Volume of hyperbolic tetrahedron *

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Abstract

We show the simple formula for volume of compact hyperbolic tetrahedron as a function of its dihedral angles.
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1 Introduction

The calculation of volume of polyhedron in the three dimensional space is very old and difficult problem. Probably, the first result in this direction belongs to Tartaglia (1494) who find the volume of Euclidean tetrahedron. Now this result is know as Cayley-Menger formula. In hyperbolic and spherical the situation is more complicated. The volume formula for bi-orthogonal tetrahedron (so called orthoscheme) was obtained by Lobachevsky and Schläfli for hyperbolic and spherical cases respectively. The volume of hyperbolic tetrahedron in many particular cases was investigated by Vinberg [V]. The comlpete solution of the problem was given just recently by nunber of authors Yu. Cho, H. Kim [ChK], J. Murakami, U. Yano [MY] and A.Ushijima [U]. All results deals with analytical formulae which express the volume in terms of 16 Dilogarithm or Lobachevsky functions contributing the dihedral angles of tetrahedron and some addition parameter which can be find as a root of some complicated quadratic equation.

Let $T(A, B, C, D, E, F)$ be a compact hyperbolic tetrahedron with dihedral angles A, B, C, D, E, F. The main result of our paper is the following theorem.

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Theorem 1. *The volume of a hyperbolic tetrahedron $T = T(A, B, C, D, E, F)$ is given by the following formula*

$$Vol(T) = -\frac{1}{4}\int_{z_1}^{z_2} \log \frac{\cos\frac{A+B+C+z}{2}\cos\frac{A+E+F+z}{2}\cos\frac{B+D+F+z}{2}\cos\frac{C+D+E+z}{2}}{\sin\frac{A+B+D+E+z}{2}\sin\frac{A+C+D+F+z}{2}\sin\frac{B+C+E+F+z}{2}\sin\frac{z}{2}}\,dz,$$

where z_1 and z_2 are roots of the integrand defined by conditions $0 < z_2 - z_1 < \pi$.

More precisely,

$$z_1 = \arctan\frac{k_3}{k_4} - \arctan\frac{k_1}{k_2},$$
$$z_2 = \pi - \arctan\frac{k_3}{k_4} - \arctan\frac{k_1}{k_2},$$

where

$$\begin{aligned}
k_1 &= -(\cos S + \cos(A+D) + \cos(B+E) + \cos(C+F) + \cos(D+E+F) \\
&\quad + \cos(D+B+C) + \cos(A+E+C) + \cos(A+B+F)), \\
k_2 &= \sin S + \sin(A+D) + \sin(B+E) + \sin(C+F) + \sin(D+E+F) + \sin(D+B+C) \\
&\quad + \sin(A+E+C) + \sin(A+B+F), \\
k_3 &= 2\,(\sin A \sin D + \sin B \sin E + \sin C \sin F), \\
k_4 &= \sqrt{k_1^2 + k_2^2 - k_3^2}.
\end{aligned}$$

Notice that the sums of dihedral angles $V_1 = A+B+C$, $V_2 = A+E+F$, $V_3 = B+D+F$, $V_4 = C+D+E$ in the numerator of the volume formula and $H_1 = A+B+D+E$, $H_2 = A+C+D+F$, $H_3 = B+C+E+F$ in the denominator have the following geometrical sense. V_i is a sum of dihedral angles at the edges meeting in the vertex v_i (see Fig. 1)and H_i is a sum of dihedral angles along three Hamiltonian cycles of T. With use of these notations the volume formula will have more short and convenient form

$$Vol(T) = -\frac{1}{4}\int_{z_1}^{z_2} \log \frac{\cos\frac{V_1+z}{2}\cos\frac{V_2+z}{2}\cos\frac{V_3+z}{2}\cos\frac{V_4+z}{2}}{\sin\frac{H_1+z}{2}\sin\frac{H_2+z}{2}\sin\frac{H_3+z}{2}\sin\frac{z}{2}}\,dz.$$

Recall that the Dilogarithm function is defined by the integral

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt,$$

where $x \in \mathsf{C}\setminus[1,\infty)$ and a continuous branch of the $\log\xi = \log|\xi| + i\,\arg\xi$ is determined by conditions $-\pi < \arg\xi < \pi$. We set also $l(z) = \mathrm{Li}_2\,(e^{i\,z})$.

