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The η-invariant of hyperbolic 3-manifolds*

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Introduction

Let M^3 be a compact oriented Riemannian manifold of dimension 3. Consider the operator A on even forms on M, $\Omega^0 + \Omega^2$, defined on Ω^{2p} $(p=0,1)$ by

$$A = (-1)^p (*d - d*).$$

The eigenvalues $\{\lambda\}$ of A are all real and can be either positive or negative. In [1], Atiyah, Patodi and Singer have defined

$$\eta(s) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda) |\lambda|^{-s}.$$

In [1] it is proved that $\eta(s)$ has a meromorphic continuation to the entire complex plane and does not have a pole at zero. It follows that $\eta(0)$ is well-defined. This value is the η-invariant $\eta(M)$ of M.

Theorem (Atiyah-Patodi-Singer [1]). *Let W be a 4-dimensional compact oriented Riemannian manifold with boundary M and assume that, near M, it is isometric to a product. Then*

$$\eta(M) = \tfrac{1}{3} \int_W P_1 - \operatorname{Sign}(W),$$

where $\operatorname{Sign}(W)$ *is the signature of the non-degenerate quadratic form defined by the cup product on the image $H^2(W, M)$ in $H^2(W)$ and P_1 is the first Pontrjagin form of the Riemannian metric.*

In this paper, we use the formula in the above theorem to study $\eta(M)$ for hyperbolic 3-manifolds and calculate the right-hand side of the formula explicitly, in a special case. Our main task is to represent the integral of P_1 by some more tractible ones.

In Sect. 1, we deal with general compact oriented Riemannian manifolds M of dimension 3. Let $F(M)$ be the $SO(3)$ oriented frame bundle of M. Let L be a

* Dedicated to Professor M. Nakaoka on his sixtieth birthday

link in M. Let α be an orthonormal framing on M such that at each point of L the first vector of α is tangent to L. Let $\mathscr{F}$ be an orthonormal framing defined on $M-L$ which has a special singularity at L (see Sect. 1, this notion is due to Meyerhoff [10]). We define the torsion number of α along L, $\tau(L,\alpha)$, and the difference degree, $d(\mathscr{F},\alpha)$. The former is a real number and the latter an integer. Let Q be the Chern-Simons form on $F(M)$ of the Riemannian metric. Let $s\colon M-L\to F(M)$ be the section defined by $\mathscr{F}$. Then the integral $\int_{s(M-L)} Q$ is defined. Finally let $\delta(M,\alpha)$ be the Hirzebruch invariant of the framed 3-manifold (M,α) defined by $\delta(M,\alpha)=\frac{1}{3}P_1[W]-\mathrm{Sign}(W)$, where W is a compact oriented 4-manifold with boundary M and $P_1[W]$ is the relative Pontrjagin number of W with respect to the framing α [9].

Using these notions, we prove in Sect. 1,

Theorem 1. *Let M be a compact oriented Riemannian manifold of dimension* 3. *Let L be a link in M. Let α be an orthonormal framing on M such that, at each point of L, the first vector of it is tangent to L. Let $\mathscr{F}$ be an orthonormal framing on $M-L$ having a special singularity at L. Then*

$$\eta(M)=\frac{1}{3}\int_{s(M-L)} Q-\frac{1}{6\pi}\tau(L,\alpha)+\frac{2}{3}d(\mathscr{F},\alpha)+\delta(M,\alpha).$$

The precise definitions of all the terms in the right hand side of the above equation will be given in Sect. 1. We note that the last two terms in the equation have nothing to do with the metric.

From Sect. 2 on, we restrict our attention to hyperbolic 3-manifolds. Our method is as follows. Fix an oriented complete hyperbolic 3-manifold N of finite volume with h cusps $(h\geqq 1)$. By Thurston [14], deforming suitably the hyperbolic structure on N and completing it, we obtain a family of infinitely many closed hyperbolic 3-manifolds $\{M\}$ (hyperbolic Dehn surgery). Topologically each M is obtained from N by attaching h solid tori to the h ends of N, and the corresponding h core curves of the solid tori are short simple geodesic loops in M and form a link γ in M. We want to apply Theorem 1 to the η-invariant of M. At first we must choose a link L in M and a framing $\mathscr{F}$ on $M-L$ having a special singularity at L. It would seem natural to set $L=\gamma$. Unfortunately in general, $M-\gamma$ does not admit such a framing. However adding some extra loops $m=\bigcup m_i$ to γ and setting $L=\gamma\cup m$, we can construct such a framing $\mathscr{F}$ on $M-L$. Then choosing a suitable framing α on M we can apply Theorem 1 to $\eta(M)$. As noted before, the last two terms of the equation in Theorem 1 are purely topological and may be calculated by the functorial method (see Sect. 5, for example). The calculation of $\tau(L,\alpha)$ is local and comparatively easy. What is $\int_{s(M-L)} Q$?

In Sects. 3 and 4, we study this integral. $M-\gamma$ is N with deformed hyperbolic structure, $M-L=N-m$ and $\int_{s(M-L)} Q=\int_{s(N-m)} Q$. We show that the link m in N can be chosen independently of M (Proposition 3.1, Sect. 3). Let U be the deformation space of the hyperbolic structure on N. For $u\in U$, we denote the corresponding hyperbolic manifold by N_u. If we choose a framing $\mathscr{F}_u$ on N_u-m, the integral $\int_{s(N_u-m)} Q$ is defined, where $s\colon N_u-m \to F(N_u)$ is the section defined by $\mathscr{F}_u$. We choose a family of framings $\{\mathscr{F}_u\}_{u\in U}$ such that $\mathscr{F}_u$ varies in a good manner on a neighborhood of the end of N when u varies through U, each $\mathscr{F}_u$ has a special singularity at m and $\int_{s(N_u-m)} Q$ defines a real-valued smooth function on U. Here 'a good manner' means the following: if N_u can be completed to a closed hyperbolic 3-manifold M_u by adjoining h geodesic loops γ to the end of N_u, then the framing $\mathscr{F}_u$ on $N_u-m=M_u-(\gamma\cup m)$ has a special singularity at $L=\gamma\cup m$.

The deformation space U has a natural complex structure (Sect. 2 for a brief summary). We express the function $\int_{s(N_u-m)} Q$ on U as the imaginary part of some analytic function $f(u)$ on U. There is a bi-invariant closed analytic differential form C of degree 3 on the complex Lie group $PSL_2(\mathbb{C})$ such that the imaginary part of C is the Chern-Simons form Q (regarding $PSL_2(\mathbb{C})$ as the $SO(3)$ frame bundle of hyperbolic 3-space H^3) and the real part of C is the volume form plus an exact form up to scalar multiplication (Def. 3.1., Sect. 3). Using a developing map of N_u into H^3, C can be pulled back to $F(N_u)$, and the integral $\int_{s(N_u-m)} C$ is defined and its imaginary part is $\int_{s(N_u-m)} Q$. For a technical reason, to define $f(u)$, we must subtract from the integral of C a term arising from the extra link m, and we obtain a complex-valued function $f(u)$ on U whose imaginary part contains $\int_{s(N_u-m)} Q$ (Def. 3.2, Sect. 3). The analyticity of $f(u)$ is stated in Theorem 3.1 in Sect. 3 and proved in Sect. 4.

Taking the exponential of $2\pi f(u)$, $F(u)=\exp(2\pi f(u))$, and calculating it, we prove the following in Sect. 3,

Theorem 2. *Let N be an oriented complete hyperbolic 3-manifold of finite volume with h cusps. Let U be the deformation space of the hyperbolic structure on N. Let u^0 be the point of U representing the original complete hyperbolic structure on N. Then there is a complex analytic function $F(u)$ on a neighborhood V of u^0 in U such that if $u\in V$ represents the hyperbolic manifold N_u which can be completed to a closed hyperbolic manifold M_u by adjoining h geodesic loops $\gamma=\bigcup\gamma_i$ to the h ends of N_u, then*

$$|F(u)|=\exp\left(\frac{2}{\pi}\operatorname{vol}(M_u)+\Sigma_i\ length(\gamma_i)\right),$$

$$\arg F(u)=(4\pi\ CS(M_u)+\Sigma_i\ torsion(\gamma_i)) \bmod 2\pi\mathbf{Z},$$

where $\operatorname{vol}(M_u)$ is the volume of M_u, $CS(M_u)$ is the Chern-Simons invariant of M_u and torsion (γ_i) is the torsion of the geodesic loop γ_i (Def. 1.2, Sect. 1).

The above theorem was conjectured in [13].

In Sect. 5, using the results in the preceeding sections, we calculate the η-invariant of the hyperbolic manifold $M_{p,q}$ obtained by performing Dehn surgery of type (p,q) along the figure-eight knot K in S^3. By [14], $N=S^3-K$ has a complete hyperbolic structure of finite volume with one cusp, and the points of the deformation space of the hyperbolic structure on N are parametrized by pairs (z,w) of complex numbers in the upper half plane satisfying the equation

$$\log z+\log(1-z)+\log w+\log(1-w)=0, \tag{I}$$

where log is taken with $-\pi<\arg<\pi$.

In addition, if $u=(z,w)$ satisfies the following equation for a coprime pair of integers (p,q),

$$p\log w(1-z)+q\log z^2(1-z)^2=2\pi i, \tag{II}$$

then the corresponding hyperbolic manifold N_u can be completed to a closed hyperbolic manifold $M_{p,q}$, and for each coprime pair of integers (p,q) such that $|p|\geqq 5$ if $|q|=1$, there is such a pair (z,w) ([14], §4).

We prove the following in Sect. 5.

Theorem 3. *Let $M_{p,q}$ be the hyperbolic manifold obtained by performing Dehn surgery of type (p,q) along the figure-eight knot in S^3, where (p,q) is a coprime pair of integers such that $|p|\geqq 5$ if $|q|=1$. Let (z,w) be the pair of complex numbers in the upper half plane satisfying* (I) *and* (II) *in the above.* Then,

$$\eta(M_{p,q})=-\frac{1}{3\pi^2}\operatorname{Re}\left(R(z)+R(w)-\frac{\pi^2}{6}\right)+\frac{1}{3p\pi}\arg z(1-z)$$
$$+\frac{1}{p}\operatorname{def}(p;q,1)+\frac{q}{3p}$$

where

(i) $R(x)$ is the function on the upper half plane defined by

$$R(x)=\tfrac{1}{2}\log x\log(1-x)-\int_0^x\log(1-t)\,d\log t$$

and

(ii) $\operatorname{def}(p;q,1)=-\sum_{k=1}^{p-1}\cot\frac{k}{p}\pi\cot\frac{k}{p}q\pi$ is the Hirzebruch defect [7].

If $|p|$ or $|q|$ increases to $+\infty$, then (z,w) converges to $(e^{\pi i/3}, e^{\pi i/3})$ and the terms except the last two terms of the right-hand side of the above equation converge to zero. Hence, if p is fixed and $|q|$ is sufficiently large, then $\eta(M_{p,q})$ is nearly equal to $a(p,q)=\frac{1}{p}\operatorname{def}(p;q,1)+\frac{q}{3p}$. By definition, $\operatorname{def}(p;q,1)$ depends only on $q \bmod p$ and $\frac{1}{p}\operatorname{def}(p;q,1)=-\frac{1}{3p}(q+r)\bmod\frac{1}{3}\mathbf{Z}$, where $qr=1\bmod p$. Hence $a(p,q)=-\frac{r}{3p}\bmod\frac{1}{3}\mathbf{Z}$. If $q\neq q'\bmod p$, it follows $|a(p,q)-a(p,q')|\geqq\frac{1}{3p}$. If $q=q'\bmod p$ and $q\neq q'$, then $\operatorname{def}(p;q,1)=\operatorname{def}(p;q',1)$ and we have $|a(p,q)-a(p,q')|\geqq\frac{1}{3}$.

Corollary 1. *If p is fixed, then $\eta(M_{p,q})\neq\eta(M_{p,q'})$ for $q\neq q'$ and $|q|, |q'|\gg 0$. In particular, for $q\neq q'\gg 0$, $\pi_1(M_{p,q})$ is not isomorphic to $\pi_1(M_{p,q'})$.*

The last statement of the above corollary follows from the rigidity theorem of Mostow [12].

If a closed hyperbolic manifold admits a self-homotopy equivalence of degree -1, then it admits an orientation-reversing isometry [12]. This implies that its η-invariant vanishes [1].

Corollary 2. *If p is fixed, then for all sufficiently large q, $M_{p,q}$ admits no self-homotopy equivalence of degree -1.*

1. A splitting of the η-invariant of 3-dimensional Riemannian manifolds

In this section, we consider a general 3-dimensional compact oriented Riemannian manifold M with Riemannian metric g. Let $F(M)$ be its $SO(3)$ oriented frame bundle. Let (θ_i) and (θ_{ij}) be the fundamental form and the connection form respectively of the Riemannian connection on $F(M)$, that is, (θ_{ij}) is a matrix of 1-forms on $F(M)$ such that $\theta_{ij} = -\theta_{ji}$ and $d\theta_i = -\Sigma\,\theta_{ij} \wedge \theta_j$ $(i,j = 1,2,3)$. If α is an orthonormal framing defined on a subset A in M, it defines the section s: $B \to F(M)$ for each subset $B \subset A$. Let L be a link in M, that is, $L = L_1 \cup \ldots \cup L_k$ is a finite union of smoothly embedded disjoint circles $L_1, \ldots, L_k$ in M.

Definition 1.1. Let $\alpha = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$ be an orthonormal framing defined on a subset of M containing L. Assume that $\bar{e}_1(y)$ is tangent to L at each $y \in L$. Then the torsion number of α along L, $\tau(L, \alpha)$, is defined by

$$\tau(L, \alpha) = -\int_{s(L)} \theta_{23},$$

where s: $L \to F(M)$ is the section defined by α and the orientation of L is given by $\bar{e}_1$. Clearly

$$\tau(L, \alpha) = \sum_{i=1}^{k} \tau(L_i, \alpha).$$

Lemma 1.1. *Let $\alpha = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$ and $\alpha' = (\bar{e}'_1, \bar{e}'_2, \bar{e}'_3)$ be two orthonormal framings on a subset of M containing L such that $\bar{e}_1(y)$ and $\bar{e}'_1(y)$ are tangent to L at each $y \in L$. Then,*

$$\tau(L, \alpha) - \tau(L, \alpha') \in 2\pi\mathbb{Z},$$

where $\mathbb{Z}$ denotes the ring of rational integers.

Proof. It suffices to show the lemma in the case that L is a simple closed curve in M. By assumption, for each $y \in L$,

$$\begin{aligned}
\bar{e}'_1(y) &= \nu\,\bar{e}_1(y),\\
\bar{e}'_2(y) &= \nu(\cos v(y))\,\bar{e}_2(y) + \nu(\sin v(y))\,\bar{e}_3(y),\\
\bar{e}'_3(y) &= -(\sin v(y))\,\bar{e}_2(y) + (\cos v(y))\,\bar{e}_3(y),
\end{aligned}$$

where $\nu = \pm 1$ and v: $L \to \mathbb{R}/2\pi\mathbb{Z}$ is a smooth map. Let s and s' be the sections $L \to F(M)$ defined by α and α' respectively. Then $s'^*\theta_{23} = s^*\theta_{23} - \nu\, dv$ and

$$\int_L s'^*\theta_{23} = \int_L s^*\theta_{23} - \nu \int_L dv.$$

In the right hand side L is oriented by $\bar{e}'_1$. Hence,

$$\tau(L,\alpha')=\tau(L,\alpha)+v\int_L dv.$$

Since $\int_L dv$ is an integral multiple of 2π, the lemma follows. q.e.d.

