\xi \neq 0$. Since each edge has two ends, we have for each j either

$$x_{i_1(j)j} = x_{i_2(j)j} = 1, \quad x_{ij} = 0 \quad \text{for } i \notin \{i_1(j), i_2(j)\} \text{ with } i_1(j) \neq i_2(j)$$

or

$$x_{i_1(j)j} = 2, \quad x_{ij} = 0 \quad \text{for } i \neq i_1(j).$$

In the second case we write $i_2(j) = i_1(j)$. Then the right hand side of (23) equals

$$\sum_{j=1}^{n} (\xi_{i_1(j)} + \xi_{i_2(j)})^2$$

since i and i' range over $\{i_1(j), i_2(j)\}$. This can be zero only if ξ_i has opposite sign at the two ends of every edge, and looking at any 2-simplex of the triangulation we see that this implies $\xi_i = 0$ for all i. ■

§3. PROOF OF THEOREM 2.2

The matrix U has four kinds of rows; the L-rows, the M-rows, the R-rows indexed by $j > n$, and the R-rows indexed by $j \leq n$. If a is a row of one of the first three kinds, then

$$a = ((a'_\nu)_{1 \leq \nu \leq n}, (a''_\nu)_{1 \leq \nu \leq n})$$

where a'_ν and a''_ν are the exponents of z_ν and $1 - z_\nu$ in the μ-invariant of a closed oriented simplicial curve α in the torus T associated to some end of N (namely a representative of ℓ_i or m_i or the class killed by Dehn surgery at the $(j-n)$-th cusp for some $j > n$). Let b denote another such row, associated to a simplicial curve β. Then the Theorem follows from the three formulae

$$\sum_{\nu=1}^{n} (r'_{j_1\nu} r''_{j_2\nu} - r''_{j_1\nu} r'_{j_2\nu}) = 0, \quad j_1, j_2 = 1, \ldots, n \tag{24}$$

$$\sum_{\nu=1}^{n} (r'_{j\nu} a''_\nu - r''_{j\nu} a'_\nu) = 0, \quad j = 1, \ldots, n \tag{25}$$

$$\sum_{\nu=1}^{n} (a'_\nu b''_\nu - a''_\nu b'_\nu) = 2[\alpha] \cdot [\beta], \tag{26}$$

where in the last formula the intersection number $[\alpha] \cdot [\beta]$ is zero if α and β are at different ends of N.

We start with the proof of (24). Let Λ be the space $\mathbb{Z}^2$ with the skew symmetric pairing

$$(r', r'') \wedge (s', s'') = r's'' - r''s'.$$

In Λ we have vectors

$$\rho_1 = (1, 0), \quad \rho_2 = (-1, 1), \quad \rho_3 = (0, -1)$$

with

$$\rho_i \wedge \rho_j = \begin{cases} 0 & j = i \\ \pm 1 & j \equiv i \pm 1 \pmod 3. \end{cases}$$

Moreover, (r'_{jv}, r''_{jv}) is the sum of (at most six) contributions, each equal to one of ρ_1, ρ_2 or ρ_3, corresponding to the edges of S_v which coincide with the j-th 1-simplex e_j of N. Hence the left-hand side of (24) counts the number of triples (v, E_1, E_2) where $1 \leq v \leq n$ and E_1 and E_2 are adjacent edges of S_v which get identified with 1-simplices e_{j_1} and e_{j_2} in N, the triple being counted positively or negatively according to the relative orientations of E_1 and E_2 on S_v (see Fig. 6). These contributions cancel in pairs, because each triple determines a 2-simplex of N (the face of S_v containing E_1 and E_2) and this 2-simplex has a 3-simplex on each side of it with opposite relative orientations.

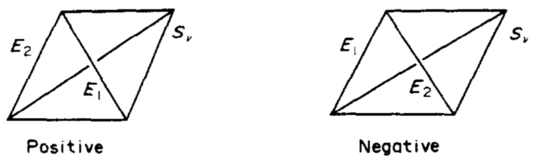

Fig. 6.

Before we prove (25) and (26) we give a heuristic reason why they should be valid. Note that changing the paths α and β within their homology classes changes a and b by linear combinations of rows of R (cf. the proof of 2.1 and the remark following equation (19) in §2). Therefore (24) implies that the left side of equation (25) depends only on $[\alpha]$, and (24) and (25) imply the left side of (26) depends only on $[\alpha]$ and $[\beta]$. If there are sufficiently many simplices in our triangulation of the torus T containing α, then we can move α within its homology class to lie "far" from edge e_j (since e_j meets T in at most 2 points and α can be deformed to avoid the star of these points); then every term of (25) will be zero. A similar heuristic would indicate that the left side of (26) should equal some multiple of $[\alpha] \cdot [\beta]$; the exact coefficient could be determined from one example. However, it appears hard to translate this argument into a precise proof, so we shall need to look in more detail at the possible geometric configurations. In view of these remarks it is, however, sufficient to prove (25) and (26) in the case that α and β are simple closed curves. We therefore assume this in the following.

The proof of (25) is similar to the proof of (24) but the combinatorics will involve counting oriented edges (with both orientations of any edge allowed). We shall denote oriented edges of simplices by F, F_1, F_2, etc. The vector (a'_v, a''_v) is the sum of contributions, each equal to ρ_1, ρ_2 or ρ_3, coming from oriented edges F of S_v which begin at a vertex of T on the path α and such that the simplex S_v intersects T in a triangle to the right of α at this vertex. The left-hand side of (25) is thus the sum of contributions ± 1 from triples (v, E, F) where E and F are adjacent edges of S_v, E is identified with e_j in N, and F is oriented and begins at a vertex of α with S_v on the right of α as above. These contributions cancel in pairs; we write the two members of a cancelling pair as (v, E, F), (v', E', F') and tabulate them in Fig. 7 according to the answers to the following two questions (in each case the answers are the same for (v, E, F) and (v', E', F')):

(i) Are E and F adjacent at the beginning point of F?
(ii) Does the 2-simplex determined by E and F meet T in an edge of α?

In the top left case of Fig. 7 S_v and $S_{v'}$ need not be distinct; we have just drawn the typical case. The notation $E \equiv E'$ means that the edge E of S_v is identified in N with the edge E' of $S_{v'}$; similarly for $F \equiv F'$.

Finally, for (26) a similar argument to those for (24) and (23) shows that the left-hand side is the sum of contributions ± 1 coming from triples (v, F_α, F_β) where F_α and F_β are adjacent oriented edges of S_v, F_α begins at a vertex of the path α with S_v on the right of α at this vertex, and similarly for F_β and β. We shall see that most of these contributions cancel and the remaining ones sum to $2[\alpha] \cdot [\beta]$. We must again distinguish several cases, and again cancellation will always be between pairs of similar type.

Case 1. *F_α and F_β do not begin at the same vertex.* We subdivide this case according to whether the 2-simplex Δ determined by F_α and F_β meets α and β in an edge of α or β.