As an immediate consequence of the Theorem 1, we have the following result obtained earlier by [MY] and [U].

Corollary 1. *In the above notations the hyperbolic volume of tetrahedron is given by the formula*

$$Vol(T) = \frac{1}{2}\Im(U(z_1,\,T) - U(z_2,\,T)),$$

where

$$\begin{array}{l} U(z,\,T)=(l(z)+l(A+B+D+E+z)+l(A+C+D+F+z)\\ +l(B+C+E+F+z)-l(\pi+A+B+C+z)-l(\pi+A+E+F+z)\\ -l(\pi+B+D+F+z)-l(\pi+C+D+E+z))/2 \end{array}.$$

To simplify Murakami-Yano-Ushijima formula we note that

$$\Im(l(z))=\Im(\mathrm{Li}_2(\mathrm{e}^{\mathrm{i}\,\mathrm{z}}))=2\,\Lambda(\frac{\mathrm{z}}{2}),$$

where $\Lambda(z)$is the Lobachevsky function defined by the integral

$$\Lambda(z)=-\int_0^z \log|2\,\sin t|\,dt.$$

Hence, the volume of tetrahedron is an algebraic sum of sixteen Lobachevsky functions.

Corollary 2. *In the above notations the hyperbolic volume of tetrahedron is given by the formula*

$$\begin{array}{rcl} Vol(T) &=& \frac{1}{2}(\Lambda(\frac{z_1}{2})+\Lambda(\frac{A+B+D+E+z_1}{2})+\Lambda(\frac{A+C+D+F+z_1}{2})\\ && +\Lambda(\frac{B+C+E+F+z_1}{2})-\Lambda(\frac{\pi+A+B+C+z_1}{2})-\Lambda(\frac{\pi+A+E+F+z_1}{2})\\ && -\Lambda(\frac{\pi+B+D+F+z_1}{2})-\Lambda(\frac{\pi+C+D+E+z_1}{2})\\ && -\Lambda(\frac{z_2}{2})-\Lambda(\frac{A+B+D+E+z_2}{2})-\Lambda(\frac{A+C+D+F+z_2}{2})\\ && -\Lambda(\frac{B+C+E+F+z_2}{2})+\Lambda(\frac{\pi+A+B+C+z_2}{2})+\Lambda(\frac{\pi+A+E+F+z_2}{2})\\ && +\Lambda(\frac{\pi+B+D+F+z_2}{2})+\Lambda(\frac{\pi+C+D+E+z_2}{2})). \end{array}$$

2 Preliminaries

Let compact hyperbolic tetrahedron $T=(A,B,C,D,E,F)\in\mathbb{H}^3$ have vertices v_1,v_2,v_3,v_4 and dihedral angles A,B,C,D,E,F with edge lengths a,b,c,d,e,f respectively (see Fig. 1).

Denote by

$$G=\langle-\cos\alpha_{ij}\rangle_{i,j=1,2,3,4}=\begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F\\ -\cos A & 1 & -\cos C & -\cos E\\ -\cos B & -\cos C & 1 & -\cos D\\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}$$

the Gram matrix of T. Let $H=\langle c_{ij}\rangle_{i,j=1,2,3,4}$ be the associated with G matrix formed by $c_{ij}=(-1)^{i+j}M_{ij}$, where M_{ij} is $(i,j)-th$ minor of G. In the following proposition we collect some known results about hyperbolic tetrahedron (see, for instance, [U]).