Definition 1.2. The torsion of a link L in M, torsion (L), is defined by

$$\operatorname{torsion}(L)=\tau(L,\alpha)\bmod 2\pi\mathbb{Z},$$

where α is an orthonormal framing defined on a subset of M containing L satisfying the condition of Definition 1.1.

Let $d: M\times M\to\mathbb{R}$ be the distance function on M induced by the Riemannian metric. For a link L in M, set $N_\varepsilon(L)=\{x\in M\,|\,d(x,L)\leqq\varepsilon\}$, where $\varepsilon>0$ and $d(x,L)=\inf\{d(x,y)\,|\,y\in L\}$. For a sufficiently small $\varepsilon>0$, $N_\varepsilon(L)$ is diffeomorphic to $L\times D^2$, where D^2 is the 2-disc, and each $x\in N_\varepsilon(L)$ can be joined to a unique point $y\in L$ by a unique geodesic $\gamma(x,y)$ in $N_\varepsilon(L)$ such that $d(x,y)=d(x,L)=$ length $(\gamma(x,y))$. Moreover for each $y\in L$, $S_\delta(y)=\{x\in N_\varepsilon(L)\,|\,d(x,L)=d(x,y)=\delta\leqq\varepsilon\}$ is a smooth circle in $N_\varepsilon(L)$.

The following notion is due to Meyerhoff [10].

Definition 1.3. Let L be a link in M. Let $\mathscr{F}=(e_1,e_2,e_3)$ be an orthonormal framing defined on $M-L$. Assume that $\mathscr{F}$ satisfies the following properties in $N_\varepsilon(L)$ for a sufficiently small $\varepsilon>0$: For each $x\in N_\varepsilon(L)-L$, taking $y\in L$ as above,

(i) $e_3(x)$ is tangent to $\gamma(x,y)$ and it has the direction opposite to y, and

(ii) $e_2(x)$ is tangent to $S_\delta(y)$, here $d(x,y)=\delta$. Then we say that $\mathscr{F}$ has a special singularity at L.

Note that e_1 has the direction along L in $N_\varepsilon(L)-L$. $\mathscr{F}$ looks like this near L.

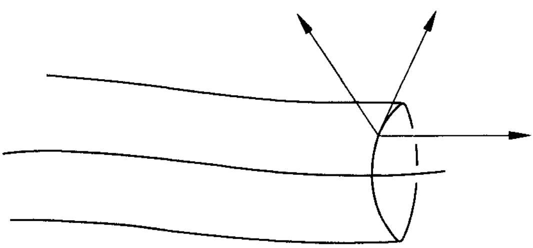

Fig. 1

Lemma 1.2. *Let $\mathscr{F}$ be an orthonormal framing on $M-L$ having a special singularity at L. Let s: $M-L\to F(M)$ be the section defined by $\mathscr{F}$. Set $\bar{M}=$ the closure of $s(M-L)$ in $F(M)$. Then $\bar{M}$ is a 3-dimensional compact manifold with boundary $\partial\bar{M}$. $\partial\bar{M}$ is diffeomorphic to $S^1\times L$ and it is mapped onto L by the bundle projection $F(M)\to M$. Moreover $\mathscr{F}$ extends uniquely in a smooth manner to a framing $\bar{\mathscr{F}}$ on $\bar{M}$, where we identify $M-L$ with $s(M-L)$.*

Proof. Let $\alpha=(\bar{e}_1,\bar{e}_2,\bar{e}_3)$ be an orthonormal framing defined on a subset of M containing L such that $\bar{e}_1(y)$ is tangent to L at each $y\in L$ and it has the same direction as e_1 of $\mathscr{F}$ near y. Then $S(y)=\lim_{\delta\to 0} s(S_\delta(y))$ is a smooth circle in $F(M)$ represented by the rotation of the framing $\alpha(y)=(\bar{e}_1(y),\bar{e}_2(y),\bar{e}_3(y))$ about $\bar{e}_1(y)$. We have $\partial\overline{M}=\bigcup_{y\in L} S(y)$ and identifying the 1-sphere S^1 with $\mathbb{R}/2\pi\mathbb{Z}$, we define a map ψ: $S^1\times L\to\partial\overline{M}$ by

$$\psi(v,y)=(\bar{e}_1(y),(\cos v)\bar{e}_2(y)-(\sin v)\bar{e}_3(y),(\sin v)\bar{e}_2(y)+(\cos v)\bar{e}_3(y)),$$

for $0\leqq v<2\pi$ and $y\in L$, where the right hand side represents a framing at y, hence a point of $F(M)$. Then ψ is a diffeomorphism. Since $e_2(x)\to\partial/\partial v$ as $x\to y$ for $x\in L$ and $y\in L$, $\mathscr{F}$ extends naturally to a framing $\overline{\mathscr{F}}$ which is given on $\partial\overline{M}$ by

$e_1=$the parallel lifts of the unit tangent vectors of L,

$e_2=\partial/\partial v$,

$e_3=$inward normal vectors at ∂M in M. q.e.d.

In the rest of this section, we fix a link L in M and an orthonormal framing $\mathscr{F}=(e_1,e_2,e_3)$ on $M-L$ having a special singularity at L.

Let $\alpha=(\bar{e}_1,\bar{e}_2,\bar{e}_3)$ be an orthonormal framing on M. We assume that α satisfies the following condition, (*) for each point $y\in L$, $\bar{e}_1(y)$ is tangent to L and it has the same direction with e_1 of $\mathscr{F}$ near y.

Any framing on M can be deformed by homotopy so that it satisfies (*). We note that the latter condition about the direction of $\bar{e}_1$ is technical and not essential.

Let W be a 4-dimensional compact oriented Riemannian manifold with boundary M. We assume that W is isometric to a product $M\times[0,1]$ near M, where $M=M\times 0$. We set $W_0=W-M\times[0,1)$. Let $F(W)$ be the $SO(4)$ oriented frame bundle of W, and let p: $F(W)\to W$ be the bundle projection. We calculate the integral of the first Pontrjagin form by using a suitable connection c on $F(W)$ instead of the Riemannian connection on it. The connection c is defined as follows. Let c_g be the Riemannian connection on $F(M)$. Let c_α be the connection on $F(M)$ defined by the framing α. Let $\mu(t)$ be a smooth monotone increasing function defined on $[0,1]$ such that $\mu(t)=0$ $(0\leqq t\leqq 1/3)$ and $\mu(t)=1$ $(2/3\leqq t\leqq 1)$. For $t\in[0,1]$, let c_t be the connection on $F(M)$ defined by $c_t=(1-\mu(t))c_g+\mu(t)c_\alpha$, where $+$ is taken in the convex linear space of all the smooth connections on $F(M)$. Then $c_0=c_g$ and $c_1=c_\alpha$. Let c be the connection on $F(M\times[0,1])$ such that $c=c_t$ on $F(M\times t)$ and c is trivial in the direction of t. Extend c to a smooth connection on $F(W)$ in an arbitrary way on $F(W_0)$ and we get a smooth connection c on $F(W)$.

Let P_1 be the first Pontrjagin form of the connection c. Then,

$$\int_W P_1=\int_{M\times[0,1]} P_1+\int_{W_0} P_1.$$

At first we consider $\int_{W_0} P_1$. The framing α induces a map h: $W\to BSO(4)$ such that $h(M)=$ a point and h classifies the tangent bundle of W. Let $[P_1]\in H^4(BSO(4),\mathbb{Z})$ be the universal first Pontrjagin class. Then the relative Pontrjagin number with respect to α is defined as the evaluation $\langle h*[P_1],$

$[W]\rangle$, where $[W]$ denotes the orientation class of $H_4(W, M, \mathbb{Z})$ and h^*: $H^4(BSO(4), \mathbb{Z}) \to H^4(W, M, \mathbb{Z})$ is the map induced by h. We denote this number by $P_1[W]$. The following lemma is a standard fact from the theory of characteristic classes.

Lemma 1.3. $\int_{W_0} P_1 = P_1[W]$.

Next we consider $\int_{M\times[0,1]} P_1$. The framing $\mathscr{F} = (e_1, e_2, e_3)$ on $M-L$ and the unit tangent vector $\partial/\partial t$ of $[0,1]$ define the orthonormal framing $(e_1, e_2, e_3, \partial/\partial t)$ on $(M-L)\times[0,1]$, and it defines the section s: $(M-L)\times[0,1] \to F((M-L)\times[0,1])$. Let X be the closure of the image $s((M-L)\times[0,1])$ in $F((M-L)\times[0,1])$. As in Lemma 1.2, X is a 4-manifold with boundary and is diffeomorphic to $\overline{M}\times[0,1]$, where $\overline{M}$ is the 3-manifold defined in the lemma. The pull back of the first Pontrjagin form P_1, p^*P_1, is an exact form and by the Chern-Simons theory [5], there is a differential form of degree 3, Q^c, on $F(W)$ such that $p^*P_1 = dQ^c$ and Q^c is defined canonically by the connection c. The explicit form of Q^c on $F(M\times[0,1])$ is as follows. Let (θ^c_{ij}) and (Ω^c_{ij}) be the connection form and the curvature form respectively of the connection c $(i,j=1,2,3,4)$. Then $d\theta^c_{ij} = -\sum_{k=1}^{4} \theta^c_{ik}\wedge\theta^c_{kj} + \Omega^c_{ij}$. Since $\{\partial/\partial t\}$ is a parallel vector field on $\overline{M}\times[0,1]$ with respect to c, $\theta^c_{i4}(\partial/\partial t)=0$ on $X=\overline{M}\times[0,1]$ $(1\leqq i\leqq 4)$. Hence on X, we have

$$Q^c = \frac{1}{4\pi^2}(\theta^c_{12}\wedge\theta^c_{13}\wedge\theta^c_{23} + \theta^c_{12}\wedge\Omega^c_{12} + \theta^c_{13}\wedge\Omega^c_{13} + \theta^c_{23}\wedge\Omega^c_{23})$$

(see [5], §6).

By Stokes's theorem,

$$\int_{M\times[0,1]} P_1 = \int_X p^*P_1 = \int_X dQ^c = \int_{\partial X} Q^c.$$

The boundary of X, $\partial X = \partial(\overline{M}\times[0,1])$, consists of three parts, $\overline{M}\times 0$, $\overline{M}\times 1$ and $\partial\overline{M}\times[0,1]$. We consider the integral of Q^c on the three parts separately.

(i) $\int_{\overline{M}\times 0} Q^c$.

Let (θ_{ij}) and (Ω_{ij}) be the connection form and the curvature form respectively of the Riemannian connection on $F(M)$ as before. The Chern-Simons form Q on $F(M)$ is defined by

$$Q = \frac{1}{4\pi^2}(\theta_{12}\wedge\theta_{13}\wedge\theta_{23} + \theta_{12}\wedge\Omega_{12} + \theta_{13}\wedge\Omega_{13} + \theta_{23}\wedge\Omega_{23}).$$

The connection c is the product connection of the Riemannian connection c_g on $F(M)$ with the trivial connection in the direction of $[0,1]$ near $M\times 0$. It follows that

$$\int_{\overline{M}\times 0} Q^c = \int_{s(M-L)} Q,$$

where s: $M-L \to F(M)$ is the section defined by $\mathscr{F}$.

(ii) $\int_{\bar{M}\times 1} Q^c$.

The restriction of α to $M-L$ extends uniquely in a smooth manner to a framing $\bar{\alpha}$ on $\bar{M}$, where we identify $M-L$ with $s(M-L)$ (s is the section defined by $\mathscr{F}$), such that $\bar{\alpha}|\partial\bar{M}$ is the parallel lift of $\alpha|L$. On $\bar{M}$, there are two orthonormal framings $\bar{\alpha}=(\bar{e}_1,\bar{e}_2,\bar{e}_3)$ and $\bar{\mathscr{F}}=(e_1,e_2,e_3)$ which is defined in Lemma 1.2. Since α satisfies the condition $(*)$, $\bar{e}_1=e_1$ on $\partial\bar{M}$. Define the difference map f: $\bar{M}\to SO(3)$ by

$$(e_1(x),e_2(x),e_3(x))=(\bar{e}_1(x),\bar{e}_2(x),\bar{e}_3(x))f(x)$$

for $x\in\bar{M}$. Then $f(\partial\bar{M})\in SO(2)$. Hence f induces the homomorphism f_*: $H_3(\bar{M},\partial\bar{M},\mathbb{Z})\to H_3(SO(3),\ SO(2),\mathbb{Z})$. Let $[\bar{M}]$ be the orientation class in $H_3(\bar{M},\partial\bar{M},Z)=\mathbb{Z}$. Let $[SO(3)]$ be the orientation class in $H_3(SO(3),\mathbb{Z})=H_3(SO(3),SO(2),\mathbb{Z})=\mathbb{Z}$, where $SO(3)$ is canonically oriented.

Definition 1.4. The difference degree, $d(\mathscr{F},\alpha)$, is defined by $f_*[\bar{M}]=d(\mathscr{F},\alpha)[SO(3)]$.

Lemma 1.4. $\int_{\bar{M}\times 1} Q^c=2d(\mathscr{F},\alpha)$.

Proof. On a neighborhood of $M\times 1$, the connection c is the product of the connection c_α with the trivial connection in the direction of $[0,1]$. Since c_α is flat, its curvature forms vanish, and the connection form of c_α is given by the skew symmetric matrix of 1-forms $(f(x)^{-1}df(x))$ for $x\in\bar{M}$, where df is the differential of f and $(f(x)^{-1}df(x))$ is considered as an element of the Lie algebra of $SO(3)$. It follows that, on $\bar{M}\times 1$, $Q^c=\frac{1}{4\pi^2}\theta^c_{12}\wedge\theta^c_{13}\wedge\theta^c_{23}$ is equal to $f^*(-2\omega)$, where ω is the normalized invariant measure on $SO(3)$. Since $\bar{M}\times 1$ is $\bar{M}$ with opposite orientation, it follows

$$\int_{\bar{M}\times 1} Q^c=\int_{-\bar{M}} f^*(-2\omega)=d(\mathscr{F},\alpha)\int_{SO(3)} 2\omega=2d(\mathscr{F},\alpha).\quad \text{q.e.d.}$$

(iii) $\int_{\partial\bar{M}\times[0,1]} Q^c$

Lemma 1.5. $\int_{\partial\bar{M}\times[0,1]} Q^c=-\frac{1}{2\pi}\tau(L,\alpha)$.