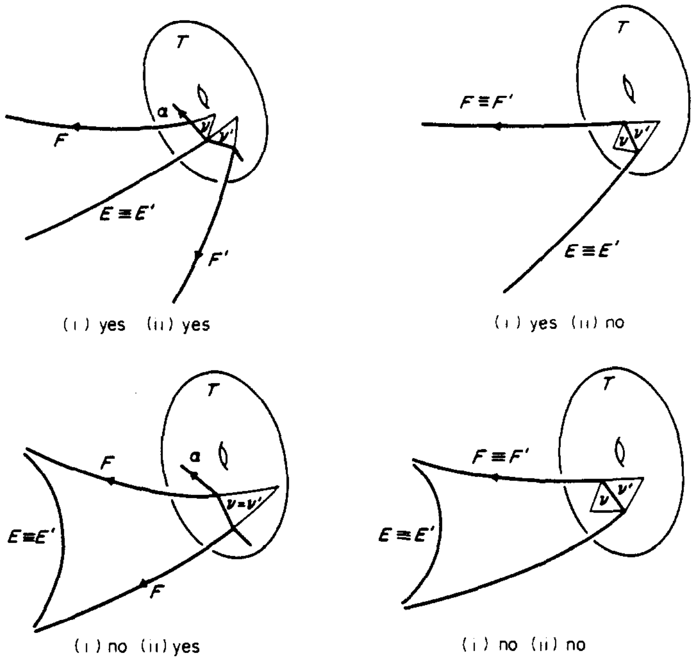

Fig. 7.

Case 1.1. *Δ does not meet α or β in an edge of α or β.* In this case there is a cancelling pair $\{(v, F_\alpha, F_\beta), (v', F'_\alpha, F'_\beta)\}$ with $F_\alpha \equiv F'_\alpha$ and $F_\beta \equiv F'_\beta$ as in Fig. 8a, Fig. 8a with α and β exchanged, or Fig. 8b.

Case 1.2. *Δ meets α or β in an edge of α or β.* Assume Δ meets α in an edge of α. Thus two edges of Δ meet α and F_β may or may not be the third edge of Δ.

Case 1.2.1. *F_β is the third edge.* Then there is a cancelling pair $\{(v, F_\alpha, F_\beta), (v, F'_\alpha, F'_\beta)\}$ as in Fig. 9a or b.

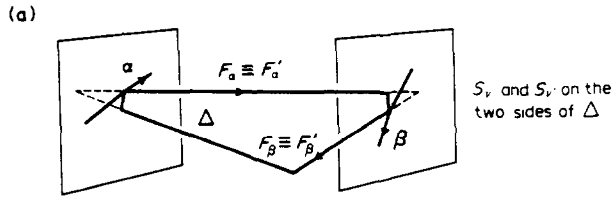

Fig. 8a.

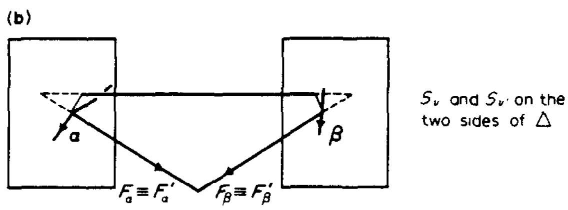

Fig. 8b.

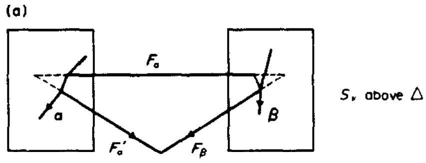

Fig. 9a.

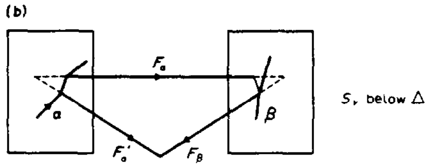

Fig. 9b.

Case 1.2.2. *F_β is not the third edge of* Δ. In this case there is a cancelling pair $\{(v, F_\alpha, F_\beta), (v', F'_\alpha, F'_\beta)\}$ with $F_\beta \equiv F'_\beta$ as in Fig. 10a or with $F_\beta \equiv$ (F'_α reversed) as in Fig. 10b. In the case of Fig. 10b there are also analogous pictures with the directions of α or β reversed. Note also that

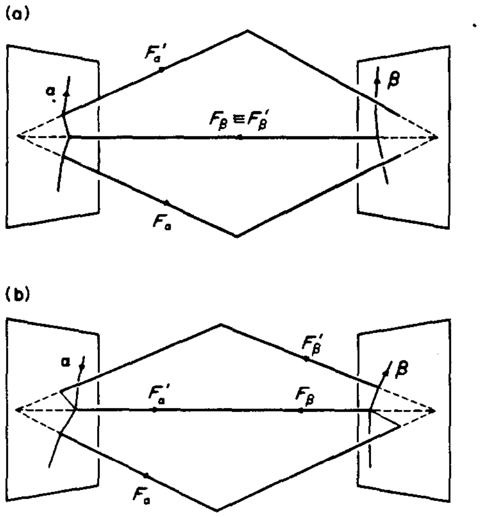

Fig. 10.

Cases 1.2.1 and 1.2.2 can occur simultaneously, as for example in Fig. 11; this does not affect the argument.

Case 2. *F_α and F_β begin at the same vertex.* In this case α and β must be on the same torus T and F_α and F_β correspond to vertices of α and β which are connected by a 1-simplex $e \subset \Delta \cap T$ of the triangulation of T.

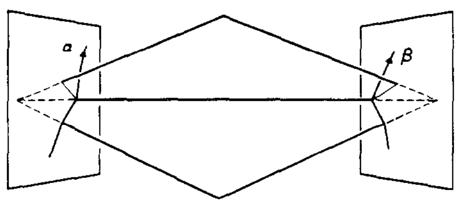

Fig. 11.

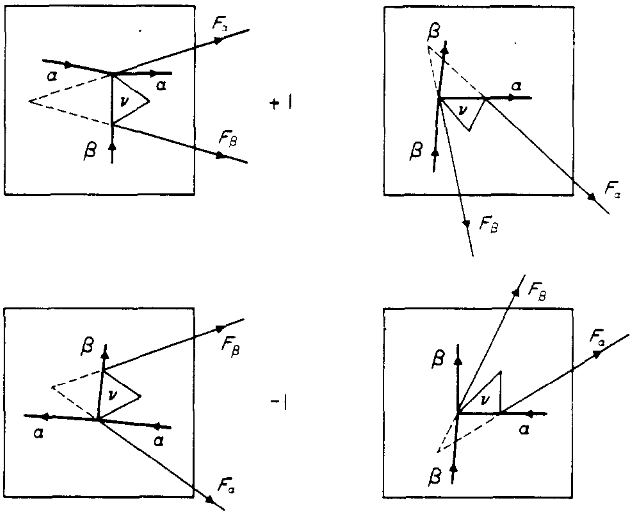

Fig.12.

Case 2.1. *This* 1-*simplex e is not part of* α *or* β. Then there is a cancelling pair $\{(v, F_\alpha, F_\beta), (v', F'_\alpha, F'_\beta)\}$ with $F_\alpha \equiv F'_\alpha$, $F_\beta \equiv F'_\beta$, and s_v and $s_{v'}$, the two 3-simplices on each side of Δ.

Case 2.2. *The* 1-*simplex e is part of both* α *and* β. Then $\{(v, F_\alpha, F_\beta), (v, F_\beta, F_\alpha)\}$ is a cancelling pair.

Case 2.3. *The* 1-*simplex e is part of* α *or of* β *but not both.* In this case we get a contribution of ± 1 to the left-hand side of (26) as in Fig. 12. That is, we get a contribution of ± 1, with sign appropriate to the intersection number, for each instance of α and β touching at a vertex with β on the right of α and a similar contribution for each instance of α and β touching at a vertex with α on the right of β. The former contributions sum to $[\alpha]\cdot[\beta]$ and the latter contributions also do, so the total is $2[\alpha]\cdot[\beta]$, completing the proof.