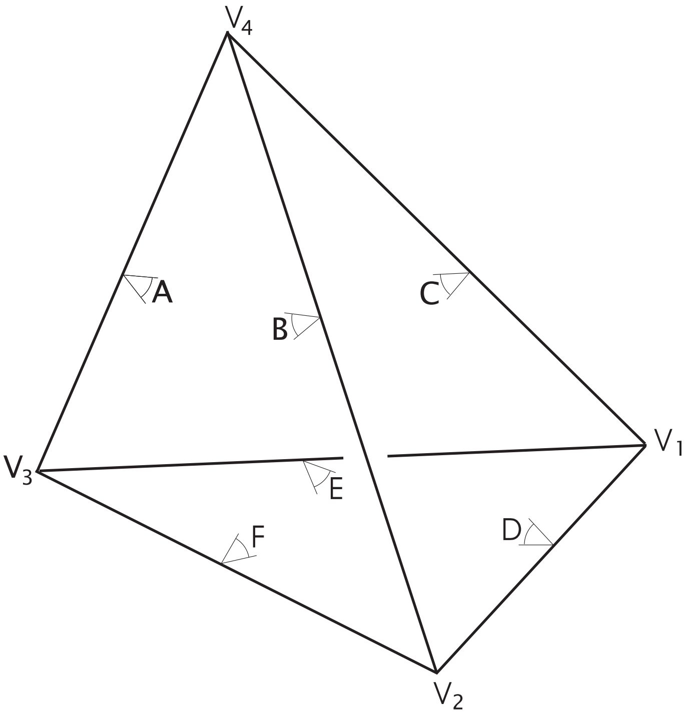

Figure 1: The tetrahedron T

Proposition 1. *Let T be a proper hyperbolic tetrahedron. Then*

$$\begin{aligned} &\text{(i)} && \det G < 0 \\ &\text{(ii)} && c_{ii} > 0, i = 1,2,3,4 \\ &\text{(iii)} && \cosh l_{ij} = \frac{c_{ij}}{\sqrt{c_{ii}c_{jj}}}, \end{aligned} \tag{1}$$

where l_{ij} is a hyperbolic length of the edge joining vertices v_i and v_j.

Further, we need the following assertion due to Jacobi ([P], Theorem 2.5.1, p.12).

Proposition 2 (Jacobi). *Let $A = \langle a_{ij} \rangle_{i,j=1,\dots,n}$ be a matrix and $\Delta = \det A$ is determinant of A. Denote by $C = \langle c_{ij} \rangle_{i,j=1,\dots,n}$ the matrix formed by elements $c_{ij} = (-1)^{i+j} \det A_{ij}$, where A_{ij} is $(n-1) \times (n-1)$ minor obtained by removing $i-th$ line and $j-th$ column of the matrix A. Then for any k, $1 \le k \le n-1$ we have*

$$det\langle c_{ij} \rangle_{i,j=1,\dots,k} = \Delta^{k-1} det\langle a_{ij} \rangle_{i,j=k+1,\dots,n}$$

Our calculation of volume of tetrahedron will be based on the following Schläfli formula (see, for instance [Sh],[Hd],[K]).

Proposition 3 (The Schläfli volume formula). *Let compact simplex $S \in \mathbb{H}^n$ $(n \ge 2)$ have vertices $P_o, \dots, P_n$ and dihedral angles $\alpha_{jk} = \angle(S_j, S_k)$, $0 \le j < k \le n$, of order*

$n-1$ formed by the faces S_j, S_k of S with apex $S_{jk} := S_j \cap S_k$. Then the differential of the volume function V_n on the set of all simplices in $\mathbb{X}^n$ can be represented by

$$-dV(S) = \frac{1}{(n-1)} \sum_{\substack{j,k=1 \\ j<k}}^{n+1} Vol_{n-2}(S_{jk}) d\alpha_{jk} \qquad (V_o(S_{jk}) := 1).$$

The Schläfli formula for hyperbolic 3-spaces can be reduced to

$$-dV = \frac{1}{2} \sum_{\substack{j,k=1 \\ j<k}}^{4} l_{jk} d\alpha_{jk},$$

where l_{jk} are the lengths of the correspondent edges of the tetrahedron.

3 Some elementary calculations

Let $T(A,B,C,D,E,F)$ be a hyperbolic tetrahedron with dihedral angles A,B,C,D,E,F and edge lengths a,b,c,d,e,f respectively (see Fig.1) In this section we prove two pure technical results.

Lemma 1. *Let A,B,C,D,E,F be dihedral angles of a hyperbolic tetrahedron. Then*

$$z_1 = \arctan \frac{k_3}{k_4} - \arctan \frac{k_1}{k_2},$$

$$z_2 = \pi - \arctan \frac{k_3}{k_4} - \arctan \frac{k_1}{k_2},$$

where

$$\begin{aligned}
k_1 &= -(\cos(A+B+C+D+E+F) + \cos(A+D) + \cos(B+E) + \cos(C+F) \\
&\quad + \cos(D+E+F) + \cos(D+B+C) + \cos(A+E+C) + \cos(A+B+F)), \\
k_2 &= \sin(A+B+C+D+E+F) + \sin(A+D) + \sin(B+E) + \sin(C+F) \\
&\quad + \sin(D+E+F) + \sin(D+B+C) + \sin(A+E+C) + \sin(A+B+F), \\
k_3 &= 2\,(\sin A \sin D + \sin B \sin E + \sin C \sin F), \\
k_4 &= \sqrt{k_1^2 + k_2^2 - k_3^2}
\end{aligned}$$