Proof. Define a diffeomorphism ψ: $S^1\times L\times[0,1]\to\partial\bar{M}\times[0,1]$ by

$$\psi(v,y,t)=(\bar{e}_1(y),(\cos v)\bar{e}_2(y)-(\sin v)\bar{e}_3(y),(\sin v)\bar{e}_2(y)+(\cos v)\bar{e}_3(y),t)$$

where $v\in S^1=\mathbb{R}/2\pi\mathbb{Z}$, $y\in L$, $t\in[0,1]$ and $\alpha(y)=(\bar{e}_1(y),\bar{e}_2(y),\bar{e}_3(y))$. The orientation of $\bar{M}\times[0,1]$ is given by $(e_1,e_2,e_3,\partial/\partial t)$, where $\bar{\mathscr{F}}=(e_1,e_2,e_3)$. Since $e_3(x)$ is an inward normal vector at each $x\in\partial\bar{M}\times[0,1]$, the orientation of $S^1\times L\times[0,1]$ is given by $(\partial/\partial v,e_1,\partial/\partial t)$, where e_1 is the unit tangent vector of L. $\{\psi_*\,\partial/\partial v\}$ is a vertical vector field and ψ_*e_1 is parallel along the integral curves of that vector field. Hence $(\psi^*\Omega^c_{ij})(\partial/\partial v,\cdot)=0$ $(i,j=1,2,3)$ and $(\psi^*\theta^c_{12})(\partial/\partial v)=(\psi^*\theta^c_{13})(\partial/\partial v)=0$. Moreover by definition, $(\psi^*\theta^c_{23})(\partial/\partial v)=1$. Let q: $S^1\times L$

$\times[0,1]\to L\times[0,1]$ be the projection onto the last two factors. Then $q=p\psi$, where ψ: $S^1\times L\times[0,1]\to\partial\overline{M}\times[0,1]\subset F(M\times[0,1])$ and p: $F(M\times[0,1])\to M\times[0,1]$ is the bundle projection. Let s: $L\times[0,1]\to F(M\times[0,1])$ be the section defined by the framing $(\bar{e}_1,\bar{e}_2,\bar{e}_3,\partial/\partial t)$, where $\alpha=(\bar{e}_1,\bar{e}_2,\bar{e}_3)$. Then from the above facts, we have

$$\begin{aligned}\psi^* Q^c &= \frac{1}{4\pi^2}\psi^*(\theta^c_{12}\wedge\theta^c_{13}\wedge\theta^c_{23}+\theta^c_{23}\wedge\Omega^c_{23})\\ &= \frac{1}{4\pi^2}\psi^*(\theta^c_{23}\wedge d\theta^c_{23})\end{aligned}$$

and

$$\psi^*\theta^c_{23}=dv+q^*s^*\theta^c_{23}.$$

It follows that $\psi^*Q^c=\frac{1}{4\pi^2}dv\wedge d(q^*s^*\theta^c_{23})$. From the above orientation convention, by partial integration along S^1 and Stokes's theorem, we have

$$\begin{aligned}\int_{\partial\overline{M}\times[0,1]} Q^c &= \int_{S^1\times L\times[0,1]}\psi^*Q^c\\ &= \frac{1}{4\pi^2}\int_{S^1\times L\times[0,1]} dv\wedge d(q^*s^*\theta^c_{23})\\ &= \frac{1}{2\pi}\int_{L\times[0,1]} d(s^*\theta^c_{23})\\ &= \frac{1}{2\pi}\Big(\int_{L\times 0} s^*\theta^c_{23}-\int_{L\times 1} s^*\theta^c_{23}\Big).\end{aligned}$$

Since the connection c is c_g on $F(M\times 0)$ and c_α on $F(M\times 1)$, we have $s^*\theta^c_{23}=s^*\theta_{23}$ on $L\times 0$ and $s^*\theta^c_{23}=0$ on $L\times 1$. Hence the last expression of the above integral equals to $\frac{1}{2\pi}\int_L s^*\theta_{23}=-\frac{1}{2\pi}\tau(L,\alpha)$ by definition. q.e.d.

The calculations in the above proof will be frequently used in the subsequent sections, and we will omit the details of such calculations.

From (i), (ii) and (iii) in the above, we have

Theorem 1.1. *Let M be a 3-dimensional compact oriented Riemannian manifold. Let L be a link in M. Let $\mathscr{F}$ be an orthonormal framing on $M-L$ having a special singularity at L. Let α be an orthonormal framing on M satisfying (*). Let W be a 4-dimensional compact oriented Riemannian manifold with boundary M, and assume that it is isometric to a product near M. Then*

$$\int_W P_1=\int_{s(M-L)} Q-\frac{1}{2\pi}\tau(L,\alpha)+2d(\mathscr{F},\alpha)+P_1[W],$$

where s: $M-L\to F(M)$ is the section defined by $\mathscr{F}$.

Since $d(\mathscr{F},\alpha)$ and $P_1[W]$ are integers and $\int_W P_1 \bmod \mathbf{Z}$ is an invariant of M, we obtain from Lemma 1.1 and Definition 1.2:

Corollary 1.1. *In the situation in Theorem* 1.1,

$$\int_{s(M-L)} Q - \frac{1}{2\pi}\, \textit{torsion}\, (L) \bmod \mathbb{Z}$$

is an invariant of M.

This corollary is proved in [10] by a different method.

From Theorem 1.1, we obtain immediately Theorem 1 in the Introduction.

2. The deformation space of hyperbolic structure

We give a very brief account: see [13] and [14].

We use the upper half space model of hyperbolic 3-space, $H^3 = \{(c,t) | c \in \mathbb{C}$ and $t > 0\}$ with metric $ds^2 = t^{-2}(|dc|^2 + dt^2)$, which is bounded by the extended complex plane $\mathbb{C} \cup \infty$. If we write the points of H^3 in quaternion form $q = c + tj$, the orientation-preserving isometries of H^3 are complex Möbius transformations $q \to (\alpha q + \beta)(\gamma q + \delta)^{-1}$, where $\alpha\delta - \beta\gamma = 1$ and the computation is carried out within the algebra of quaternions. The group of orientation-preserving isometries may be identified with $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm \mathrm{id}\}$. The action of $PSL_2(\mathbb{C})$ on H^3 is transitive and the isotropy subgroup of the point $(0,1) \in H^3$ is $SO(3)$, its maximal compact subgroup. The correspondence $g \to g(0,1)$ for $g \in PSL_2(\mathbb{C})$ induces a diffeomorphism $PSL_2(\mathbb{C})/SO(3) \to H^3$, and the natural projection, $PSL_2(\mathbb{C}) \to H^3$ is considered as the $SO(3)$ oriented frame bundle of H^3. Thus $PSL_2(\mathbb{C}) = F(H^3)$.

Let N be a complete hyperbolic 3-manifold of finite volume with h cusps ($h \geqq 1$). There is a holonomy representation, ρ: $\Gamma = \pi_1(N, x^0) \to PSL_2(\mathbb{C})$ ($x^0 \in N$), which is unique up to equivalence, and N is identified with the quotient space $\rho(\Gamma)\backslash H^3$. Corresponding to the h cusps, N has h ends $\{\varepsilon_i\}_{i=1,\ldots,h}$ each of which is diffeomorphic to $T^2 \times (0, \infty)$, where T^2 denotes the 2-torus.

An ideal tetrahedron Δ in H^3 is a geodesic tetrahedron with all vertices at infinity $= \partial H^3$. An ideal tetrahedron is described (up to isometry) by a single complex number z in the upper half plane such that the euclidean triangle cut out of any vertex of Δ by a horosphere section is similar to the triangle with vertices 0, 1 and z. We write $\Delta = \Delta(z)$. The numbers z, $1 - 1/z$ and $1/1 - z$ give the same tetrahedron: to specify z uniquely, we must pick an edge of Δ (the dihedral angle at this edge will be $\arg(z)$).

We assume that N is decomposed into a finite union of ideal tetrahedra $\Delta_1 \cup \ldots \cup \Delta_n$, where the vertices of each Δ_i are deleted. For each Δ_i ($i = 1, \ldots, n$), we make a choice of an edge of Δ_i and write $\Delta_i = \Delta_i(z_i^0)$. Then to each edge of Δ_i is associated one of the three numbers z_i^0, $1 - 1/z_i^0$, $1/1 - z_i^0$ the modulus of the edge. We write

$$N = \Delta(z_1^0) \cup \ldots \cup \Delta(z_n^0).$$

Any hyperbolic 3-manifold is obtainable from an ideally triangulated one by Dehn surgeries on some cusps. Hence by the above assumption on N, we do not lose any generality in subsequent arguments.

If we replace $u=(z_1^0,\dots,z_n^0)$ by $u=(z_1,\dots,z_n)$ ($\operatorname{im} z_i>0$, $i=1,\dots,n$), we obtain a complex $N_u=\Delta(z_1)\cup\dots\cup\Delta(z_n)$ with the same gluing pattern as N. The necessary and sufficient condition that N_u gives a smooth (not necessarily complete) hyperbolic manifold is that at each edge e of N_u the tetrahedra Δ_i abutting e close up as one goes around e, and thus the product of the corresponding moduli of Δ_i at e is $\exp(2\pi i)$ (the product is taken in the universal cover $\tilde{\mathbb{C}}^*$ of $\mathbb{C}^*$). The consistency condition at e is written as

$$\prod_{i=1}^{n} z_i^{r_i}(1-z_i)^{r_i'}=\pm 1$$

for some integers r_i, r_i' depending on e [14]. Once we have chosen the numbers z_i (satisfying the consistency conditions), N acquires a smooth hyperbolic structure, in general incomplete. The deformation space U of the hyperbolic structure on N is the variety of $u=(z_1,\dots,z_n)\in\mathbb{C}^n$ which satisfies the consistency relations. For $u\in U$, we denote the corresponding hyperbolic manifold by N_u.

Choose a pair of simple closed curves (m_i,l_i) on each torus section T_i of ε_i ($i=1,\dots,h$) which forms a basis of $H_1(T_i)$. For each $u\in U$, let $\rho_u\colon \Gamma\to PSL_2(\mathbb{C})$ be a holonomy representation. After suitable choice of a base point, we consider (m_i,l_i) as elements of Γ. If $\rho_u(m_i)$ and $\rho_u(l_i)$ are not parabolic, they have two fixed points in $\mathbb{C}\cup\infty$ which we can put at 0 and ∞, so as Möbius transformations on $\mathbb{C}\cup\infty$,

$$\rho_u(m_i)\colon c\to a_i c,\qquad \rho_u(l_i)\colon c\to b_i c$$

for some $a_i,b_i\in\mathbb{C}^*$. Set $u_i=\log a_i$ and $v_i=\log b_i$. If $\rho_u(m_i)$ and $\rho_u(l_i)$ are parabolic, we set $u_i=v_i=0$.

Theorem (Thurston [14], Neumann-Zagier [13]). *The deformation space U of the hyperbolic structure on N has complex dimension h and can be holomorphically parametrized by points $(u_1,\dots,u_h)\in\mathbb{C}^h$ in a neighborhood of $u^0\in U$.*

We will need the following fact in the proof of Theorem 3.1 in Sect. 4. Let P be the subset of U defined by $P=\{u\in U\mid N_u \text{ has at least one cusp}\}$.

Proposition 2.1. *In a neighborhood V of u^0 in U, P is a proper algebraic subset.*

Proof. In a neighborhood V of u^0 in U, the points of V are parametrized by $(u_1,\dots,u_h)$ as in the above theorem, and $P\cap V$ is precisely the set $\{u\in V\mid \text{some } u_i=0\}$. q.e.d.

If $u=(u_1,\dots,u_h)\in V-P$, for each $i=1,\dots,h$, there is a unique pair $(p_i,q_i)\in\mathbb{R}^2\cup\infty$, such that $p_i u_i+q_i v_i=2\pi i$. This collection of pairs is called the generalized Dehn surgery invariant, and if each (p_i,q_i) is a pair of coprime integers, N_u can be completed to a closed hyperbolic manifold denoted by $M_{(p_1,q_1),\dots,(p_h,q_h)}$ by adjoining a closed geodisc γ_i to each end ε_i. Topologically this manifold is obtained from N by performing Dehn surgeries which kill the homotopy classes represented by $p_i m_i+q_i l_i$, $i=1,\dots,h$ ([14], §4).

3. An analytic function on the deformation space and the η-invariant of manifolds obtained by hyperbolic Dehn surgery

The Lie algebra of $PSL_2(\mathbb{C})$, $\mathfrak{g}$, is the complex Lie algebra consisting of all 2×2-complex matrices of trace zero. We regard $\mathfrak{g}$ as the tangent space of

$PSL_2(\mathbb{C})$ at the identity. Let

$$h=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad e=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad f=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Then $\{h, e, f\}$ form a base of $\mathfrak{g}$ over $\mathbb{C}$. Let $\{h^*_{\mathbb{C}}, e^*_{\mathbb{C}}, f^*_{\mathbb{C}}\}$ be its dual base of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$.

Definition 3.1. The complex differential form of degree 3 on $PSL_2(\mathbb{C})$, C, is defined as the left-invariant differential form on $PSL_2(\mathbb{C})$ whose value at the identity is given by $\frac{i}{\pi^2} h^*_{\mathbb{C}} \wedge e^*_{\mathbb{C}} \wedge f^*_{\mathbb{C}}$.

We can easily check that C is a complex analytic form on $PSL_2(\mathbb{C})$ which is closed and bi-invariant. Now $\{h, e, f, ih, ie, if\}$ form a base of $\mathfrak{g}$ over $\mathbb{R}$. Let $\{h^*, e^*, f^*, (ih)^*, (ie)^*, (if)^*\}$ be the dual base of $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$. Let (θ_i) and (θ_{ij}) be the fundamental form and the connection form respectively on $PSL_2(\mathbb{C}) = F(H^3)$ of the Riemannian connection of H^3. Then θ_i and θ_{ij} are the left-invariant smooth forms on $PSL_2(\mathbb{C})$ whose value at the identity are given by

$$\theta_1 = 2h^*, \qquad \theta_2 = e^* + f^*, \qquad \theta_3 = (ie)^* - (if)^*$$
$$\theta_{23} = -2(ih)^*, \qquad \theta_{13} = (ie)^* + (if)^*, \qquad \theta_{12} = e^* - f^*.$$

From these, we have

$$h^*_{\mathbb{C}} = \tfrac{1}{2}(\theta_1 - i\theta_{23}),$$
$$e^*_{\mathbb{C}} = \tfrac{1}{2}((\theta_2 + \theta_{12}) + i(\theta_3 + \theta_{13})),$$
$$f^*_{\mathbb{C}} = \tfrac{1}{2}((\theta_2 - \theta_{12}) - i(\theta_3 - \theta_{13})).$$

We have

Lemma 3.1. *C is written as*

$$C = \frac{1}{4\pi^2}(4\theta_1 \wedge \theta_2 \wedge \theta_3 - d(\theta_1 \wedge \theta_{23} + \theta_2 \wedge \theta_{31} + \theta_3 \wedge \theta_{12}))$$
$$+ \frac{i}{4\pi^2}(\theta_{12} \wedge \theta_{13} \wedge \theta_{23} - \theta_{12} \wedge \theta_1 \wedge \theta_2 - \theta_{13} \wedge \theta_1 \wedge \theta_3 - \theta_{23} \wedge \theta_2 \wedge \theta_3).$$

Since H^3 has the constant sectional curvature -1, $\Omega_{ij} = -\theta_i \wedge \theta_j$ $(i, j = 1, 2, 3)$. Thus C is a complex analytic form on $PSL_2(\mathbb{C})$ whose real part is the volume form plus an exact form (up to scalar multiplication) and whose imaginary part is the Chern-Simons form Q.