§4. GOOD PARAMETERS FOR DEFORMATION

We have described how the hyperbolic structure on M can be varied in dependence on the parameter $\mathbf{z} = (z_1, \ldots, z_n)$. This parameter is somewhat arbitrary in that it depended on a choice of ideal triangulation; moreover, it contains redundant information: $\mathbf{z}$ is constrained by the "consistency relations" (equations (17) of §2) of which precisely $n - h$ are independent (Proposition 2.3). In this section we shall describe a more natural and intrinsic parametrization of the deformations, due to Thurston [7, §5.8], and discuss the relation between the parameters used and the geometry of the cusps.

The parametrization to be described is an easy consequence of the fact, which we proved in section 2, that the deformation space has dimension precisely h. Thurston's hyperbolic Dehn surgery theorem also follows easily. Thus our analysis gives a proof of these facts using only Mostow rigidity plus as much of the combinatorics of ideal triangulations as is expressed by Proposition 2.2. This seems worth noting.

In §2 we introduced the number

$$\prod_{v=1}^{m} z_v^{m'_{iv}}(1 - z_v)^{m''_{iv}} \tag{27}$$

associated to the chosen "meridian" m_i in the torus T_i corresponding to the i-th unsurgered end of N. Up to a constant ± 1 this number is the derivative of the holonomy of m_i in the affine structure on T_i. This constant ± 1 is the value of formula (27) at $\mathbf{z} = \mathbf{z}^0 = (z_1^0, \ldots, z_n^0)$.

Thus

$$\prod_{\nu=1}^{n}\left(\frac{z_\nu}{z_\nu^0}\right)^{m'_{\nu i}}\left(\frac{1-z_\nu}{1-z_\nu^0}\right)^{m''_{\nu i}}$$

is the exact derivative of the holonomy. We denote its logarithm by u_i so

$$u_i = \sum_{\nu=1}^{n}\left(m'_{\nu i}\log\frac{z_\nu}{z_\nu^0} + m''_{\nu i}\log\frac{1-z_\nu}{1-z_\nu^0}\right) \tag{28}$$

is an analytic function of $\mathbf{z}$ which vanishes at $\mathbf{z}^0$. Similarly

$$v_i = \sum_{\nu=1}^{n}\left(l'_{\nu i}\log\frac{z_\nu}{z_\nu^0} + l''_{\nu i}\log\frac{1-z_\nu}{1-z_\nu^0}\right) \tag{29}$$

is the logarithm of the holonomy for the "longitude" $\ell_i \subset T_i$.

The promised parameters are $u_1, \ldots, u_h$. Denote by $\mathscr{D}_0$ the variety of $\mathbf{z} \in \mathbb{C}^r$ which satisfy the consistency relations. Then $\mathbf{u} = (u_1, \ldots, u_h)$ maps a neighborhood of $\mathbf{z}^0 \in \mathscr{D}_0$ biholomorphically onto a neighborhood $\mathscr{D}$ of $0 \in \mathbb{C}^h$. The bijectivity is not obvious at first sight; it would be plausible for $\mathbf{u}$ to be a branched covering, with the several inverse images of a given point corresponding to several ideal triangulations of the same Dehn surgered version of M. We shall prove the bijectivity of $\mathbf{u}$ later.

The parameter $\mathbf{u}$ is still not completely natural, in that it depends on the choice of a primitive element m_i in the homology of each cusp. We could, for instance, equally well have chosen $\mathbf{v} = (v_1, \ldots, v_h)$ as the parameter. We will, however, stick to our choice and thus consider the v_i as dependent variables. We next discuss this dependence and draw some elementary consequences.

LEMMA 4.1. *In a neighborhood of the origin*
$v_i = u_i \cdot \tau_i(u_1, \ldots, u_h)$, $i = 1, \ldots, h$ *for analytic functions* $\tau_i(u)$ *which satisfy:*

(a) $\tau_i(0, \ldots, 0)$ *is in the upper half plane and is the modulus of the euclidean structure on the torus* T_i *associated to the i-th cusp of* M *(with respect to* m_i, ℓ_i*).*

(b) $\tau_i(u_1, \ldots, u_h)$ *is an even function of each of its arguments.*

Part (a) is implicit in Thurston [7, §5.8], but we sketch the argument since we need the geometry. Consider m_i and ℓ_i as elements of $\pi_1(M)$ after suitable choice of basepoint and let $H(m_i)$ and $H(\ell_i)$ be their respective holonomies.

We use the upper half space model

$$\mathbb{H}^3 = \{(z, r) \in \mathbb{C} \times \mathbb{R} \mid r > 0\}$$

so an element of $\mathrm{Isom}^+(\mathbb{H}^3)$ corresponds to a Mobius transformation $z \mapsto \dfrac{az+b}{cz+d}$ of $(\mathbb{C} \times \{0\}) \cup \{\infty\}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$. The elements $H(m_i)$ and $H(\ell_i)$ are commuting parabolic elements, so by a conjugation we may assume they are the following translations in the $\mathbb{C}$-plane:

$$H(m_i): z \mapsto z + 1, \quad H(\ell_i): z \mapsto z + \tau. \tag{30}$$

Then τ is the modulus of the euclidean structure on T_i and with our orientation conventions $\tau \in \mathbb{H}$ (upper half-plane).

We now deform the hyperbolic structure slightly to a parameter value $\mathbf{u} = (u_1, \ldots, u_h)$ and denote the deformed holonomy by $H_{\mathbf{u}}$. *Since* $H_{\mathbf{u}}(m_i)$ and $H_{\mathbf{u}}(\ell_i)$ commute, they have the same fixed points on the sphere $\mathbb{C} \cup \{\infty\}$ at infinity. If they are parabolic then the cusp of M has remained a cusp and $u_i = v_i = 0$. Otherwise, $H_{\mathbf{u}}(m_i)$ and $H_{\mathbf{u}}(\ell_i)$ have two fixed points, which we can put at $(0, 0)$ and $(0, \infty)$ so

$$H_{\mathbf{u}}(m_i): z \mapsto az, \quad H_{\mathbf{u}}(\ell_i): z \mapsto bz \tag{31}$$

for some $a, b \in \mathbb{C}^*$. By definition, $\mathbf{u}_i = \log a$ and $v_i = \log b$. Moreover, since $\mathbf{u}$ is small, a fundamental domain for (30) must be almost similar to a fundamental domain for (31) (see Fig. 13). Thus, noting that a and b are close to 1, we have

$$v_i = \log b \approx b - 1 \approx \tau(a-1) \approx \tau \log a = \tau u_i,$$

where $\approx$ means first order equality. In particular, for $\mathbf{u}$ near 0 it follows that v_i vanishes if and only if u_i does, so v_i/u_i is analytic and, moreover, the value of v_i/u_i at $\mathbf{u} = 0$ is τ. Thus part (a) of the Lemma is proven. Before proving part (b) we need to recall some geometry.