are roots of equation

$$\frac{\cos\frac{A+B+C+z}{2}\cos\frac{A+E+F+z}{2}\cos\frac{B+D+F+z}{2}\cos\frac{C+D+E+z}{2}}{\sin\frac{A+B+D+E+z}{2}\sin\frac{A+C+D+F+z}{2}\sin\frac{B+C+E+F+z}{2}\sin\frac{z}{2}} = 1.$$

Moreover $0 < z_2 - z_1 < \pi$.

Proof. Direct calculations show that our equation is equivalent to

$$k_1 \cos z + k_2 \sin z = k_3, \tag{2}$$

where $k_1,\ k_2, k_3$ are the same as it pointed above.

The dihedral angles of hyperbolic tetrahedron are less then π and consequently $k_3 > 0$. We note that the following identity is hold

$$k_1^2 + k_2^2 - k_3^2 = -4 \det G, \tag{3}$$

where G is Gramm matrix. Since $\det G < 0$, for compact hyperbolic tetrahedron, we have

$$k_1^2 + k_2^2 > 0, \tag{4}$$

$$0 < k_3, k_4 < \sqrt{k_1^2 + k_2^2}. \tag{5}$$

Hence, the equation (2) is equivalent to

$$\sin(z + x) = \sin \alpha, \tag{6}$$

where x and α are defined by equalities

$$\sin x = \frac{k_1}{\sqrt{k_1^2 + k_2^2}}, \qquad \cos x = \frac{k_2}{\sqrt{k_1^2+k_2^2}}, \tag{7}$$

$$\sin \alpha = \frac{k_3}{\sqrt{k_1^2 + k_2^2}} > 0. \tag{8}$$

Consider the following two the roots of (6)

$$z_1 = \alpha - x = \arctan \frac{k_3}{k_4} - \arctan \frac{k_1}{k_2} \tag{9}$$

$$z_2 = \pi - \alpha - x = \pi - \arctan \frac{k_3}{k_4} - \arctan \frac{k_1}{k_2}, \tag{10}$$

From (5) we have inequality $0 < \alpha < \pi/2$. Hence, $0 < z_2 - z_1 < \pi$ and the proof is complete.

□

Lemma 2. *Let*

$$\Phi(z) = -\frac{1}{4} \log \frac{\cos \frac{V_1+z}{2} \cos \frac{V_2+z}{2} \cos \frac{V_3+z}{2} \cos \frac{V_4+z}{2}}{\sin \frac{H_1+z}{2} \sin \frac{H_2+z}{2} \sin \frac{H_3+z}{2} \sin \frac{z}{2}},$$

where $V_1 = A + B + C$, $V_2 = A + E + F$, $V_3 = B + D + F$, $V_4 = C + D + E$ *and* $H_1 = A + B + D + E$, $H_2 = A + C + D + F$, $H_3 = B + C + E + F$.

Then

$$\int_{z_1}^{z_2} \frac{\partial \Phi}{\partial A} \, dz = -\frac{a}{2}.$$

Proof. Since

$$\frac{\partial\Phi}{\partial A} = -\frac{1}{8}(\tan\frac{V_1+z}{2} + \tan\frac{V_2+z}{2} + \cot\frac{H_1+z}{2} + \cot\frac{H_2+z}{2}),$$

we obtain

$$\int_{z_1}^{z_2} \frac{\partial\Phi}{\partial A}\, dz = -\frac{1}{4}\log\frac{\cos\frac{V_1+z_2}{2}\cos\frac{V_2+z_2}{2}\sin\frac{H_1+z_1}{2}\sin\frac{H_2+z_1}{2}}{\cos\frac{V_1+z_1}{2}\cos\frac{V_2+z_1}{2}\sin\frac{H_1+z_2}{2}\sin\frac{H_2+z_2}{2}}.$$