Let M be an oriented smooth hyperbolic manifold (complete or incomplete) of dimension 3. Let $F(M)$ be its $SO(3)$ oriented frame bundle. Choose a base point $x_0 \in M$ and set $\Gamma = \pi_1(M, x_0)$. Let $\tilde{M}$ be the universal cover of M. Let d: $\tilde{M} \to H^3$ be a developing map. Let ρ: $\Gamma \to PSL_2(\mathbb{C})$ be the holonomy representation defined by $d(g\tilde{x}) = \rho(g) d(\tilde{x})$, where $g \in \Gamma$ and $\tilde{x} \in \tilde{M}$. Taking the differential of d, we obtain the $SO(3)$-bundle map $\tilde{d}$: $F(\tilde{M}) \to PSL_2(\mathbb{C})$. Since the form C is left invariant, $\tilde{d}^* C$ projects to a closed form on $F(M) = \Gamma \backslash F(\tilde{M})$ which we denote also by C. Let $\mathscr{F}$ be an orthonormal framing on M and let s: $M \to F(M)$

be the section defined by $\mathscr{F}$. Then $s^* C$ is a complex 3-form on M and $\int_M s^* C = \int_{s(M)} C$ is defined.

We define the notion of a simple framing. As before, $H^3 = \{(c,t) | c \in \mathbb{C}$ and $t > 0\}$. Let γ be an oriented geodesic in H^3 and let $\gamma(-\infty)$ and $\gamma(+\infty)$ be its initial and terminal endpoints in ∂H^3 respectively. There is an element $g \in PSL_2(\mathbb{C})$ such that $g\gamma = t$-axis and $g(\gamma(-\infty)) = 0$ and $g(\gamma(+\infty)) = \infty$.

Let (r, β, ϕ) be the polar coordinate of H^3 defined by

$c = r(\sin\beta)(\cos\phi + i\sin\phi)$, $t = r\cos\beta$, where $r > 0$, $0 \leqq \beta < \pi/2$ and $0 \leqq \phi < 2\pi$.

Let $\mathscr{F}(t\text{-axis}) = (e_1, e_2, e_3)$ be the framing defined on $H^3 - \{t\text{-axis}\}$ by

$$e_1 = r(\cos\beta)\,\partial/\partial r, \qquad e_2 = -(\cot\beta)\,\partial/\partial\phi, \qquad e_3 = (\cos\beta)\,\partial/\partial\beta$$

for a point (r, β, ϕ). We define the framing $\mathscr{F}(\gamma)$ on $H^3 - \gamma$ by $\mathscr{F}(\gamma) = (g^{-1})_* \mathscr{F}(t\text{-axis})$. Then $\mathscr{F}(\gamma)$ does not depend on the choice of g. It can be seen that $\mathscr{F}(\gamma)$ is invariant under the action of the subgroup of $PSL_2(\mathbb{C})$ which leaves γ invariant. Next let w be a point of ∂H^3. There is an element $g \in PSL_2(\mathbb{C})$ such that $gw = \infty$. Let $\mathscr{F}(\infty) = (e_1, e_2, e_3)$ be the framing on H^3 defined by

$$e_1 = t(\partial/\partial x), \qquad e_2 = -t(\partial/\partial y), \qquad e_3 = -t(\partial/\partial t),$$

for a point $(c, t) = (x + iy, t) \in H^3$. We define the framing $\mathscr{F}(w)$ on H^3 by $\mathscr{F}(w) = (g^{-1})_* \mathscr{F}(\infty)$. In this case, the definition of $\mathscr{F}(w)$ includes an ambiguity of rotations about the e_3-vectors. However $\{e_1\}$ and $\{e_2\}$ form two parallel vector fields on each horosphere with center w with respect to the euclidean structure on it. It can be seen that $\mathscr{F}(w)$ is invariant up to rotations about its e_3-vectors under the action of the subgroup of $PSL_2(\mathbb{C})$ which leaves w fixed. We call each of the framings $\mathscr{F}(\gamma)$ and $\mathscr{F}(w)$ a simple framing.

Lemma 3.2. *For a simple framing $\mathscr{F}(\gamma)$ (resp. $\mathscr{F}(w)$), let s: $H^3 - \gamma$ (resp. H^3) $\to PSL_2(\mathbb{C})$ be the section defined by it. Then $s^* C = 0$ (pointwise).*

Proof. Since C is left invariant, it suffices to show the lemma for $\mathscr{F}(t\text{-axis})$ and $\mathscr{F}(\infty)$. For $\mathscr{F}(t\text{-axis})$, using the above polar coordinate (r, β, ϕ),

$$s^*\theta_1 = (1/\cos\beta)\,d\log r, \qquad s^*\theta_2 = (-\tan\beta)\,d\phi, \qquad s^*\theta_3 = (1/\cos\beta)\,d\beta,$$
$$s^*\theta_{12} = 0, \qquad s^*\theta_{13} = (\tan\beta)\,d\log r, \qquad s^*\theta_{23} = (-1/\cos\beta)\,d\phi.$$

For $\mathscr{F}(\infty)$, using the coordinate $(x + iy, t)$,

$$s^*\theta_1 = (1/t)\,dx, \qquad s^*\theta_2 = (-1/t)\,dy, \qquad s^*\theta_3 = (-1/t)\,dt,$$
$$s^*\theta_{12} = 0, \qquad s^*\theta_{13} = (1/t)\,dx, \qquad s^*\theta_{23} = (-1/t)\,dy.$$

In both cases, we obtain $s^* C = 0$. q.e.d.

Lemma 3.3. *Let $\mathscr{F}(\gamma) = (e_1, e_2, e_3)$ be a simple framing on $H^3 - \gamma$. Let v: $H^3 - \gamma \to \mathbb{R}/2\pi\mathbb{Z}$ be a smooth map. Define a new framing on $H^3 - \gamma$, $\mathscr{F}'(\gamma) = (e'_1, e'_2, e'_3)$, by*

$$e'_1(x)=(\cos v(x))\,e_1(x)+(\sin v(x))\,e_2(x)$$
$$e'_2(x)=-(\sin v(x))\,e_1(x)+(\cos v(x))\,e_2(x)$$
$$e'_3(x)=e_3(x)$$

for $x\in H^3-\gamma$. *Let* s': $H^3-\gamma\to PSL_2(\mathbb{C})$ *be the section defined by* $\mathscr{F}'(\gamma)$. *Then* $s'^*C=0$.

The same result holds for a simple framing $\mathscr{F}(w)$.

Proof. Let s: $H^3-\gamma\to PSL_2(\mathbb{C})$ be the section defined by $\mathscr{F}(\gamma)$. Then the following relations hold,

$$s'^*\theta_1=(\cos v)\,s^*\theta_1+(\sin v)\,s^*\theta_2,$$
$$s'^*\theta_2=-(\sin v)\,s^*\theta_1+(\cos v)\,s^*\theta_2,$$
$$s'^*\theta_3=s^*\theta_3,$$
$$s'^*\theta_{12}=s^*\theta_{12}-dv,$$
$$s'^*\theta_{13}=(\cos v)\,s^*\theta_{13}+(\sin v)\,s^*\theta_{23},$$
$$s'^*\theta_{23}=-(\sin v)\,s^*\theta_{13}+(\cos v)\,s^*\theta_{23}.$$

We have

$$s'^*C=s^*C-dv\wedge d(s^*\theta_3)-i\,dv\wedge d(s^*\theta_{12}).$$

By Lemma 3.2 and its proof, $s^*C=d(s^*\theta_3)=s^*\theta_{12}=0$. q.e.d.

Let N be an oriented complete hyperbolic 3-manifold of finite volume with h cusps ($h\geqq 1$). We assume that N has an ideal triangulation, $N=\Delta(z_1)\cup\ldots\cup\Delta(z_n)$. N has h ends $\varepsilon_1,\ldots,\varepsilon_h$ and we set $\varepsilon=\varepsilon_1\cup\ldots\cup\varepsilon_h$. Take a base point x_0 in N and a point x_i in a torus section of ε_i ($i=1,\ldots,h$). Let $q_i(t)$ ($0\leqq t\leqq 1$) be a path in N with $q_i(0)=x_0$ and $q_i(1)=x_i$. Set $\Gamma=\pi_1(N,x_0)$ and $\Gamma_i=(q_i)_\#\,\pi_1(T^2,x_i)$, where $(q_i)_\#$ is the homomorphism: $\pi_1(T^2,x_i)\to\pi_1(N,x_0)$ induced by q_i.

Let U be the deformation space of the hyperbolic structure on N. Let P be the subset of U defined by $P=\{u\in U\mid N_u$ has at least one cusp$\}$. For $u\in U$, let ρ_u: $\Gamma\to PSL_2(\mathbb{C})$ be a holonomy representation. If $u\in U-P$, then for each $i=1,\ldots,h$, $\rho_u(\Gamma_i)$ is an abelian subgroup of $PSL_2(\mathbb{C})$ consisting of loxodromic elements and it leaves a unique geodesic in H^3 invariant. We orient this geodesic (arbitrarily) and denote it by γ_i. For $\delta>0$, we set $E_\delta(\gamma_i)=\{x\in H^3\mid 0<d(x,\gamma_i)\leqq\delta\}$ and $T_\delta(\gamma_i)=\{x\in H^3\mid d(x,\gamma_i)=\delta\}$, where d denotes the hyperbolic distance. Let $\widetilde{E_\delta(\gamma_i)}$ and $\widetilde{T_\delta(\gamma_i)}$ be the universal cover of $E_\delta(\gamma_i)$ and $T_\delta(\gamma_i)$ respectively. Then the action of $\rho_u(\Gamma_i)$ on $E_\delta(\gamma_i)$ (resp. $T_\delta(\gamma_i)$) is covered by a free action of Γ_i on $\widetilde{E_\delta(\gamma_i)}$ (resp. $\widetilde{T_\delta(\gamma_i)}$). The induced metric on $\widetilde{T_\delta(\gamma_i)}$ from the hyperbolic metric on H^3 lifts to a metric on $\widetilde{T_\delta(\gamma_i)}$ and it gives $\widetilde{T_\delta(\gamma_i)}$ the structure of the euclidean plane. The free action of Γ_i on $T_\delta(\gamma_i)$ is generated by parallel translations by two independent vectors of this euclidean plane [14]. We set

$$E_\delta(\varepsilon_i)=\Gamma_i\backslash\widetilde{E_\delta(\gamma_i)}$$

and

$$T_\delta(\varepsilon_i)=\Gamma_i\backslash\widetilde{T_\delta(\gamma_i)}.$$

For any sufficiently small $\delta>0$, $E_\delta(\varepsilon_i)$ gives a neighborhood of the end ε_i of N_u which is diffeomorphic to $T^2\times[0,\infty)$ and $T_\delta(\varepsilon_i)$ gives a torus section of ε_i. Since the simple framing $\mathscr{F}(\gamma_i)$ is invariant under the action of $\rho_u(\Gamma_i)$, it defines a framing on $E_\delta(\varepsilon_i)$ which we denote by $\mathscr{F}(\varepsilon_i)$. We call $E_\delta(\varepsilon_i)$ the δ-neighborhood of ε_i, $T_\delta(\varepsilon_i)$ the δ-torus section of ε_i and $\mathscr{F}(\varepsilon_i)$ a simple framing on $E_\delta(\varepsilon_i)$.

If $u\in P$, then some of the ends of N_u are cusps. If the i-th end is a cusp, $\rho_u(\Gamma_i)$ is a free abelian subgroup of rank 2 of $PSL_2(\mathbb{C})$ consisting of parabolic elements. It fixes a unique point $w\in\partial H^3$. $\rho_u(\Gamma_i)$ acts freely on each horoball neighborhood of w, and for a sufficiently small horoball neighborhood $E(w)$ of w, the orbit space $\rho_u(\Gamma_i)\backslash E(w)$ gives a neighborhood of the cusp ε_i. We denote $\rho_u(\Gamma_i)\backslash E(w)$ by $E(\varepsilon_i)$. Then $E(\varepsilon_i)$ is a flat torus section of ε_i. Since $\rho_u(\Gamma_i)$ acts on each horosphere with center w by parallel translations with respect to the euclidean structure on it, any simple framing $\mathscr{F}(w)$ is invariant under the action of $\rho_u(\Gamma_i)$ and it defines a framing $\mathscr{F}(\varepsilon_i)$ on $E(\varepsilon_i)$. We call this framing on $E(\varepsilon_i)$ a simple framing.