By Lemma 4.1 (a), if $\mathbf{u}$ is small and $u_i \neq 0$ then v_i is not a real multiple of u_i. Hence there is a unique solution $(p_i, q_i) \in \mathbb{R}^2 \cup \{\infty\}$ to

$$p_i u_i + q_i v_i = 2\pi i \tag{32}$$

(we take $(p_i, q_i) = \infty$ if $u_i = 0$). This (p_i, q_i) is called the generalized *Dehn surgery coefficient* by Thurston. It's geometric significance is as follows. Let $M_\mathbf{u}$ be the completion of M with the hyperbolic metric given by parameter value $\mathbf{u}$. Thus $M_\mathbf{u}$ differs topologically from M by the addition of a set γ_i of limit points at the i-th cusp for each i. If $(p_i, q_i) = \infty$ then $\gamma_i = \varnothing$ and $M_\mathbf{u}$ still has a cusp here. If (p_i, q_i) is a coprime pair of integers then γ_i is a circle and at the i-th cusp $M_\mathbf{u}$ is the result of hyperbolic (p_i, q_i)-Dehn surgery on M. In all other cases $M_\mathbf{u}$ is either metrically singular or not even a manifold (γ_i a circle or a point respectively) depending on whether or not p_i and q_i are rationally dependent.

Denote by κ the map which assigns to a parameter value $\mathbf{u}$ near 0 the corresponding generalized Dehn surgery invariant $((p_1, q_1), \ldots, (p_h, q_h)) \in (\mathbb{R}^2 \cup \{\infty\})^h$. Then κ maps a neighborhood of $0 \in \mathbb{C}^h$ homeomorphically to a neighborhood of $\infty \in (\mathbb{R}^2 \cup \{\infty\})^h$ (to see this observe that if

$$\varphi((p_1, q_1), \ldots, (p_h, q_h)) = 2\pi i \left(\frac{1}{p_i + q_i \tau_i(\mathbf{0})}, \ldots, \frac{1}{p_h + q_h \tau_h(\mathbf{0})} \right)$$

then $\varphi \circ \kappa$ has Jacobian equal to the identity at $\mathbf{0}$). This is Thurston's hyperbolic Dehn surgery theorem and is how he proved it.

Now suppose (p_i, q_i) is a coprime integer pair, so at the i-th cusp $M_\mathbf{u}$ has resulted by true hyperbolic Dehn surgery from M. We want to compute the length and torsion of the new geodesic γ_i added at this cusp.

LEMMA 4.2. *Choose integers r_i, s_i such that* $\det \begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix} = 1$. *Then*

$$\textit{length}\,(\gamma_i) + i\,.\,\textit{torsion}\,(\gamma_i) = -(r_i u_i + s_i v_i) \quad (\textit{mod } 2\pi i).$$

Proof. On T_i the classes $p_i m_i + q_i \ell_i$ and $r_i m_i + s_i \ell_i$ form a basis for homology. Since $p_i m_i + q_i \ell_i$ is the class killed by Dehn surgery, $r_i m_i + s_i \ell_i$ represents the new geodesic γ_i in $M_\mathbf{u}$. Thus the holonomy of γ_i (which is well defined up to conjugation) is given by

$$H(\gamma_i) = H(r_i m_i + s_i \ell_i) = (z \mapsto a^{r_i} b^{s_i} z)$$

in the notation of equation (31), and therefore

$$\begin{aligned} \text{length}\,(\gamma_i) + i \cdot \text{torsion}\,(\gamma_i) &= \pm \log (a^{r_i} b^{s_i}) \\ &= \pm (r_i u_i + s_i v_i), \end{aligned}$$

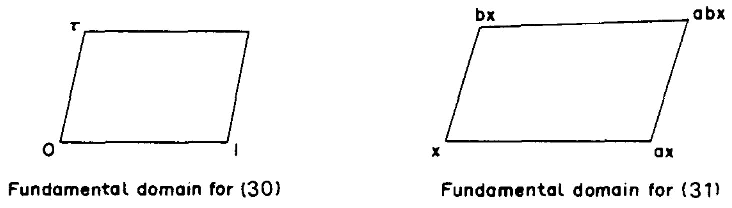

Fig. 13.

where the sign must be chosen to give a positive real part. To verify this sign we rewrite the equations

$$\begin{aligned} p_i u_i + q_i v_i &= 2\pi i \\ r_i u_i + s_i v_i &= \varepsilon\,(\text{length } (\gamma_i) + i \cdot \text{torsion}\,(\gamma_i)) \end{aligned} \tag{33}$$

with $\varepsilon = \pm 1$ as

$$\begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix} \begin{pmatrix} \operatorname{Re}(u_i) & \operatorname{Im}(u_i) \\ \operatorname{Re}(v_i) & \operatorname{Im}(v_i) \end{pmatrix} = \begin{pmatrix} 0 & 2\pi \\ \varepsilon \cdot \text{length}\,(\gamma_i) & \varepsilon \cdot \text{torsion}\,(\gamma_i) \end{pmatrix}$$

Then taking determinants gives

$$\operatorname{Im}(u_i \bar{v}_i) = 2\pi\varepsilon \cdot \text{length}\,(\gamma_i) \tag{34}$$

But $v_i = \tau_i(\mathbf{u}) u_i$ with $\tau_i(\mathbf{u})$ in the upper half plane (Lemma 4.1a), so $\operatorname{Im}(u_i \bar{v}_i)$ is negative. Hence $\varepsilon = -1$.

We can now finally prove part (b) of Lemma 4.1. Choose $\mathbf{u}$ near the origin in $\mathbb{C}^h$ such that $M_{\mathbf{u}}$ is obtained by true hyperbolic Dehn surgery at every cusp, i.e. each pair (p_i, q_i) is either a coprime integer pair or ∞. Assume $(p_1, q_1) \neq \infty$ and replace (p_1, q_1) by $(-p_1, -q_1)$ leaving the other (p_i, q_i)'s unchanged. Let $\mathbf{u}'$ be the corresponding parameter value. Then $M_{\mathbf{u}}$ and $M_{\mathbf{u}'}$ are the same topologically, so by Mostow rigidity they are isometric. In particular the right hand sides of equations (33) are unchanged. We can thus solve (33) for the new u_1 and v_1 to see that u_1 and v_1 have been replaced by $-u_1$ and $-v_1$ while the remaining u_i and v_i are unchanged. Thus for each i the ratio $v_i/u_i = \tau_i(\mathbf{u})$ is an even function of u_1 and similarly of $u_2, \ldots, u_n$, at least at parameter values $\mathbf{u}$ corresponding to true hyperbolic Dehn surgery. But any analytic set containing these parameter values has a dense set of tangent directions at the origin and thus includes a neighborhood of $0 \in \mathbb{C}^h$. Thus $\tau_i(\mathbf{u})$ is an even function of its arguments on all of $\mathscr{D}$.

Replacing one coordinate of u by its negative gives isometric $M_{\mathbf{u}}$ and $M_{\mathbf{u}'}$ (this is so also for parameter values which do not correspond to hyperbolic Dehn surgeries by the continuation argument just used; think in terms of the holonomy homomorphism $\pi_1(M) \to PSL(2, \mathbb{C})$). This involution will not, in general, have a simple expression in terms of $\mathbf{z} = (z_1, \ldots, z_n)$. In fact, from Thurston's description of hyperbolic Dehn surgery it is clear that what is happening geometrically is that we have two ideal triangulations of $M_{\mathbf{u}}$ distinguished by the direction in which edges of the triangulation spiral in towards the geodesic γ_i, and these may well have very different $\mathbf{z}$ values. Note that outside a tubular neighborhood of the γ_i the combinatorics of the triangulation of $M_{\mathbf{u}}$ are the same as for M, while inside a tubular neighborhood of γ_i the only choice is the direction of spiral, which corresponds to a choice of sign for u_i. Thus the parameter $\mathbf{u}$ determines both $M_{\mathbf{u}}$ and its ideal triangulation, so the change of parameters from $\mathbf{z}$ to $\mathbf{u}$ is bijective, as mentioned earlier. Again, we use the above continuation argument to deduce this for general $\mathbf{z}$ and $\mathbf{u}$ from the hyperbolic Dehn surgery case.