Consider separately the numerator Q_1 of the fraction

$$\frac{Q_1}{Q_2} = \frac{\cos\frac{V_1+z_2}{2}\cos\frac{V_2+z_2}{2}\sin\frac{H_1+z_1}{2}\sin\frac{H_2+z_1}{2}}{\cos\frac{V_1+z_1}{2}\cos\frac{V_2+z_1}{2}\sin\frac{H_1+z_2}{2}\sin\frac{H_2+z_2}{2}}$$

under the logarithm. We have

$$\begin{aligned}
Q_1 &= \frac{1}{4}(\sin\frac{H_1+V_1}{2}\cos\frac{z_1+z_2}{2} + \cos\frac{H_1+V_1}{2}\sin\frac{z_1+z_2}{2} + \sin\frac{H_1-V_1}{2}\cos\frac{z_1-z_2}{2} \\
&\quad + \cos\frac{H_1-V_1}{2}\sin\frac{z_1-z_2}{2})(\sin\frac{H_2+V_2}{2}\cos\frac{z_1+z_2}{2} + \cos\frac{H_2+V_2}{2}\sin\frac{z_1+z_2}{2} \\
&\quad + \sin\frac{H_2-V_2}{2}\cos\frac{z_1-z_2}{2} + \cos\frac{H_2-V_2}{2}\sin\frac{z_1-z_2}{2}) \\
&= \frac{1}{4}(q_{11}\cos^2\frac{z_1+z_2}{2} + q_{22}\sin^2\frac{z_1+z_2}{2} + q_{33}\cos^2\frac{z_1-z_2}{2} + q_{44}\sin^2\frac{z_1-z_2}{2} \\
&\quad + q_{12}\cos\frac{z_1+z_2}{2}\sin\frac{z_1+z_2}{2} + q_{13}\cos\frac{z_1+z_2}{2}\cos\frac{z_1-z_2}{2} + q_{14}\cos\frac{z_1+z_2}{2}\sin\frac{z_1-z_2}{2} \\
&\quad + q_{23}\cos\frac{z_1+z_2}{2}\cos\frac{z_1-z_2}{2} + q_{24}\sin\frac{z_1+z_2}{2}\sin\frac{z_1-z_2}{2} + q_{34}\cos\frac{z_1-z_2}{2}\sin\frac{z_1-z_2}{2}),
\end{aligned}$$

where $q_{ii} = f_i(H_1,V_1)f_i(H_2,V_2)$, $q_{ij} = f_i(H_1,V_1)f_j(H_2,V_2) + f_j(H_1,V_1)f_i(H_2,V_2)$, $i \neq j$ and $f_1(x,y) = \cos\frac{x+y}{2}$, $f_2(x,y) = \sin\frac{x+y}{2}$, $f_3(x,y) = \cos\frac{x-y}{2}$, $f_4(x,y) = \sin\frac{x-y}{2}$.

From (7), (8) and (9), (10) we have

$$\cos\frac{z_1+z_2}{2} = \frac{k_1}{\sqrt{k_1^2+k_2^2}}, \quad \sin\frac{z_1+z_2}{2} = \frac{k_2}{\sqrt{k_1^2+k_2^2}},$$
$$\cos\frac{z_1-z_2}{2} = \frac{k_3}{\sqrt{k_1^2+k_2^2}}, \quad \sin\frac{z_1-z_2}{2} = \frac{-k_4}{\sqrt{k_1^2+k_2^2}},$$

where $k_4^2 = k_1^2 + k_2^2 - k_3^2 = -4\det G$. Hence

$$\begin{aligned}
Q_1 &= \frac{1}{4\,(k_1^2+k_2^2)}(q_{11}k_1^2 + q_{22}k_2^2 + q_{33}k_3^2 + q_{44}k_4^2 + q_{12}k_1k_2 + q_{13}k_1k_3 + q_{23}k_2k_3 \\
&\quad -(q_{14}k_1 + q_{24}k_2 + q_{34}k_3)k_4).
\end{aligned}$$

Substituting k_1, k_2 and k_3 in terms of dihedral angles we obtain after routine simplifications

$$Q_1 = \frac{-2}{k_1^2+k_2^2}(\cos D + \cos(C-E))(\cos D + \cos(B-F))(2c_{34} + k_4\sin A).$$

By the similar way we obtain

$$Q_2 = \frac{-2}{k_1^2 + k_2^2}(\cos D + \cos(C - E))(\cos D + \cos(B - F))(2c_{34} - k_4 \sin A).$$