Proposition 3.1. *There are a link m in N (possibly empty) and an orthonormal framing $\mathscr{F}$ defined on $N-m$ such that $\mathscr{F}$ is a simple framing on a neighborhood of the ends (=cusps) ε of N and $\mathscr{F}$ has a special singularity at m. The link m can be taken so that m is contained in an arbitrarily small neighborhood of the cusps and each cusp contains at most one connected component of m.*

Proof. Let $E=E_1\cup\ldots\cup E_h$ be a neighborhood of the cusps $\varepsilon=\bigcup\varepsilon_i$ such that $E_i=E(\varepsilon_i)$ is an orbit space of a horoball by the action of Γ_i, and $E_i\cap E_j=\phi$ if $i\neq j$ for $i,j=1,\ldots,h$. Set $T=\partial E$. Then T consists of h disjoint 2-tori, $T=T_1\cup\ldots\cup T_h$, where $T_i=\partial E_i$. We choose arbitrarily a family of simple framings $\mathscr{F}(\varepsilon)=\{\mathscr{F}(\varepsilon_i)\}$ on E $(i=1,\ldots,h)$. We consider the obstruction to extending $\mathscr{F}(\varepsilon)$ to an orthonormal framing on N. This is a purely topological problem. By definition, any two simple framings on E are homotopic by rotations about e_3-vectors. Hence the obstruction is independent of the choice of simple framings $\mathscr{F}(\varepsilon)$. Set $N_0=N-\mathring{E}$. Then N_0 is a 3-manifold with boundary $\partial N_0=T$. The only obstruction to extending $\mathscr{F}(\varepsilon)$ over N is characterized by an element o of $H^2(N,\ E,\ \pi_1(SO(3))=H^2(N_0,T,\mathbb{Z}_2)$. Consider the following commutative diagram,

$$\begin{array}{ccccc}
H^1(T,\mathbb{Z}_2) & \xrightarrow{\delta^*} & H^2(N_0,T,\mathbb{Z}_2) & \xrightarrow{k^*} & H^2(N_0,\mathbb{Z}_2) \\
\downarrow D & & \downarrow D & & \downarrow D \\
H_1(T,\mathbb{Z}_2) & \xrightarrow{j_*} & H_1(N_0,\mathbb{Z}_2) & \longrightarrow & H_1(N_0,T,\mathbb{Z}_2),
\end{array}$$

where the upper (resp. lower) row is the cohomology (resp. homology) exact sequence of the pair (N_0,T) with $\mathbb{Z}_2$-coefficients, D denotes the Poincaré duality isomorphisms and j and k are the inclusions. The class $k^* o$ represents the obstruction to putting a framing on N_0, whence $k^* o = 0$. By exactness $o = k^* o'$ for some $o' \in H^1(T, \mathbb{Z}_2)$. Put $\bar{o} = D o' \in H_1(T, \mathbb{Z}_2) = \bigoplus_{i=1}^{h} H_1(T_i, \mathbb{Z}_2)$. $\bar{o}$ is written as $\bar{o} = \Sigma \bar{o}_i$, $\bar{o}_i \in H_1(T_i, \mathbb{Z}_2)$. If $\bar{o}_i \neq 0$, $\bar{o}_i$ is represented by a simple closed curve m'_i on T_i. Set $m' = \bigcup m'_i$ if some $\bar{o}_i \neq 0$ and $m' = \phi$ if $\bar{o} = 0$. We identify E with $T \times [0, \infty)$, where $T \times 0 = T$. Then m' lies on $T \times 0$. For each i such that $m'_i \neq \phi$, let m_i be a simple closed curve on $T_i \times 1 \subset E_i$ which is isotopic to m'_i in E_i. Set $m = U m_i$. It can be seen that there is a framing $\mathscr{F}_1$ on $E - m$ such that $\mathscr{F}_1 = \mathscr{F}(\varepsilon)$ on $T \times [2, \infty) \subset E$ and $\mathscr{F}_1$ has a special singularity at m. We compare $\mathscr{F}_1$ and $\mathscr{F}(\varepsilon)$ on $T \times 0 = T$. The difference between these two framings on T defines a map from T to $SO(3)$, and its homotopy class is represented by an element $f = \Sigma f_i \in H^1(T, \mathbb{Z}_2) = \oplus H^1(T_i, \mathbb{Z}_2)$. If $\bar{o}_i = 0$, we may assume that $\mathscr{F}_1 = \mathscr{F}(\varepsilon)$ on E_i, whence $f_i = 0$. If $\bar{o}_i \neq 0$, let l'_i be a simple closed curve on $T_i = T_i \times 0$ such that $l'_i \cap m'_i =$ one point. The pair (m'_i, l'_i) forms a base of $H_1(T_i, \mathbb{Z}_2)$ and f_i is characterized by its values on them. The curve m'_i is isotopic to a curve in $T_i \times 2 \subset E_i$ by an isotopy in $E_i - m$. On $T_i \times 2$, $\mathscr{F}_1 = \mathscr{F}(\varepsilon)$ and it follows that $f_i([m'_i]) = 0$, where $[m'_i]$ is the homology class in $H_1(T_i, \mathbb{Z}_2)$ represented by m'_i. Modulo 1-chains in $T_i \times [2, \infty)$, the curve l'_i is homologous in $E_i - m$ to a curve which is the boundary of a small 2-disc in E_i intersecting with m_i at exactly one point. Since $\mathscr{F}_1 = \mathscr{F}(\varepsilon)$ on $T_i \times [2, \infty)$ and $\mathscr{F}_1$ has a special singularity at m, we have $f_i([l'_i]) \neq 0$ in $\mathbb{Z}_2$. Therefore the Poincaré dual of f_i is represented by m'_i and that of $f = \Sigma f_i$ by $m' = \bigcup m'_i$. On the other hand, the obstruction to extending $\mathscr{F}(\varepsilon)$ over N is also represented by m'. It follows that $\mathscr{F}_1$ extends to a framing $\mathscr{F}$ on $N - m$, and $\mathscr{F}$ has the desired properties. q.e.d.

Proposition 3.2. *Let m be a link in N given in Proposition* 3.1. *Then there is a family of orthonormal framings $\{\mathscr{F}_u\}_{u \in U}$ such that, for each $u \in U$,* (i) *$\mathscr{F}_u$ is defined on $N_u - m$ and it has a special singularity at m,* (ii) *$\mathscr{F}_u$ is a simple framing $\mathscr{F}(\varepsilon)$ on a neighborhood of the end of N_u and* (iii) *$\{\mathscr{F}_u\}_{u \in U - P}$ depends on $u \in U - P$ in a smooth manner and the integral $\int_{s(N_u - m)} C$ defines a smooth function on U, where $s\colon N_u - m \to F(N_u)$ is the section defined by $\mathscr{F}_u$.*

Proof. Let $\mathscr{F}$ be an orthonormal framing on $N - m$ which is a simple framing on a neighborhood of the cusps ε and has a special singularity at m. For $u \in U$, N_u has a different hyperbolic structure from N. By Schmidt's orthonormalization, orthonormalizing $\mathscr{F}$ with respect to this new hyperbolic structure, we obtain an orthonormal framing $\mathscr{F}'_u$ on $N_u - m$. Near m, $\mathscr{F}'_u$ is homotopic to a framing having a special singularity at m. By deforming in a neighborhood of m if necessary, we may assume that $\mathscr{F}'_u$ has a special singularity at m and the family of framings $\{\mathscr{F}'_u\}_{u \in U}$ depends on u in a smooth manner. Let $s'\colon N_u - m \to F(N_u)$ be the section defined by $\mathscr{F}'_u$. Then the integral $\int_{s'(N_u - m)} C$ defines a complex-valued smooth function on U. Let $u \in U - P$. Each flat torus section of a cusp of N gives a δ-torus section of the corresponding end of N_u for some $\delta > 0$. Hence the e_3-vectors of $\mathscr{F}'_u$ coincide with the e_3-vectors of a simple framing $\mathscr{F}(\varepsilon)$ on a neighborhood of that end. Therefore deforming $\mathscr{F}'_u$ by rotations about its e_3-vectors in a small neighborhood of the end of N_u, we obtain an orthonormal framing $\mathscr{F}_u$ such that $\mathscr{F}_u$ is a simple framing $\mathscr{F}(\varepsilon)$ on a neighborhood of the end of N_u and it has a special singularity at m. These deformations can be carried out in a smooth manner with respect to $u \in U - P$. By Lemma 3.3, these deformations have no affect on the integrals of C. Thus $\int_{s'(N_u - m)} C = \int_{s(N_u - m)} C$, where s: $N_u - m \to F(N_u)$ is the section defined by $\mathscr{F}_u$. The family of framings $\{\mathscr{F}_u\}_{u \in U}$ gives the desired one. q.e.d.

Let $\{\mathscr{F}_u\}_{u \in U}$ be a family of framings given in Proposition 3.2. Let $\kappa = (f_1, f_2, f_3)$ be an orthonormal framing defined on a subset of N containing m such that $f_1(y)$ is tangent to m at each $y \in m$ and it has the same direction as the e_1-vectors of $\mathscr{F}$ near y. For $u \in U$, we orthonormalize κ with respect to the hyperbolic metric of N_u, and the resulting orthonormal framing is denoted by κ_u. Clearly the first vectors of κ_u are tangent to m at any points of m.

Now we define the function $f(u)$.

Definition 3.2. Let $\{\mathscr{F}_u\}_{u \in U}$ and $\{\kappa_u\}_{u \in U}$ be as above. We define the complex valued function on U by

$$f(u; \mathscr{F}_u, \kappa_u) = \int_{s(N_u - m)} C - \frac{1}{2\pi} \int_{s(m)} (\theta_1 - i\,\theta_{23}),$$

where s: $N_u - m \to F(N_u)$ and s: $m \to F(N_u)$ are the sections defined by $\mathscr{F}_u$ and κ_u respectively. If $\mathscr{F}_u$ and κ_u are prescribed, we write $f(u; \mathscr{F}_u, \kappa_u)$ simply as $f(u)$.

The following theorem will be proved in the next section.

Theorem 3.1. *For each prescribed family* $\{\mathscr{F}_u, \kappa_u\}$, $f(u)$ *is a complex analytic function on a neighborhood* V *of* u^0 *in* U, *where* u^0 *represents the original complete hyperbolic structure on* N.

Let $u \in U$ be a point such that N_u can be completed to a closed hyperbolic manifold M_u by adjoining h closed geodesics $\{\gamma_i\}$ to the ends $\{\varepsilon_i\}$ of N_u. Set $\gamma = \bigcup \gamma_i$ and $L = \gamma \cup m$. Then L is a link in M_u and $M_u - L = N_u - m$. The section s: $N_u - m \to F(N_u)$ defined by $\mathscr{F}_u$ is considered as the section s: $M_u - L \to F(M_u)$. Note that $\mathscr{F}_u$ is an orthonormal framing on $M_u - L$ having a special singularity at L by conditions (i) and (ii) of Proposition 3.2. Let $\overline{M}_u$ be the closure of $s(M_u - L)$ in $F(M_u)$. Then $\overline{M}_u$ is a 3-manifold with boundary $\partial \overline{M}_u = S \cup R$, where S and R are mapped onto γ and m respectively by the bundle projection $F(M_u) \to M_u$. Let $\alpha = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$ be an orthonormal framing on a subset of M_u containing γ such that $e_1(y)$ is tangent to γ at each $y \in \gamma$ and it has the same direction as the first vectors of $\mathscr{F}_u$ near y. Then α and κ_u define the diffeomorphisms, ψ_α: $S^1 \times \gamma \to S$ and ψ_κ: $S^1 \times m \to R$ by

$$\psi_\alpha(v, y) = (\bar{e}_1(y), (\cos v)\,\bar{e}_2(y) - (\sin v)\,\bar{e}_3(y), \quad (\sin v)\,\bar{e}_2(y) + (\cos v)\,\bar{e}_3(y))$$

and

$$\psi_\kappa(v, y) = (f_1(y), (\cos v) f_2(y) - (\sin v) f_3(y), \quad (\sin v) f_2(y) + (\cos v) f_3(y))$$

for $v \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $y \in \gamma \cup m$. Set $\psi = \psi_\alpha \cup \psi_\kappa$: $S^1 \times L \to S \cup R$. Let q: $S^1 \times L \to L$ be the projection onto the second factor. Let s: $L \to F(M_u)$ be the section defined by α and κ_u. Then as in the proof of Lemma 1.5, we have $\psi^*\theta_i(\partial/\partial v)=\psi^*\theta_{i1}(\partial/\partial v)=0$ $(i=1,2,3)$ and $\psi^*\theta_{23}(\partial/\partial v)=1$. It follows that $\psi^*(\theta_2\wedge\theta_{31})=\psi^*(\theta_3\wedge\theta_{12})=0$ and $\psi^*(\theta_1\wedge\theta_{23})=q^*s^*\theta_1\wedge(dv+q^*s^*\theta_{23})$ $=q^*s^*\theta_1\wedge dv$. The orientations of $S^1\times\gamma$ and $S^1\times m$ are given by $(e_1,-\partial/\partial v)$ and $(f_1,-\partial/\partial v)$ respectively. By calculations similar to those in the proof of Lemma 1.5, we have

$$\begin{aligned}\operatorname{Re} f(u)&=\frac{1}{\pi^2}\int_{M_u}\theta_1\wedge\theta_2\wedge\theta_3-\frac{1}{4\pi^2}\int_{S^1\times L}\psi^*(\theta_1\wedge\theta_{23})-\frac{1}{2\pi}\int_{s(m)}\theta_1\\&=\frac{1}{\pi^2}\operatorname{vol}(M_u)+\frac{1}{2\pi}\int_{s(\gamma)}\theta_1+\frac{1}{2\pi}\int_{s(m)}\theta_1-\frac{1}{2\pi}\int_{s(m)}\theta_1\\&=\frac{1}{\pi^2}\operatorname{vol}(M_u)+\frac{1}{2\pi}\int_{s(\gamma)}\theta_1\\&=\frac{1}{\pi^2}\operatorname{vol}(M_u)+\frac{1}{2\pi}\sum_i\operatorname{length}(\gamma_i).\end{aligned}$$

Next we consider $\operatorname{Im} f(u)$. Following [10], the Chern-Simons invariant $CS(M_u)$ of M_u is defined by

$$CS(M_u)=\int_{s(M_u-L)}\tfrac{1}{2}Q-\frac{1}{4\pi}(\tau(\gamma,\alpha)+\tau(m,\kappa_u)) \bmod \tfrac{1}{2}\mathbb{Z},$$

where $s\colon M_u-L\to F(M_u)$ is the section defined by $\mathcal{F}_u$. The fact that this is actually an invariant of M_u follows from Corollary 1.1 in Sect. 1. Using this invariant we have

$$\begin{aligned}\operatorname{Im} f(u)&=\int_{s(M_u-L)}Q-\frac{1}{2\pi}\tau(m,\kappa_u)\\&=2\,CS(M_u)+\frac{1}{2\pi}\tau(\gamma,\alpha) \bmod \mathbb{Z}\\&=2\,CS(M_u)+\frac{1}{2\pi}\sum_i\operatorname{torsion}(\gamma_i) \bmod \mathbb{Z}.\end{aligned}$$

We set $F(u)=\exp(2\pi f(u))$. Then $F(u)$ is a complex analytic function on a neighborhood V of u^0 in U by Theorem 3.1. For $u\in U$ such that N_u can be completed to a closed hyperbolic manifold M_u by adjoining h simple closed geodesics $\{\gamma_i\}$ to the h ends of N_u,

$$F(u)=\exp\left(\frac{2}{\pi}\operatorname{vol}(M_u)+i4\pi\,CS(M_u)\right)\prod_{i=1}^{h}\exp(\operatorname{length}(\gamma_i)+i\operatorname{torsion}(\gamma_i)).$$

This proves Theorem 2 in Introduction.

Remark. These computations show that $F(u)$ is independent of the choices of $\{\mathcal{F}_u\}$, $\{\kappa_u\}$ and m; in particular, $f(u)$ differs only by an integral multiple of i when different choices of them are made. However this can be proved directly using the closedness of C.

The η-invariant of M_u.

We consider $\eta(M_u)$, where $u \in V$ on which $f(u)$ is analytic and N_u is completed to the closed hyperbolic manifold M_u. Set $L = \gamma \cup m$ as above. Let $\alpha_u = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$ be an orthonormal framing on M_u such that $\bar{e}_1(y)$ is tangent to L at each $y \in L$ and it has the same direction as the first vectors of $\mathscr{F}_u$ near y. As noted before, $\mathscr{F}_u$ is an orthonormal framing on $M_u - L$ having a special singularity at L. Applying Theorem 1 to this case, we obtain

Theorem 3.2.

$$\eta(M_u) = \tfrac{1}{3} \operatorname{Im} f(u) - \frac{1}{6\pi} \tau(\gamma, \alpha_u) + \tfrac{2}{3} d(\mathscr{F}_u, \alpha_u) + \delta(M_u, \alpha_u)$$

$$+ \frac{1}{6\pi} (\tau(m, \kappa_u) - \tau(m, \alpha_u)),$$

where $f(u)$ is the analytic function on a neighborhood V of u^0 in U defined in Definition 3.2 using the framings $\{\mathscr{F}_u\}$ and $\{\kappa_u\}$, and $u \in V$.