To end this section we give two more formulae for the length L_i of the geodesic γ_i created by Dehn surgery. Though elementary, they are important ingredients of the volume computation. The second one is equation (2) of the introduction which proves the equivalence of Theorems 1A and 1B.

PROPOSITION 4.3.

$$L_i = -\frac{1}{2\pi} \operatorname{Im}(u_i \bar{v}_i) \tag{35}$$

$$L_i = \frac{2\pi}{Q_i(p_i, q_i)} + \sum_{j=1}^{h} 0\left(\frac{1}{p_j^4 + q_j^4}\right) \tag{36}$$

where Q_i is as described in the introduction. In particular length (γ_i) is the restriction of the analytic function $L_i(u) = -\frac{1}{2\pi} \operatorname{Im}(u_i \bar{v}_i)$ defined in a whole neighborhood of $0 \in \mathbb{C}^h$ (rather than just at the Dehn surgery points).

Proof. (35) is just a restatement of (34). To see (36) note first that

$$v_i = \tau_i(0)\cdot u_i + (\text{higher order}).$$

By equation (32) (with apologies for the double use of i)

$$u_i = \frac{2\pi i}{p_i + \tau_i(0)\,q_i} + (\text{higher order}).$$

Hence

$$-\frac{1}{2\pi}\operatorname{Im}(u_i\bar{v}_i) = 2\pi\frac{\operatorname{Im}\tau_i(0)}{|p_i+\tau_i(0)q_i|^2} + (\text{higher order})$$
$$= 2\pi\frac{\operatorname{Vol}(T_i)}{\operatorname{length}(p_i m_i + q_i \ell_i)} + (\text{higher order}).$$

The higher order terms must start at fourth order, since they are even in each (p_i, q_i).

§5. VOLUME COMPUTATIONS

Let S be an ideal tetrahedron and α, β, γ its dihedral angles (each of which occurs at two opposite edges), with $\alpha+\beta+\gamma=\pi$. These are the angles of the euclidean triangle cut off T by a horosphere section at a vertex. If $z\in\mathbb{C}$ is the parameter of T then (after possible reordering)

$$\alpha = \arg(z),\quad \beta = \arg\left(1-\frac{1}{z}\right),\quad \gamma = \arg\left(\frac{1}{1-z}\right).$$

In Chapter 7 of Thurston's Notes [7] Milnor proves the formula, essentially due to Lobachevsky,

$$\operatorname{Vol}(S(z)) = Л(\alpha) + Л(\beta) + Л(\gamma), \tag{37}$$

where

$$Л(\theta) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin 2n\theta}{n^2}.$$

The function $Л(\theta)$ equals $\frac{1}{2}\operatorname{Im} Li_2(e^{2i\theta})$, where Li_2 is the dilogarithm function

$$Li_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2}\quad (|z|\le 1).$$

The function $Li_2(z)$ can be holomorphically extended to the cut plane $\mathbb{C}\backslash[1,\infty]$. Its real part cannot be expressed in terms of functions of a single variable, but a formula due to Kummer and quoted in Lewin's book on the dilogarithm function ([4], p. 121, eq. (5.5)) says that

$$\operatorname{Im} Li_2(z) = Л(\arg z) + Л\left(\arg\frac{z-1}{z}\right) + Л\left(\arg\frac{1}{1-z}\right) - \log|z|\arg(1-z)$$

for z in the upper half-plane. Thus (37) can be rewritten

$$\operatorname{Vol}(S(z)) = D(z) \tag{38}$$

where

$$D(z) = \operatorname{Im} Li_2(z) + \log|z|\arg(1-z) \tag{39}$$

The function $D(z)$ is single-valued, continuous, real analytic except at 0 and 1, and satisfies

$$D(z) = D\left(1-\frac{1}{z}\right) = D\left(\frac{1}{1-z}\right),\quad D(z) = -D(1-z) = -D(\bar{z}).$$

It has occurred in several places in the literature, in particular in the work of Spencer Bloch

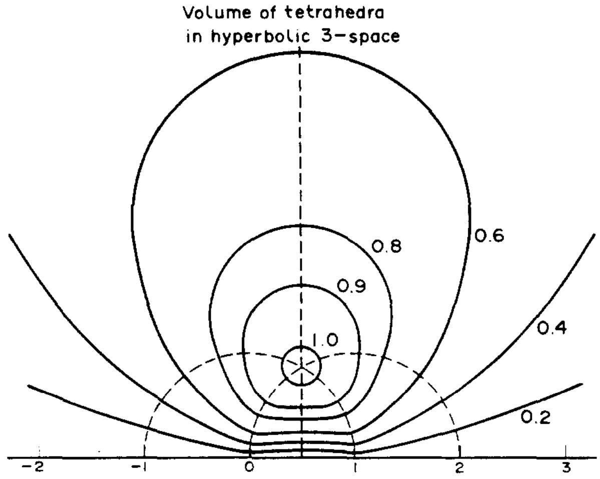

Fig. 14.

[1] (whose notation $D(z)$ we have adopted) on K-theory and in work of Wigner [10]. The fact that Vol $(S(z))$ equals $D(z)$ was also observed by Dupont and Sah [3].

The level curves of $D(z)$ are shown in Fig. 14; the function reaches its maximum value 1.0149 . . . at the point $\frac{1+i\sqrt{3}}{2}$, corresponding to the regular ideal tetrahedron.

Returning to our hyperbolic manifold, we have from (38) the formula

$$\operatorname{Vol} M_{\mathbf{u}} = \sum_{\nu=1}^{n} D(z_\nu), \tag{40}$$

where $\mathbf{u} = (u_1, \ldots, u_h)$ is the $\mathbf{u}$-parameter value corresponding to $\mathbf{z} = (z_1, \ldots, z_n)$. We shall analyze (40) by differentiating it, since the derivatives of $D(z)$ are elementary functions:

$$\frac{\partial D}{\partial z} = \frac{i}{2}\left(\frac{\log|1-z|}{z} + \frac{\log|z|}{1-z}\right), \quad \frac{\partial D}{\partial \bar{z}} = -\frac{i}{2}\left(\frac{\log|1-z|}{\bar{z}} + \frac{\log|z|}{1-\bar{z}}\right). \tag{41}$$

Assume therefore that the parameter value $\mathbf{u}$ (and hence also $\mathbf{z}$) is varying in dependence on a single variable ξ. Then by (40) and (41),

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\operatorname{Vol}(M_{\mathbf{u}}) = \frac{i}{2}\sum_{\nu=1}^{n}\left[\left(\frac{\log|1-z_\nu|}{z_\nu} + \frac{\log|z_\nu|}{1-z_\nu}\right)\frac{\mathrm{d}z_\nu}{\mathrm{d}\xi} - \left(\frac{\log|1-z_\nu|}{\bar{z}_\nu} + \frac{\log|z_\nu|}{1-\bar{z}_\nu}\right)\frac{\mathrm{d}\bar{z}_\nu}{\mathrm{d}\xi}\right]. \tag{42}$$