Hence

$$\begin{aligned} \int_{z_1}^{z_2} \frac{\partial \Phi}{\partial A}\, dz &= -\frac{1}{4} \log \frac{Q_1}{Q_2} && (11) \\ &= -\frac{1}{4} \log \frac{c_{34} + \sqrt{-\det(G)} \sin A}{c_{34} - \sqrt{-\det(G)} \sin A} = -\frac{1}{2}\mathrm{Arctanh}\, \frac{\sqrt{-\det(G)} \sin A}{c_{34}}. \end{aligned}$$

By the proposition 1 we have

$$\cosh^2 a = \frac{{c_{34}}^2}{c_{33} c_{44}}, \tag{12}$$

and, consequently,

$$\sinh^2 a = \frac{c_{33} c_{44} - {c_{34}}^2}{c_{33} c_{44}}.$$

From the other hand Proposition 2 gives

$$\begin{vmatrix} c_{33} & c_{34} \\ c_{43} & c_{44} \end{vmatrix} = \det(G) \begin{vmatrix} 1 & -\cos A \\ \cos A & -1 \end{vmatrix}.$$

Therefore

$$c_{33} c_{44} - {c_{34}}^2 = -\det(G)(1 - \cos^2 A),$$

and we obtain

$$\sinh^2 a = \frac{c_{33} c_{44} - {c_{34}}^2}{c_{33} c_{44}} = \frac{-\det(G)(1 - \cos^2 A)}{c_{33} c_{44}} = -\det(G) \frac{\sin^2 A}{c_{33} c_{44}}. \tag{13}$$

Combining (13) and (12) we deduce

$$\tanh^2 a = -\det(G) \frac{\sin^2 A}{{c_{34}}^2}. \tag{14}$$

Our assertion now follows from (11) and (14). □

4 The proof of the main theorem

Proof. (of Theorem 1) Let $T = T(A, B, C, D, E, F)$ be a hyperbolic tetrahedron. Denote by $V = V(A, B, C, D, E, F)$ the hyperbolic volume of T. Set

$$\widetilde{V} = \int_{z_1}^{z_2} \Phi(A, B, C, D, E, F, z)\, dz,$$

where

$$\Phi(A,B,C,D,E,F,z) = -\frac{1}{4}\log\frac{\cos\frac{A+B+C+z}{2}\cos\frac{A+E+F+z}{2}\cos\frac{B+D+F+z}{2}\cos\frac{C+D+E+z}{2}}{\sin\frac{A+B+D+E+z}{2}\sin\frac{A+C+D+F+z}{2}\sin\frac{B+C+E+F+z}{2}\sin\frac{z}{2}}.$$

The function V satisfies to the Schläfli differential equation with the following initial condition: for $A = B = C = D = E = F \to \pi/3$, T tends to regular Euclidean tetrahedron and, consequently, $V \to 0$.

To prove $\widetilde{V} = V$ we shall show that $\widetilde{V}$ satisfies to the Schläfli differential equation, and $\widetilde{V}$ satisfies to the same initial condition as V do. By the Leibniz Rule we have

$$\frac{\partial \widetilde{V}}{\partial A} = \Phi(A,B,C,D,E,F,z_2)\frac{\partial z_2}{\partial A} - \Phi(A,B,C,D,E,F,z_1)\frac{\partial z_1}{\partial A} + \int_{z_1}^{z_2}\frac{\partial \Phi}{\partial A}\,dz.$$

Since z_1, z_2 are the roots of the equation $\Phi = 0$ we obtain by use of Lemma 2

$$\frac{\partial \widetilde{V}}{\partial A} = -\frac{a}{2}.$$

The equalities

$$\frac{\partial \widetilde{V}}{\partial B} = -\frac{b}{2},\quad \frac{\partial \widetilde{V}}{\partial C} = -\frac{c}{2},\quad \frac{\partial \widetilde{V}}{\partial D} = -\frac{d}{2},\quad \frac{\partial \widetilde{V}}{\partial E} = -\frac{e}{2},\quad \frac{\partial \widetilde{V}}{\partial F} = -\frac{f}{2},$$

can be obtained by the similar way.

To finish the proof we notice that for $A = B = C = D = E = F \to \pi/3$, $\Phi \to 0$ and, consequently, $\widetilde{V} \to 0$. □

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