4. Proof of Theorem 3.1

Let $N = \Delta(z_1^0) \cup \ldots \cup \Delta(z_n^0)$ be the ideal triangulation of the complete hyperbolic manifold N. For each point $u = (z_1, \ldots, z_n)$ of the deformation space U of the hyperbolic structure on N, let $N_u = \Delta(z_1) \cup \ldots \cup \Delta(z_n)$ be the ideal triangulation of N_u. For each $u \in U$, starting at the ideal tetrahedron in H^3 with vertices $\{0, 1, z_1, \infty\}$, by analytic continuation, we obtain a developing map $d_u \colon \tilde{N}_u \to H^3$, where $\tilde{N}_u$ is the universal cover of N_u. Thus we obtain a family of the developing maps $\{d_u\}_{u \in U}$ which depends smoothly on u.

Let $\rho_u \colon \Gamma = \pi_1(N, x_0) \to PSL_2(\mathbb{C})$ be the holonomy representation defined by $d_u(g\tilde{x}) = \rho_u(g) d_u(\tilde{x})$, where $\tilde{x} \in \tilde{N}_u$ and $g \in \Gamma$.

$\tilde{N}_u$ has the ideal triangulation consisting of the infinite ideal tetrahedra each of which is a lift of some $\Delta(z_i)$, $i = 1, \ldots, n$. Let 0 be the set of all the vertices of those infinite ideal tetrahedra. Then we can define the image set $d_u(0)$ in $\partial H^3 = \mathbb{C} \cup \infty$. $d_u(0)$ is a subset of ∂H^3 consisting of all the points each of which is a fixed point of the subgroup $\rho_u(g \Gamma_i g^{-1})$ for some $g \in \Gamma$ and $\Gamma_i = \pi_1(\varepsilon_i)$ $(i = 1, \ldots, h)$. For each $o \in 0$, the coordinate of $d_u(o)$ in $\mathbb{C} \cup \infty$ can be written as a meromorphic function of $(z_1, \ldots, z_n)$. Hence it gives an analytic map from U to the Riemann sphere $\mathbb{C} \cup \infty$.

Lemma 4.1. *Let $\{o_1, o_2, o_3\}$ be a set of three distinct points in 0. Let $a \in U$ and let U_a be a neighborhood of a in U. Assume that $d_u(o_i) \neq d_u(o_j)$ $(i \neq j,\ i, j = 1, 2, 3)$ for each $u \in U_a$. Then for each $u \in U_a$, there is a unique element $g(u) \in PSL_2(\mathbb{C})$ such that $g(u) d_a(o_i) = d_u(o_i)$ $(i = 1, 2, 3)$ and the map $U_a \ni u \to g(u) \in PSL_2(\mathbb{C})$ is an analytic map.*

Proof. If we represent $g(u)$ by a matrix in $SL_2(\mathbb{C})$, the components of the matrix are given by the solutions of linear equations whose coefficients are polynomials of the coordinates of $d_a(o_i)$ and $d_u(o_i)$ in $\mathbb{C} \cup \infty$ $(i = 1, 2, 3)$. From this, the lemma follows. q.e.d.

For each subset A of N_u, we denote by $\tilde{A}$ the inverse image of A of the covering map $\tilde{N}_u \to N_u$. Let m be a link in N given in Proposition 3.1 and let $\{\mathcal{F}_u\}_{u\in U}$ be a family of framings given in Proposition 3.2. Then the lift of $\mathcal{F}_u$ to $\tilde{N}_u - \tilde{m}$ defines the map $\tilde{s}$: $\tilde{N}_u - \tilde{m} \to PSL_2(\mathbb{C})$ such that the following diagram commutes

$$\begin{array}{ccc} \tilde{N}_u - \tilde{m} & \xrightarrow{\tilde{s}} & PSL_2(\mathbb{C}) \\ & \searrow^{d_u} & \downarrow \\ & & H^3 \end{array}$$

where d_u is the restriction of the developing map and the right vertical map is the bundle projection of $PSL_2(\mathbb{C}) = F(H^3)$ as in Sect. 2.

Choose lifts in $\tilde{N}_u$ of the n ideal tetrahedra of N_u, $\tilde{\Delta}(z_1), \ldots, \tilde{\Delta}(z_n)$, so that

$$D_u = \tilde{\Delta}(z_1) \cup \ldots \cup \tilde{\Delta}(z_n)$$

is a connected fundamental domain in $\tilde{N}_u$ with respect to the covering transformation group. Then $D_u \cap \tilde{m}$ consists of finite arcs. Set $D'_u = D_u - D_u \cap \tilde{m}$. By definition, $\int_{s(N_u - m)} C = \int_{D'_u} \tilde{s}^* C$, where s: $N_u - m \to F(N_u)$ is the section defined by $\mathcal{F}_u$.

As in Sect. 2, let P be the subset of U defined by $P = \{u \in U \mid N_u$ has at least one cusp$\}$. By Proposition 2.1, there is a neighborhood V of $u^0 = (z_1^0, \ldots, z_n^0)$ in U such that $P \cap V$ is a proper algebraic subset in V. We prove the analyticity of $f(u)$ on V.

Since $f(u)$ is a smooth function on V and $V \cap P$ is represented as the zero set of some non-trivial analytic functions, by a fundamental theorem of the complex function theory of several variables, it suffices to show that $f(u)$ is analytic on $V - P$ (e.g. [3]). Let us fix an (arbitrary) point $a \in V - P$. Take an open neighborhood of a in $V - P$, V_a, such that V_a is analytically equivalent to the unit ball in $\mathbb{C}^h$ (see Sect. 2) and $\overline{V_a} \subset V - P$. On V_a, we will express $f(u)$ by a path-integral of an analytic closed 1-form.

Let $\{o_k\}_{k=1,\ldots,r}$ be all the vertices of the ideal polyhedron D_u, where $r = \#\{$the vertices of $D_u\}$. For each $u \in V_a$, each vertex o_k is one of the two fixed points of $\rho_u(g_k \Gamma_i g_k^{-1})$ in ∂H^3, which are the end points of the geodesic γ_k in H^3 left invariant by $\rho_u(g_k \Gamma_i g_k^{-1})$, for some g_k and $\Gamma_i = \pi_1(\varepsilon_i)$ $(i = 1, \ldots, h)$. Set for $\delta > 0$,

$$D_\delta(\gamma_k) = \{x \in H^3 \mid d(x, \gamma_k) \leqq \delta\},$$

where d denotes the hyperbolic distance. Since the framing $\mathcal{F}_u$ on $N_u - m$ is a simple framing on a neighborhood of the end of N_u and $\overline{V_a}$ is compact, we may choose $\delta > 0$ to be so small that $\tilde{m} \cap \bigcup_k D_\delta(\gamma_k) = \phi$ and $\tilde{s}(D_u \cap D_\delta(\gamma_k))$ $(\subset PSL_2(\mathbb{C}) = F(H^3))$ is a part of the simple framing $\mathcal{F}(\gamma_k)$ for each $u \in V_a$ and $k = 1, \ldots, r$. Set $D''_u = D_u - \tilde{m} - \bigcup_k \mathring{D}_\delta(\gamma_k) = D'_u - \bigcup_k \mathring{D}_\delta(\gamma_k)$. By Lemma 3.2, $\tilde{s}^* C = 0$ on $\bigcup_k D_\delta(\gamma_k)$ and we have $\int_{D'_u} \tilde{s}^* C = \int_{D''_u} \tilde{s}^* C$. The map $\tilde{s}$: $D''_u \to PSL_2(\mathbb{C})$ is an immersion, and by taking the closure of the image $\tilde{s}(D''_u)$ in $PSL_2(\mathbb{C})$, we can compactify D''_u and we denote the resulting compact polyhedron by X_u. Since $\mathscr{F}_u$ has a special singularity at m, by the same argument as in the proof of Lemma 1.2 (using the lift of the framing κ_u on m to $\tilde{m}$), we see that X_u is obtained by attaching a finite union of cylinders diffeomorphic to $S^1 \times (D_u \cap \tilde{m})$ after deleting $D_u \cap \tilde{m}$ from $D_u - \bigcup_k \mathring{D}_\delta(\gamma_k)$.

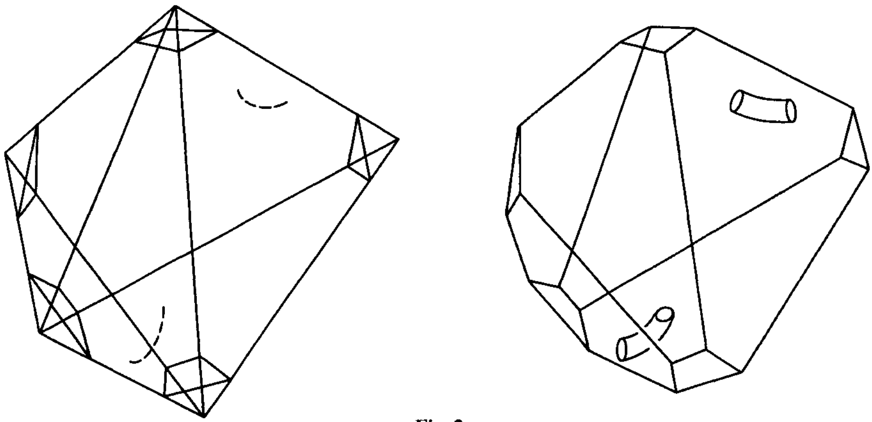

Fig. 2

The immersion $\tilde{s}$ extends naturally to an immersion $\tilde{s}$: $X_u \to PSL_2(\mathbb{C})$ and $\int_{D''_u} \tilde{s}^* C = \int_{X_u} \tilde{s}^* C$. The boundary ∂X_u consists of three parts, $\partial X_u = Y_u \cup Z_u \cup W_u$, where

$$Y_u = \partial X_u - D''_u \text{ (the set of the compactifying points)}$$
$$Z_u = D''_u \cap (\bigcup_k D_\delta(\gamma_k))$$
$$W_u = \partial X_u - (Y_u \cup Z_u).$$

For each 2-face S_u of W_u, there is another 2-face S'_u of W_u such that $S'_u = g S_u$ for some $g \in \Gamma$ uniquely determined by S_u, and W_u consists of finite pairs of such 2-faces $(S_u, g S_u)$. Now we fix $a \in V_a$ as before, and set $X = X_a$, $Y = Y_a$, $Z = Z_a$, $W = W_a$ and $W = \bigcup (S \cup g S)$. We note that the ideal triangulation of N gives a combinatorial triangulation of N and the link m is a combinatorial submanifold of N. If the hyperbolic structure on N varies, the combinatorial properties of N and m remain unchanged. Thus the combinatorial shape of X_u remains unchanged when u varies. Each X_u has the boundary pattern indicated in Fig. 2 and we have a family of diffeomorphisms

$$\{h_u : X \to X_u\}_{u \in V_a}$$

preserving this boundary pattern and sattisfying $h_u(g x) = g h_u(x)$ for $x \in S \subset W$. Define

$$H: V_a \times X \to PSL_2(\mathbb{C})$$

by $H(u, x) = \tilde{s}(h_u(x))$ for $u \in V_a$ and $x \in X$.

We define smooth 1-forms of three kinds on V_a as follows.

(i) $\omega_1(g)$.

Let $g \in \Gamma$. We set

$$\omega_1(g) = (\rho_u(g))^* \frac{1}{2\pi}(\theta_1 - i\,\theta_{23})$$

where we regard $u \to \rho_u(g)$ as a map $V_a \to PSL_2(\mathbb{C})$. $\omega_1(g)$ is an analytic 1-form on V_a.

(ii) $\omega_2(g)$.

Let G: $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ be the multiplication of $PSL_2(\mathbb{C})$, $G(g_1, g_2) = g_1 g_2 (g_1, g_2 \in PSL_2(\mathbb{C}))$. Since C is a bi-invariant form on $PSL_2(\mathbb{C})$, $G^* C$ is written as

$$G^* C = p_1^* C + \omega^{2,1} + \omega^{1,2} + p_2^* C$$

where p_j: $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ is the projection onto the j-th factor $(j=1,2)$ and $\omega^{i,j}$ is an analytic form belonging to $\overset{i}{\wedge} PSL_2(\mathbb{C}) \otimes \overset{j}{\wedge} PSL_2(\mathbb{C})$ $(i,j = 1,2)$.

Let (S, gS) be a pair of 2-faces of W. Let

$$\rho \times H\colon\ V_a \times S \to PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$$

be the map defined by $(\rho \times H)(u,x) = (\rho_u(g),\ H(u,x))$ for $u \in V_a$ and $x \in S$. Then $(\rho \times H)^* \omega^{1,2}$ is a smooth 3-form on $V_a \times S$. By partial integration along S, we define the smooth 1-form on V_a,

$$\omega_2(g) = \int_S (\rho \times H)^* \omega^{1,2}.$$

Let $(u_1, \ldots, u_h)$ be a complex coordinate on V_a. Since the map $V_a \ni u \to \rho_u(g) \in PSL_2(\mathbb{C})$ is analytic, each term in $(\rho \times H)^* \omega^{1,2}$ which involves the factor $d\bar{u}_i$ $(i = 1, \ldots, h)$ in its expression in the coordinate vanishes under the above partial integration. Hence $\omega_2(g)$ is a smooth 1-form on V_a which does not involve $d\bar{u}_i$, that is, it is of type $(1,0)$.

(iii) $\omega_3(\Delta)$.

For $u \in V_a$, let Δ_u be one of the ideal tetrahedra $\{\Delta(z_i)\}$ $(i = 1, \ldots, n)$, where $N_u = \Delta(z_1) \cup \ldots \cup \Delta(z_n)$. Let J_u be one of the four connected components of $Z_u \cap \Delta_u$. Then J_u is a triangle with 3 vertices $\{v_0(u), v_1(u), v_2(u)\}$ each of which is an intersection of J_u with an edge geodesic of the triangulation of X_u. There is a unique geodesic $\gamma(u)$ in H^3 such that $d_u(J_u) \subset T_\delta(\gamma(u)) = \{x \in H^3 \mid d(x, \gamma(u)) = \delta\}$. Moreover for each $j = 0, 1, 2$, there is a unique geodesic $\gamma_j(u)$ in H^3 such that $d_u(v_j(u)) = \gamma_j(u) \cap T_\delta(\gamma(u))$. By assumption, $\tilde{s}(J_u) \subset PSL_2(\mathbb{C}) = F(H^3)$ is a part of the simple framing $\mathscr{F}(\gamma(u))$. We regard $\tilde{s}(v_j(u))$ as a framing at $d_u(v_j(u))$. There is a unique element $g_j(u) \in PSL_2(\mathbb{C})$ such that $\tilde{s}(v_j(u)) = g_j(u)\, \tilde{s}(v_j(a))$ $(j = 0, 1, 2)$.