If we write

$$x = \left[\left(\frac{1}{z_\nu}\frac{\mathrm{d}z_\nu}{\mathrm{d}\xi}\right)_{1\le\nu\le n}, \left(-\frac{1}{1-z_\nu}\frac{\mathrm{d}z_\nu}{\mathrm{d}\xi}\right)_{1\le\nu\le n}\right]$$

$$y = [(\log|z_\nu|)_{1\le\nu\le n}, (\log|1-z_\nu|)_{1\le\nu\le n})$$

then (42) can be rewirtten

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\operatorname{Vol}(M_{\mathbf{u}}) = -\operatorname{Im}(xJ_{2n}y^t). \tag{43}$$

We can apply Corollary 2.4 to this, since the necessary relations $Rx^t = 0$ and $Ry^t = 0$ are the derivative and the real part respectively of the logarithm of the compatibility relations (47). This gives

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\operatorname{Vol}(M_{\mathbf{u}}) = -\frac{1}{2}\operatorname{Im}(xC^tJ_{2h}Cy^t). \tag{44}$$

On the other hand, by equations (28) and (29),

$$xC^t = \left[\left(\frac{\mathrm{d}v_i}{\mathrm{d}\xi}\right)_{1 \le i \le h}, \left(\frac{\mathrm{d}u_i}{\mathrm{d}\xi}\right)_{1 \le i \le h}\right],$$

$$yC^t = [(\mathrm{Re}\, v_i)_{1 \le i \le h}, (\mathrm{Re}\, u_i)_{1 \le i \le h}],$$

so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{Vol}(M_{\mathbf{u}}) &= -\frac{1}{2}\mathrm{Im}\sum_{i=1}^{h}\left[\frac{\mathrm{d}v_i}{\mathrm{d}\xi}\mathrm{Re}(u_i) - \frac{\mathrm{d}u_i}{d\xi}\mathrm{Re}(v_i)\right] \\ &= \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\xi}\sum_{i=1}^{h}\mathrm{Im}(u_i\bar{v}_i) + \frac{1}{4}\sum_{i=1}^{h}\mathrm{Im}\left(\frac{\mathrm{d}u_i}{d\xi}v_i - \frac{\mathrm{d}v_i}{\mathrm{d}\xi}u_i\right) \\ &= -\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\xi}\sum_{i=1}^{h}L_i(\mathbf{u}) + \frac{1}{4}\sum_{i=1}^{h}\mathrm{Im}\left(\frac{\mathrm{d}u_i}{\mathrm{d}\xi}v_i - \frac{\mathrm{d}v_i}{\mathrm{d}\xi}u_i\right) \quad \text{(Prop. 4.3).} \end{aligned} \tag{45}$$

Rewriting this as an equation of total differentials,

$$\mathrm{d}(\mathrm{Vol}(M_{\mathbf{u}})) = -\frac{\pi}{2}\mathrm{d}\left(\sum_{i=1}^{h} L_i(\mathbf{u})\right) + \frac{1}{4}\mathrm{Im}\sum_{i=1}^{h}(v_i\mathrm{d}u_i - u_i\mathrm{d}v_i), \tag{46}$$

shows that the holomorphic differential $\sum_{i=1}^{h}(v_i\mathrm{d}u_i - u_i\mathrm{d}v_i)$ has exact imaginary part and is hence exact. It follows that the integral

$$f(\mathbf{u}) = \frac{1}{4}\int_0^{\mathbf{u}}\sum_{i=1}^{h}(v_i\mathrm{d}u_i - u_i\mathrm{d}v_i) \tag{47}$$

is independent of the path of integration (in a simply connected neighborhood of 0) and defines a holomorphic function of $\mathbf{u}$; integrating equation (46) now gives

$$\mathrm{Vol}(M_{\mathbf{u}}) - \mathrm{Vol}(M) = -\frac{\pi}{2}\sum_{i=1}^{h} L_i(\mathbf{u}) + \mathrm{Im} f(\mathbf{u}) \tag{48}$$

(with no constant of integration since both sides vanish at 0).

By the discussion in §4, Theorems 1A, 1B, and 2 will be proved if we show that $f(\mathbf{u})$ vanishes to 4-th order at $\mathbf{u} = 0$. This is a consequence of Theorem 3, which we prove below, but it can be seen more quickly by observing that the equation $v_i = u_i\tau_i$ implies that

$$v_i\mathrm{d}u_i - u_i\mathrm{d}v_i = -u_i^2\sum_{j=1}^{h}\frac{\partial\tau_i}{\partial u_j}\mathrm{d}u_j,$$

which is of third order.

We now prove Theorem 3. Since

$$\mathrm{d}f = \frac{1}{4}\sum_{i=1}^{h}(v_i\mathrm{d}u_i - u_i\mathrm{d}v_i)$$

is an exact differential, so is

$$\mathrm{d}(4f + \sum_{i=1}^{h}u_iv_i) = 2\sum_{i=1}^{h}v_i\mathrm{d}u_i. \tag{49}$$

The fact that this form is closed is equivalent to equation (7) of Theorem 3. Equation (49) motivates the definition

$$\Phi(\mathbf{u}) := 4f(\mathbf{u}) + \sum_{i=1}^{h}u_iv_i. \tag{50}$$

The resulting equation

$$\mathrm{d}\Phi = 2\sum_{i=1}^{h}v_i\mathrm{d}u_i$$

is equation (8) of Theorem 3. Substituting $v_i = \frac{1}{2}\frac{\partial \Phi}{\partial u_i}$ into (50) gives (after trivial manipulation):

$$f(\mathbf{u}) = -\frac{1}{8}\sum_{i=1}^{h}\left(u_i \frac{\partial}{\partial u_i} - 2\right)\Phi. \tag{51}$$

Since $\left(\sum_{i=1}^{h} u_i \frac{\partial \varphi}{\partial u_i}\right) = k\Phi$ for any homogeneous polynomial φ of degree k, equation (51) gives equation (9) of Theorem 3.

§6. EXAMPLES

We shall illustrate our results on the example of the figure eight knot complement. We first describe more explicitly what our results say in the case of one cusp.

As described in §1, if the expansion of $v = v_1$ as a power series in $u = u_1$ is

$$v = \sum_{n=1}^{\infty} c_n u^n \quad (c_n = 0 \text{ for } n \text{ even}), \tag{52}$$

then the volume formula is

$$\begin{aligned} \mathrm{Vol}(M_{\mathbf{u}}) &= \mathrm{Vol}(M) + \frac{1}{4}\,\mathrm{Im}\,(u\bar{v}) + \mathrm{Im}\,(f(\mathbf{u})) \\ &= \mathrm{Vol}(M) - \frac{1}{4}\sum_{n=1}^{\infty}\mathrm{Im}\,(c_n u^n \bar{u}) - \frac{1}{4}\sum_{n=1}^{\infty}\mathrm{Im}\left(\frac{n-1}{n+1}c_n u^{n+1}\right). \end{aligned} \tag{53}$$

We can also invert series (52):

$$\begin{cases} u = \sum_{n=1}^{\infty} a_n v^n \quad (a_n = 0 \text{ for } n \text{ even}), \\ a_1 = \dfrac{1}{c_1}, \quad a_3 = -\dfrac{c_3}{c_1^4}, \quad a_5 = \dfrac{3c_3^2}{c_1^7} - \dfrac{c_5}{c_1^6}, \\ a_7 = -\left(\dfrac{12c_3^3}{c_1^{10}} - \dfrac{8c_3c_5}{c_1} + \dfrac{c_7}{c_1^8}\right), \ldots \end{cases} \tag{54}$$

and write the volume as

$$\mathrm{Vol}(M_{\mathbf{u}}) = \mathrm{Vol}(M) - \frac{1}{4}\sum_{n=1}^{\infty}\mathrm{Im}\left(a_n v^n \bar{v} + \frac{n-1}{n+1}a_n v^{n+1}\right). \tag{55}$$