Lemma 4.2. *The map g_j: $u \to g_j(u)$ is an analytic map from V_a to $PSL_2(\mathbb{C})$ $(j = 0, 1, 2)$.*

Proof. The geodesics $\gamma(u)$ and $\gamma_j(u)$ have a unique common end point $o(u) \in \partial H^3$. Let $\{o(u),\ o'(u)\}$ be the end points of $\gamma(u)$ and let $\{o(u), o''(u)\}$ be those of $\gamma_j(u)$. Then $\{o(u), o'(u), o''(u)\}$ are three distinct points in ∂H^3. Hence there is a unique element $g \in PSL_2(\mathbb{C})$ such that g maps the ordered triple $\{o(a), o'(a), o''(a)\}$ to the ordered triple $\{o(u), o'(u), o''(u)\}$. Then g maps $\gamma(a)$ to $\gamma(u)$, $\gamma_j(a)$ to $\gamma_j(u)$ and $\mathscr{F}(\gamma(a))$ to $\mathscr{F}(\gamma(u))$. Since δ is a constant, it follows that $g = g_j(u)$. By Lemma 4.1, the lemma follows. q.e.d.

There are two elements of $PSL_2(\mathbb{C})$, w_1 and w_2, such that both of them leave $\gamma(a)$ invariant and $\tilde{s}(v_1(a)) = w_1 \tilde{s}(v_0(a))$ and $\tilde{s}(v_2(a)) = w_2 \tilde{s}(v_0(a))$. Set $a_j(u) = g_j(u) w_j g_0(u)^{-1}$ $(j=1,2)$. Then $a_j(u)(\tilde{s}(v_0(u)) = \tilde{s}(v_j(u))$, and $a_j(u)$ leaves $\gamma(u)$ invariant $(j=1,2)$. The subgroup of $PSL_2(\mathbb{C})$ consisting of those elements which leave $\gamma(u)$ invariant is isomorphic to $\mathbb{C}^*$. The restriction of the simple framing $\mathscr{F}(\gamma(u))$ to $T_\delta(\gamma(u))$, $B_u = \mathscr{F}(\gamma(u)) | T_\delta(\gamma(u))$, is an orbit of the $\mathbb{C}^*$ action on $PSL_2(\mathbb{C})$ by the left multiplication of this subgroup. The universal cover $\tilde{B}_u$ of B_u has the $\mathbb{C}$-action which is the lift of the $\mathbb{C}^*$-action on B_u, and this $\mathbb{C}$-action gives an identification of $\tilde{B}_u$ with $\mathbb{C}$. The map $\tilde{s}$ immerses the triangle J_u into B_u, and the immersed image $\tilde{s}(J_u)$ lifts to an embedded triangle $\tilde{J}_u$ in $\tilde{B}_u$. With respect to the identification of $\tilde{B}_u$ with $\mathbb{C}$, $\tilde{J}_u$ is considered as an affine triangle with vertices $\{\tilde{v}_0(u), \tilde{v}_1(u), \tilde{v}_2(u)\}$ where $\tilde{v}_j(u)$ is the lift of $\tilde{s}(v_j(u))$ in B_u to $\tilde{B}_u$ $(j = 0, 1, 2)$. Let (t_0, t_1, t_2) be the barycentric coordinate of $\tilde{J}_u$ with respect to the affine structure on $\tilde{B}_u = \mathbb{C}$, and we write the points of $\tilde{J}_u$ as $\sum t_j \tilde{v}_j(u)$. Identifying J_u with $\tilde{J}_u$ by the embedding, we write the points of J_u as $\sum t_j v_j(u)$. The elements $a_1(u)$ and $a_2(u) \in PSL_2(\mathbb{C})$ are contained in the subgroup which leaves $\gamma(u)$ invariant, hence they are considered as two elements in $\mathbb{C}^*$. Hence we may write as

$$\tilde{s}(\Sigma t_j v_j(u)) = a_1(u)^{t_1} a_2(u)^{t_2} \tilde{s}(v_0(u)),$$

where $a_j(u)^t$ is defined by analytic continuation. Since $\tilde{s}(v_0(u)) = g_0(u) \tilde{s}(v_0(a))$, we have

$$\tilde{s}(\Sigma t_j v_j(u)) = a_1(u)^{t_1} a_2(u)^{t_2} g_0(u) \tilde{s}(v_0(a)).$$

Let J be the euclidean triangle in the real plane $\mathbb{R}^2$ with vertices $\{(0,0), (1,0), (0,1)\}$ and let (t_0, t_1, t_2) be its barycentric coordinate. Define the map

$$A\colon V_a \times J \to PSL_2(\mathbb{C})$$

by

$$A(u, (t_0, t_1, t_2)) = a_1(u)^{t_1} a_2(u)^{t_2} g_0(u).$$

Then A is a smooth map and, for each (t_0, t_1, t_2), the restriction of A to $V_a \times (t_0, t_1, t_2)$ gives an analytic map from V_a to $PSL_2(\mathbb{C})$ by Lemma 4.2.

We define

$$\omega_3(A) = \int_J A^* C,$$

where $\int_J$ means the partial integration along J. Then $\omega_3(A)$ defines an analytic 1-form on V_a.

Using the above smooth 1-forms on V_a of three kinds, we proceed to prove that $f(u)$ is written as a path integral of an analytic closed 1-form on V_a.

Let $u(t)$ $(0 \leqq t \leqq 1)$ be a smooth path in V_a with $u(0) = a$ and $u(1) = u$. Set

$H(t,x)=H(u(t),x)$ for $t\in[0,1]$ and $x\in X$, where $H(u,x)$: $V_a\times X\to PSL_2(\mathbb{C})$ is defined as before. $H=H(t,x)$ is a map from $[0,1]\times X$ to $PSL_2(\mathbb{C})$. Since C is a closed form, by Stokes's theorem, we have

$$\begin{aligned}0&=\int_{[0,1]\times X} dH^*C\\&=\int_{1\times X} H^*C-\int_{0\times X} H^*C+\int_{[0,1]\times\partial X} H^*C.\end{aligned}$$

Since, for $j=0,1$, $H(j,x)=H(u(j),x)=\tilde{s}\,h_{u(j)}(x)$, we have

$$\int_{1\times X} H^*C=\int_{X_u}\tilde{s}^*C \quad\text{and}\quad \int_{0\times X} H^*C=\int_X \tilde{s}^*C.$$

It follows that

$$\int_{X_u}\tilde{s}^*C-\int_X\tilde{s}^*C=-\int_{[0,1]\times\partial X} H^*C. \tag{1}$$

Since $\partial X=Y\cup Z\cup W$ as before, we have

$$\int_{[0,1]\times\partial X} H^*C=\int_{[0,1]\times Y} H^*C+\int_{[0,1]\times Z} H^*C+\int_{[0,1]\times W} H^*C. \tag{2}$$

At first we consider the integral on $[0,1]\times Y$ of H^*C. The framing $\kappa_u=(f_1,f_2,f_3)$ on m in N_u lifts to the framing $\tilde{\kappa}_u=(\tilde{f}_1,\tilde{f}_2,\tilde{f}_3)$ on $\tilde{m}$ in $\tilde{N}_u$. As in the proof of Lemma 1.2, we have the diffeomorphism

$$\psi\colon\ [0,1]\times S^1\times(D_a\cap\tilde{m})\to[0,1]\times Y$$

defined by

$$\begin{aligned}\psi(t,v,y)=\{t\}\times h_{u(t)}^{-1}(&\tilde{f}_1(y),(\cos v)\tilde{f}_2(y)-(\sin v)\tilde{f}_3(y),\\&(\sin v)\tilde{f}_2(y)+(\cos v)\tilde{f}_3(y))\end{aligned}$$

for $t\in[0,1]$, $v\in S^1=\mathbb{R}/2\pi\mathbb{Z}$ and $y\in D_a\cap\tilde{m}$, where we identify $D_a\cap\tilde{m}$ with $D_{u(t)}\cap\tilde{m}$ $(0\leqq t\leqq 1)$, $h_{u(t)}$: $Y\to Y_{u(t)}$ is the diffeomorphism defined as before and $Y_{u(t)}$ is considered as an (immersed) submanifold of $PSL_2(\mathbb{C})$. As in the proof of Lemma 1.5, we have $(\psi^*H^*\theta_i)(\partial/\partial v)=(\psi^*H^*\theta_{i1})(\partial/\partial v)=0$ $(i=1,2,3)$ and $(\psi^*H^*\theta_{23})(\partial/\partial v)=1$. It follows that

$$\psi^*H^*C=-\frac{1}{4\pi^2}\psi^*H^*(d(\theta_1\wedge\theta_{23}))+\frac{i}{4\pi^2}\psi^*H^*(\theta_{23}\wedge d\theta_{23}).$$

The orientation of $[0,1]\times S^1\times(D_a\cap\tilde{m})$ is given by $(\partial/\partial t,\ -\partial/\partial v, \tilde{f}_1)$. By calculations similar to the ones in the proof of Lemma 1.5, using the partial integration along S^1 and Stokes's theorem, we have

$$\begin{aligned}\int_{[0,1]\times Y} H^*C&=\int_{[0,1]\times S^1\times(D_a\cap\tilde{m})}\psi^*H^*C\\&=-\frac{1}{2\pi}\int_{1\times(D_a\cap\tilde{m})} s^*(\theta_1-i\theta_{23})+\frac{1}{2\pi}\int_{0\times(D_a\cap\tilde{m})} s^*(\theta_1-i\theta_{23})\\&\quad+\frac{1}{2\pi}\int_{[0,1]\times\partial(D_a\cap\tilde{m})} s^*(\theta_1-i\theta_{23}),\end{aligned}$$

where s: $[0,1]\times(D_a\cap\tilde{m})\to PSL_2(\mathbb{C})$ is defined by $s(t,y)=\tilde{\kappa}_{u(t)}(y)$ for $t\in[0,1]$ and $y\in D_a\cap\tilde{m}$. It can be easily shown that

$$\int_{1\times(D_a\cap\tilde{m})} s^*(\theta_1-i\theta_{23})=\int_{s(m)}(\theta_1-i\theta_{23}),$$

where s: $m\to F(N_u)$ is the section defined by κ_u, and

$$\int_{0\times(D_a\cap\tilde{m})} s^*(\theta_1-i\theta_{23})=\int_{s(m)}(\theta_1-i\theta_{23}),$$

where s: $m\to F(N_a)$ is the section defined by κ_a.

By definition of $f(u)$ and (1) and (2) above, we obtain

$$f(u)-f(a)=-\int_{[0,1]\times\partial(D_a\cap\tilde{m})} s^*\frac{1}{2\pi}(\theta_1-i\theta_{23}) - \int_{[0,1]\times W} H^*C-\int_{[0,1]\times Z} H^*C. \tag{3}$$

We consider the three integrals of the right hand side of (3) separately.

(i)′ $$\int_{[0,1]\times\partial(D_a\cap\tilde{m})} s^*\frac{1}{2\pi}(\theta_1-i\theta_{23})$$

$D_a\cap\tilde{m}$ consists of finite arcs and $\partial(D_a\cap\tilde{m})$ is a finite union of points. As m is a finite union of disjoint simple closed curves, $\partial(D_a\cap\tilde{m})$ consists of finite pairs of two points (y,gy), where $g\in\Gamma$ is uniquely determined by y. For simplicity, we set $\Phi=\frac{1}{2\pi}(\theta_1-i\theta_{23})$. Since y and gy have mutually opposite orientations in $\partial(D_a\cap\tilde{m})$, we have

$$\int_{[0,1]\times\partial(D_a\cap\tilde{m})} s^*\Phi=\Sigma\,\varepsilon_y\Big(\int_{[0,1]\times gy} s^*\Phi-\int_{[0,1]\times y} s^*\Phi\Big),\qquad (\varepsilon_y=\pm 1),$$

where $[0,1]\times gy$ and $[0,1]\times y$ are both oriented as $[0,1]$ with natural orientation. Since $s(t,y)=\tilde{\kappa}_{u(t)}(y)$ $(0\leqq t\leqq 1)$ and $\tilde{\kappa}_{u(t)}$ is the lift of $\kappa_{u(t)}$ to $\tilde{m}$, we have

$$s(t,gy)=\rho_{u(t)}(g)\,s(t,y).$$

Hence the map $t\to s(t,gy)$ is the composition of the maps

$$[0,1]\xrightarrow{\rho\times s} PSL_2(\mathbb{C})\times PSL_2(\mathbb{C})\xrightarrow{G} PSL_2(\mathbb{C}),$$

where $(\rho\times s)(t)=(\rho_{u(t)}(g),\ s(t,y))$. Since Φ is a bi-invariant 1-form on $PSL_2(\mathbb{C})$, we obtain $G^*\Phi=p_1^*\Phi+p_2^*\Phi$. It follows that on $[0,1]\times gy$,

$$s^*\Phi=(\rho_{u(t)}(g))^*\Phi+s(t,y)^*\Phi.$$

Hence

$$\int_{[0,1]\times gy} s^*\Phi-\int_{[0,1]\times y} s^*\Phi =\int_{[0,1]}(\rho_{u(t)}(g))^*\Phi.$$

Therefore $\int_{[0,1]\times\partial(D_a\cap\tilde{m})} s^*\Phi$ is a sum of path integrals along $u(t)$ of 1-forms of type (i).

(ii)′

$$\int_{[0,1]\times W} H^* C.$$

W is a finite union of the pairs of the 2-faces (S, gS). Choose the orientations of S and gS such that $g\colon S\to gS$ is orientation-preserving. Then the orientation of S induced from X is opposite to that of gS. Hence

$$\int_{[0,1]\times W} H^* C = \sum \varepsilon_S\Big(\int_{[0,1]\times gS} H^* C - \int_{[0,1]\times S} H^* C\Big), \qquad (\varepsilon_S = \pm 1).$$

For each $(t, gx)\in[0,1]\times gS$, $H(t, gx) = \tilde{s}\, h_{u(t)}(gx) = \tilde{s}\, g(h_{u(t)}(x)) = \rho_{u(t)}(g)\, \tilde{s}\, h_{u(t)}(x) = \rho_{u(t)}(g)\, H(t, x)$. Hence $H\colon [0,1]\times gS \to PSL_2(\mathbb{C})$ is considered as the composition of the maps

$$[0,1]\times S \xrightarrow{\rho\times H} PSL_2(\mathbb{C})\times PSL_2(\mathbb{C}) \xrightarrow{G} PSL_2(\mathbb{C}),$$

where $(\rho\times H)(t,x) = (\rho_{u(t)}(g), H(t,x))$. As before, $G^* C = p_1^* C + \omega^{2,1} + \omega^{1,2} + p_2^* C$. For dimensional reasons, $(\rho\times H)^* p_1^* C = 0$ and $(\rho\times H)^*\omega^{2,1} = 0$. Therefore

$$(G(\rho\times H))^* C = (\rho\times H)^*\omega^{1,2} + H^* C.$$

It follows that

$$\begin{aligned}\int_{[0,1]\times gS} H^* C - \int_{[0,1]\times S} H^* C &= \int_{[0,1]\times S} (\rho\times H)^*\omega^{1,2} \\ &= \int_{[0,1]} \Big(\int_S (\rho\times H)^*\omega^{1,2}\Big) \\ &= \int_{[0,1]} \omega_2(g).\end{aligned}$$

This shows that $\int_{[0,1]\times W} H^* C$ is a sum of path integrals along $u(t)$ of 1-forms of type (ii).