On the other hand the Dehn surgery coefficients (p, q) are determined by

$$pu + qv = 2\pi i. \tag{56}$$

By (52) we can write this as

$$\frac{2\pi i}{q} = \frac{A}{q}u + c_3u^3 + c_5u^5 + \ldots$$

with

$$A = p + c_1 q \tag{57}$$

and applying the inversion formula (c.f. equation (54)) to solve for u gives

$$\begin{aligned} u = \frac{2\pi i}{A} - c_3\frac{q}{A}\left(\frac{2\pi i}{A}\right)^3 + \left(3c_3^2\frac{q^2}{A^2} - c_5\frac{q}{A}\right)\left(\frac{2\pi i}{A}\right)^5 \\ - \left(12c_3^3\frac{q^3}{A^3} - 8c_3c_5\frac{q^2}{A^2} + c_7\frac{q}{A}\right)\left(\frac{2\pi i}{A}\right)^7 + \ldots. \end{aligned} \tag{58}$$

Inserting into (56) and solving for v gives

$$v = c_1\frac{2\pi i}{A} + c_3\frac{p}{A}\left(\frac{2\pi i}{A}\right)^3 - \left(3c_3^2\frac{pq}{A^2} - c_5\frac{p}{A}\right)\left(\frac{2\pi i}{A}\right)^5 + \left(12c_3^3\frac{pq^2}{A^3} - 8c_3c_5\frac{pq}{A} + c_7\frac{p}{A}\right)\left(\frac{2\pi i}{A}\right)^7 + \dots . \tag{59}$$

It is now a simple matter to express the ingredients of the volume formula (53) in terms of the Dehn surgery coefficients. Since, by (56), we have $\operatorname{Im}(u\bar{v}) = \operatorname{Im}\left(\frac{2\pi i\bar{v}}{p} - \frac{q_v}{p}\bar{v}\right) = \operatorname{Im}\left(\frac{2\pi i v}{p}\right)$, we obtain from (59):

$$\frac{1}{4}\operatorname{Im}(u\bar{v}) = -\operatorname{Im}(c_1)\frac{\pi^2}{|A|^2} + \frac{1}{4}\operatorname{Im}\left[c_3\left(\frac{2\pi}{A}\right)^4 + \left(3c_3^2\frac{q}{A} - c_5\right)\left(\frac{2\pi}{A}\right)^6 + \left(12c_3^3\frac{q^2}{A^2} - 8c_3c_5\frac{q}{A} + c_7\right)\left(\frac{2\pi}{A}\right)^8 + \dots\right]. \tag{60}$$

A similar calculation yields

$$f(\mathbf{u}) = -\frac{1}{4}\left[\frac{1}{2}c_3\left(\frac{2\pi}{A}\right)^4 + \frac{2}{3}\left(3c_3^2\frac{q}{A} - c_5\right)\left(\frac{2\pi}{A}\right)^6 + \frac{3}{4}\left(12c_3^3\frac{q^2}{A^2} - 8c_3c_5\frac{q}{A} + c_7\right)\left(\frac{2\pi}{A}\right)^8 + \dots\right] \tag{61}$$

where the term of degree $2n$ is always $\frac{n-1}{n+1}$ times the corresponding term in (60). Thus the volume formula becomes

$$\operatorname{Vol}(M_{\mathbf{u}}) = \operatorname{Vol}(M) - \operatorname{Im}(c_1)\frac{\pi^2}{|A|^2} + \frac{1}{4}\operatorname{Im}\left[\frac{1}{2}c_3\left(\frac{2\pi}{A}\right)^4 + \frac{1}{3}\left(3c_3^2\frac{q}{A} - c_5\right)\left(\frac{2\pi}{A}\right)^6 + \frac{1}{4}\left(12c_3^3\frac{q^2}{A^2} - 8c_3c_5\frac{q}{A} + c_7\right)\left(\frac{2\pi}{A}\right)^8 + \dots\right]. \tag{62}$$

To illustrate this for the figure eight knot complement (shown in Fig. 15 in white on black) we shall use the ideal triangulation described by Thurston [7]. This triangulation uses two 1-simplices, four 2-simplices, and two 3-simplices, identified as in Fig. 16.

Let z and w denote the parameters of these tetrahedra with respect to the labelled edges and

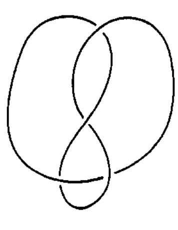

Fig. 15.

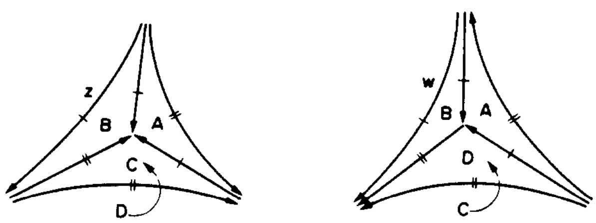

Fig. 16.

for any $x \in \mathbb{C}^*$ denote $x' = 1 - \dfrac{1}{x}$ and $x'' = (x')'$. By inspection of the same identifications on the truncated tetrahedra (Fig. 17) one sees the triangulation of the torus at infinity. It is shown in Fig. 18; the pictured fundamental domain is bounded by our choice of basis m, ℓ (this is the knot-theoretic meridian and longitude). One can read off the consistency relations

$$z^2 z' w^2 w' = 1, \quad z'(z'')^2 w'(w'')^2 = 1$$

which simplify to

$$\begin{aligned} zw(z-1)(w-1) &= 1 \\ z^{-1}w^{-1}(z-1)^{-1}(w-1)^{-1} &= 1, \end{aligned} \tag{63}$$

so the matrix R of §2 is

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

The holonomy of ℓ and m can also be read off; they simplify to

$$\begin{aligned} \ell&: \ z^2(1-z)^2 \\ m&: \ w(1-z), \end{aligned} \tag{64}$$

so the matrix C of §2 is

$$C = \begin{pmatrix} L \\ M \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The matrix X of Proposition 2.5 is $X = (2, 2)$, and the various lemmas of §2 are easily verified.

Now by (64) and by equations (28) and (29) we have

$$u = \log(w(1-z)) \tag{65}$$

$$v = 2\log(z(1-z)) = -2\log(w(1-w)) \tag{66}$$

where the branches are chosen so $u = v = 0$ near the initial value $z = w = \dfrac{1 + i\sqrt{3}}{2}$. Because

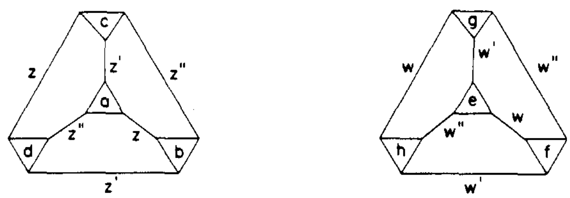

Fig. 17.

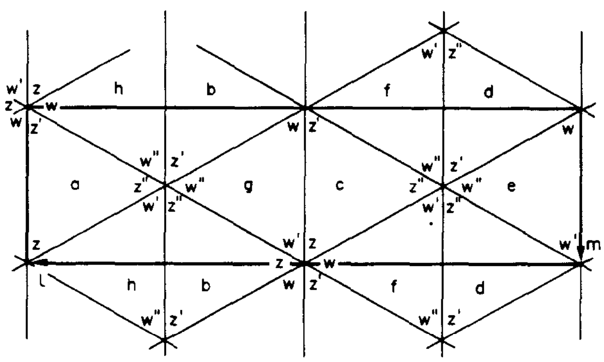

Fig. 18.

of relation (63) there is only one degree of freedom (corresponding to the one cusp of our manifold). Computations are easier in this case if we take v as our local parameter and expand u as a power series in v,

$$u = u(v) = \sum_{n=1}^{\infty} a_n v^n.$$

Thus to compute the volume of M_u as a power series in v (Equation (55)) we must know the coefficients a_n.

From (66) we get

$$z(1-z) = e^{v/2}, \quad w(1-w) = e^{-v/2},$$

or, solving the quadratic equations,

$$z = \frac{1+\sqrt{1-4e^{v/2}}}{2}, \quad w = \frac{1+\sqrt{1-4e^{-v/2}}}{2},$$

where in each case the branch of the square root is chosen which is $+i\sqrt{3}$ at $v = 0$. Hence from (65)

$$\begin{aligned}\frac{du}{dv} &= \frac{1}{1-z}\frac{d(1-z)}{dv} + \frac{1}{w}\frac{dw}{dv} \\ &= \frac{e^{v/2}}{\sqrt{1-4e^{v/2}}\,(1-\sqrt{1-4e^{v/2}})} + \frac{e^{-v/2}}{\sqrt{1-4e^{-v/2}}\,(1+\sqrt{1-4e^{-v/2}})} \\ &= \frac{1+\sqrt{1-4e^{v/2}}}{4\sqrt{1-4e^{v/2}}} + \frac{1-\sqrt{1-4e^{-v/2}}}{4\sqrt{1-4e^{-v/2}}} \\ &= \frac{1}{4\sqrt{1-4e^{v/2}}} + \frac{1}{4\sqrt{1-4e^{-v/2}}}.\end{aligned}$$

This is an even function of v, as desired. Also

$$\begin{aligned}\frac{1}{\sqrt{1-4e^{v/2}}} &= \frac{1}{\sqrt{-3}}\left(1+\frac{4}{3}(e^{v/2}-1)\right)^{-1/2} \\ &= \frac{1}{\sqrt{-3}}\sum_{m=0}^{\infty}\frac{(-1)^m}{3^m}\binom{2m}{m}(e^{v/2}-1)^m \quad \text{(binomial theorem)} \\ &= \frac{1}{\sqrt{-3}}\sum_{m=0}^{\infty}\frac{(-1)^m}{3^m}\frac{(2m)!}{m!}\sum_{n=m}^{\infty}\mathcal{S}_n^{(m)}\frac{(v/2)^n}{n!}\end{aligned}$$

where the $\mathcal{S}_n^{(m)}$ are the Stirling numbers of the second kind, defined by $(e^x-1)^m/m! = \sum_m^{\infty} \mathcal{S}_n^{(m)} x^n/n!$ or by $x^n = \sum_0^n \mathcal{S}_n^{(m)} x(x-1)\ldots(x-m+1)$ (the latter expression makes it clear that $\mathcal{S}_n^{(m)} \in \mathbb{Z}$; in fact $\mathcal{S}_n^{(m)}$ is the number of partitions of $\{1,\ldots,n\}$ into m non-empty subsets). Hence

$$\frac{du}{dv} = \frac{1}{2\sqrt{-3}}\sum_{\substack{n=0 \\ n \text{ even}}}^{\infty}\frac{v^n}{2^n n!}\sum_{m=0}^{n}\frac{(-1)^m}{3^m}\frac{(2m)!}{m!}\mathcal{S}_n^{(m)}$$

and

$$a_n = \begin{cases}\dfrac{1}{2^n n!\sqrt{-3}}\displaystyle\sum_{m=0}^{n-1}\left(\frac{-1}{3}\right)^m\frac{(2m)!}{m!}\mathcal{S}_{n-1}^{(m)} & (n \text{ odd}) \\ 0 & (n \text{ even}).\end{cases}$$

Numerically, this gives

$$u = \frac{v}{2\sqrt{-3}}\left(1 + \frac{1}{3}\frac{v^2}{12} + \frac{37}{180}\left(\frac{v^2}{12}\right)^2 + \frac{67}{360}\left(\frac{v^2}{12}\right)^3 + \dots\right). \tag{67}$$

We could also have proceeded by solving (63) and (65) to get z and w as algebraic functions of e^u, substituting into (66), and differentiating. The computation, similar to the one above but somewhat longer, gives:

$$\frac{dv}{du} = 2\frac{1 - 2e^u - 2e^{-u}}{\sqrt{e^{2u} + e^{-2u} - 2e^u - 2e^{-u} + 1}}$$

$$v = 2\sqrt{-3}\left(u + \frac{1}{3}u^3 + \frac{23}{180}u^5 + \frac{89}{1080}u^7 + \dots\right) \tag{68}$$

which is the inverse expansion of (67) and gives the coefficients c_n of equations (52) to (62). In particular, equation (53) gives

$$\operatorname{Vol}(M_{\mathbf{u}}) = \operatorname{Vol}(M) - L(\mathbf{u}) + \frac{u^4}{4\sqrt{3}} + \frac{\bar{u}^4}{4\sqrt{3}} + \frac{23u^6}{180\sqrt{3}} + \frac{23\bar{u}^6}{180\sqrt{3}} + \frac{89u^8}{960\sqrt{3}} + \frac{89\bar{u}^8}{960\sqrt{3}} + \dots$$

and equation (62) becomes:

$$\begin{aligned}\operatorname{Vol}(M_{(p,q)}) = \operatorname{Vol}(M) &- \frac{2\sqrt{3}\pi^2}{|A|^2} + \frac{1}{8i}\left[\frac{\sqrt{-3}(A^4 + \bar{A}^4)(2\pi)^4}{3|A|^8}\right.\\ &+ \left(\frac{4q(A^7 - \bar{A}^7)(2\pi)^6}{3|A|^{14}} + \frac{-23\sqrt{-3}(A^6 + \bar{A}^6)(2\pi)^6}{270|A|^{12}}\right)\\ &+ \left(\frac{-8\sqrt{-3}q^2(A^{10} + \bar{A}^{10})(2\pi)^8}{3|A|^{20}} + \frac{-46q(A^9 - \bar{A}^9)(2\pi)^8}{45|A|^{18}}\right.\\ &\left.\left. + \frac{89\sqrt{-3}(A^8 + \bar{A}^8)(2\pi)^8}{2160|A|^{16}}\right) + \dots\right]\\ = \operatorname{Vol}(M) &- \frac{2\sqrt{3}\pi^2}{p^2 + 12q^2} + \frac{4\sqrt{3}(p^4 - 72p^2q^2 + 144q^4)\pi^4}{3(p^2 + 12q^2)^4} + \dots.\end{aligned}$$

The computation of this expansion up to 2nd order originally led us to conjecture Theorem 1A, while it was an experimental computation of the above expansion in terms of u up to 6th order which first suggested Theorem 2 to us.

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