(iii)′

$$\int_{[0,1]\times Z} H^* C$$

$Z = Z_a$ is a finite union of the triangles $\{J_a\}$ described in (iii) and

$$\int_{[0,1]\times Z} H^* C = \sum \int_{[0,1]\times J_a} H^* C.$$

Each J_a has the barycentric coordinate as in (iii) and identifying J_a with the triangle J in the real plane $\mathbb{R}^2$ with vertices $\{(0,0), (1,0), (0,1)\}$, H is considered as the composition of the maps

$$u(t)\times 1\colon [0,1]\times J \to V_a\times J$$

and

$$A\colon V_a\times J \to PSL_2(\mathbb{C}),$$

where A is the map defined in (iii). It follows that $\int_{[0,1]\times J_a} H^* C$ is equal to a sum of path integrals along $u(t)$ of 1-forms of type (iii).

By (i)′, (ii)′ and (iii)′ above, we see from (3) that $(f(u)-f(a))$ can be written as a path integral along $u(t)$ of a smooth 1-form ω on V_a which is a finite sum of 1-forms defined in (i), (ii) and (iii). Let $(u_1,\ldots,u_h)$ be a complex coordinate on V_a. Then each 1-form defined in (i), (ii) and (iii) does not involve $d\bar{u}_i$ $(i=1,\ldots,h)$ in its expression, that is, it is of type (1,0). Therefore ω is written as

$$\omega=\sum \omega_i(u_1,\ldots,u_h)\,du_i$$

where $\omega_i(u_1,\ldots,u_h)$ is a smooth function of $(u_1,\ldots,u_h)$ for each $i=1,\ldots,h$. Thus

$$f(u)-f(a)=\int_{u(t)} \sum \omega_i(u_1,\ldots,u_h)\,du_i.$$

The path $u(t)$ may be arbitrarily chosen in V_a and the left-hand side of the above equation depends only on the end point $u=u(1)$. This implies that ω is a closed form. However $d\omega=0$ implies that each ω_i satisfies the Cauchy-Riemann equations at each point of V_a. Hence ω is an analytic 1-form and $f(u)$ is an analytic function on V_a. Since a is an arbitrary point in $V-P$, this proves Theorem 3.1.

5. An example Figure eight knot complement

Let S^3 be the unit sphere in $\mathbb{C}^2$, $S^3=\{(z_1,z_2)\in\mathbb{C}^2 \mid |z_1|^2+|z_2|^2=1\}$. We orient S^3 as the boundary of the unit disc in $\mathbb{C}^2$. Let K be the figure-eight knot in S^3. Then $N=S^3-K$ has a complete hyperbolic structure of finite volume with one cusp [14]. N is decomposed into two ideal terahedra $N=\Delta(e^{\pi i/3})\cup\Delta(e^{\pi i/3})$. The deformation space U of the hyperbolic structure on N has complex dimension 1 and the points of U are parametrized by pairs of complex numbers (z,w) in the upper half plane satisfying the equation (I) in Introduction. For $u=(z,w)\in U$, the corresponding hyperbolic manifold N_u is given by $N_u=\Delta(z)\cup\Delta(w)$ with same gluing pattern as N, and the equation (I) is the consistency condition in Sect. 2 ([14], §4).

For $u=(z,w)\neq u^0=(e^{\pi i/3},\ e^{\pi i/3})$, let $T_\delta(\varepsilon)$ be the δ-torus section of the end ε of N_u (δ is sufficiently small). Let (m_1,l_1) be a meridean-longitude pair on $T_\delta(\varepsilon)$: in S^3, m_1 bounds a 2-disc in a tubular neighborhood of K and l_1 is homologous to zero in Q^3-K. Taking a suitable holonomy representation ρ_u: $\pi_1(N_u)\to PSL_2(\mathbb{C})$, $\rho_u(m_1)$ and $\rho_u(l_1)$ are given as isometries of H^3 by

$$\begin{aligned}\rho_u(m_1)(c,t)&=(w(1-z)\,c, \quad |w(1-z)|\,t)\\ \rho_u(l_1)(c,t)&=(z^2(1-z)^2\,c, \quad |z^2(1-z)^2|\,t),\end{aligned} \tag{III}$$

for $(c,t)\in H^3$ ([14], §4).

For each coprime pair of integers (p,q) such that $|p|\geqq 5$ if $|q|=1$, there is a point $u(p,q)=(z,w)$ of U satisfying the equation (II) in Introduction. N_u can be completed to the closed hyperbolic manifold $M_{p,q}$ by adjoining a simple closed geodesic γ to the end of N_u ([14], §4).

We prove Theorem 3 in Introduction by applying Theorem 3.2 to $\eta(M_{p,q})$. Our method is as follows. At first we determine the analytic function $f(u) = f(z,w)$ for a suitable family of framings $\{\mathscr{F}_u\}$ and $\{\kappa_u\}$. Next we compare $\eta(M_{p,q})$ with $\eta(L(p,q))$ which is known [2], where $L(p,q)$ is the Lens space with the standard metric. Note that $M_{p,q}$ is topologically obtained by performing Dehn surgery of type (p,q) along K. If we perform Dehn surgery of type (p,q) along the trivial knot K_0 in S^3, we obtain the Lens space $L(p,q)$. From this we obtain a degree 1 map H: $M_{p,q} \to L(p,q)$, and using the functorial properties of the third and the fourth terms of the right-hand side of the equation in Theorem 3.2, we deduce the formula in Theorem 3.

We need some preliminaries.

Let ι: $S^3 \to S^3$ be the involution defined by $\iota(z_1, z_2) = (\bar{z}_1, -z_2)$ for $(z_1, z_2) \in S^3$, where $\bar{z}_1$ is the complex conjugate of z_1. Then it is well known that the figure-eight knot K can be arranged so that $\iota(K) = K$ and the two fixed points of ι lies on K. Let E be an ι-invariant tubular neighborhood of K. Then there is a 2-disc D in the interior of E, $\mathring{E}$, such that $\iota(D) = D$ and $D \cap K =$ one point. Let $m = \partial D$ and $L = K \cup m$. Let $S_1 = \{(z_1, 0) \mid |z_1| = 1\}$, $S_2 = \{(0, z_2) \mid |z_2| = 1\}$ and $L_0 = S_1 \cup S_2$. Then $\iota(S_1) = S_1$ and $\iota(S_2) = S_2$.

Lemma 5.1. *There is an orientation-preserving map k: $S^3 \to S^3$ such that*

(i) $k(K) = S_1$ and $k(m) = S_2$,

(ii) k maps the neighborhood E of $L = K \cup m$ diffeomorphically onto a neighborhood E_0 of $L_0 = S_1 \cup S_2$, and $k(S^3 - E) \subset S^3 - E_0$,

(iii) $k\iota = \iota k$.

Proof. Let $S_0 = \{(1/\sqrt{2}, z/\sqrt{2}) \mid |z| = 1\} \subset S^3$. Then $\iota(S_0) = S_0$. Let R_0 be a small closed tubular neighborhood of S_0 such that $\iota(R_0) = R_0$ and $R_0 \cap L_0 = \phi$. Let $E_0 = S^3 - \mathring{R}_0$. Then $L_0 \subset E_0$. Let l' (resp. l'_0) be a simple closed curve on ∂E (resp. ∂E_0) such that $\iota(l') \cap l' = \phi$ (resp. $\iota(l'_0) \cap l'_0 = \phi$) and it is homologous to zero in $S^3 - \mathring{E}$ (resp. $S^3 - \mathring{E}_0$). Let D^2 be the unit 2-disc in $\mathbb{R}^2$ and let $\frac{1}{2}\partial D^2$ be the 1-sphere of radius $\frac{1}{2}$ in $\mathbb{R}^2$. Then the quadruples (E, K, m, l') and (E_0, S_1, S_2, l'_0) are both equivariantly diffeomorphic to the quadrupole $(S^1 \times D^2,\ S^1 \times 0,\ 1 \times \frac{1}{2}\partial D^2,\ S^1 \times 1)$ with involution defined by $(z, y) \to (\bar{z}, -y)$, where $z \in S^1$ and $y \in D^2$ and we regard $\mathbb{R}^2$ as the complex plane $\mathbb{C}$. Hence there is an orientation-preserving diffeomorphism k': $E \to E_0$ such that $k'(K) = S_1$, $k'(m) = S_2$, $k'(l') = l'_0$ and $k'\iota = \iota k'$. The complements $S^3 - \mathring{E}$ and $S^3 - \mathring{E} = R_0$ have the free involutions (the restrictions of ι), and k' is defined on their boundaries. Let $\bar{k}'$: $\partial(S^3 - \mathring{E})/\iota \to \partial R_0/\iota$ be the quotient diffeomorphism. Since R_0/ι is homotopy equivalent to S^1, the only obstruction to extending $\bar{k}'$ to map from $(S^3 - \mathring{E})/\iota$ to R_0/ι lies in the group $H^2((S^3 - \mathring{E})/\iota,\ \partial,\ \mathbb{Z}) = \mathbb{Z}$, and it can be measured by the difference of the homology classes $\bar{k}'_*[l']$ and $[l'_0]$ in $H_1(\partial R_0/\iota, \mathbb{Z})$, where l' and l'_0 are identified with their images in the quotient spaces and $\bar{k}'(l')$ and l'_0 are considered as the curves in $\partial R_0/\iota$. Since $\bar{k}'(l') = l'_0$, this obstruction vanishes, and $\bar{k}'$ extends to a map $\bar{k}''$: $(S^3 - \mathring{E})/\iota \to R_0/\iota$. Let $\bar{k}''$: $S^3 - \mathring{E} \to R_0$ be the map which covers $\bar{k}''$ and coincides with k' on the boundary $\partial(S^3 - \mathring{E})$. Define k: $S^3 \to S^3$ by $k|E = k'$ and $k|S^3 - \mathring{E} = k''$. q.e.d.

Lemma 5.2. *The map k in Lemma 5.1 is covered by a vector bundle map Tk: $TS^3 \to TS^3$ of the tangent bundle of S^3 such that $Tk\,\iota_* = \iota_* Tk$, where ι_* is the differential of ι.*

Proof. Let E_0 be as in the proof of Lemma 5.1. By the construction of k in the proof of Lemma 5.1, $k|E$: $E \to E_0$ is an equivariant diffeomorphism. We set $Tk = k_*$, the differential of k, on E. The restrictions of ι_* to $T(S^3 - \mathring{E})$ and TR_0 are free involutions. The quotient spaces $T(S^3 - \mathring{E})/\iota_*$ and TR_0/ι_* are the tangent bundles of $(S^3 - \mathring{E})/\iota$ and R_0/ι respectively. The quotient of k_*, $\bar{k}_*$, is defined on their boundaries, $\bar{k}_*$: $T(S^3 - \mathring{E})/\iota_* | \partial(S^3 - \mathring{E})/\iota \to TR_0/\iota_* | \partial R_0/\iota$. The only obstruction to extending this bundle map to a bundle map from $T(S^3 - \mathring{E})/\iota_*$ to TR_0/ι_* lies in the cohomology group $H^2(S^3 - \mathring{E}/\iota, \partial, \pi_1(GL_3(\mathbb{R})) = \mathbb{Z}_2$. It can be seen that the obstruction lies in the subgroup $\delta^* H^1(\partial(S^3 - \mathring{E})/\iota, \mathbb{Z}_2)$, where δ^* is the coboundary homomorphism, and it can be measured by the restriction of $\bar{k}_*$, $\bar{k}_*$: $T(S^3 - \mathring{E})/\iota_* | l' \to TR_0/\iota_* | l'_0$, where l' and l'_0 are the curves defined in the proof of Lemma 5.1. Since k maps l' to l'_0, ∂E to ∂R_0 diffeomorphically and $Tk = k_*$ on E, it can be seen that this obstruction vanishes. Hence there is a bundle map extending $\bar{k}_*$ from $T(S^3 - \mathring{E})/\iota_*$ to TR_0/ι_*, and taking the bundle map from $T(S^3 - \mathring{E})$ to TR_0 which covers it, we obtain the desired bundle map Tk. q.e.d.

Lemma 5.3. *Let k and Tk be the map and the bundle map in Lemmas 5.1 and 5.2 respectively. Then*

(i) the map k: $S^3 \to S^3$ extends to a map $\tilde{k}$: $D^4 \to D^4$, and

(ii) the bundle map $Tk \oplus 1$: $TS^3 \oplus \varepsilon \to TS^3 \oplus \varepsilon$ extends to a bundle map $T\tilde{k}$: $TD^4 \to TD^4$ which covers k, where l: $\varepsilon \to \varepsilon$ is the identity map of the trivial line bundle and $TD^4 | S^3 = TS^3 \oplus \varepsilon$.

Proof. (i) Since D^4 is contractible, (i) follows by obstruction theory.

(ii) The only obstruction to extending $Tk \oplus 1$ to a bundle map over TD^4 which covers k lies in the group $H^4(D^4, \partial, \pi_3(GL_4(\mathbb{R}))$. Since $Tk \oplus 1$ preserves the subbundle ε, this obstruction is contained in the image of the composition of the maps

$$H^3(S^3, \pi_3(GL_3(\mathbb{R})) \xrightarrow{i_*} H^3(S^3, \pi_3(GL_4(\mathbb{R})) \xrightarrow{\delta^*} H^4(D^4, S^3, \pi_3(GL_4(\mathbb{R}))),$$

where i: $GL_3(\mathbb{R}) \to GL_4(\mathbb{R})$ is the canonical inclusion and δ^* is the coboundary homomorphism. Let F be a framing on S^3 and set $F(x) = (e_1(x), e_2(x), e_3(x))$ for $x \in S^3$, where $e_i(x) \in T_x S^3$. Then $(Tk)(F(x)) = (Tk(e_1(x)),\ Tk(e_2(x)),\ Tk(e_3(x))) = (e_1(k(x)),\ e_2(k(x)),\ e_3(k(x)))\,A(x)$ for some $A(x) \in GL_3^+(\mathbb{R})$, where $GL_3^+(\mathbb{R})$ denotes the connected component of $GL_3(\mathbb{R})$ containing the identity element. Thus we obtain a continuous map A: $S^3 \to GL_3^+(\mathbb{R})$ and the above obstruction can be measured by the homotopy class of A, $[A] \in \pi_3(GL_3^+(\mathbb{R})) = \mathbb{Z}$, which does not depend on the choice of the framing F. By Lemmas 5.1 and 5.2, $k\iota = \iota k$ and $Tk\,\iota_* = \iota_* Tk$. If we replace F by $\iota_* F$, $A(x)$ is replaced by $A(\iota x)$ for $x \in S^3$. Since ι is orientation reversing, we have $[A] = -[A]$. It follows that $[A] = 0$, and the obstruction vanishes. q.e.d.

Let $N = S^3 - K$ be the complete hyperbolic manifold of finite volume with one cusp. Then the involution ι can be considered as an isometric involution ι_N on N. Let m be a meridean curve of K which lies on a flat torus section of the cusp of N. We choose m so that $\iota_N(m) = m$ and m is a simple closed geodesic with respect to the euclidean structure on the flat torus section. The map k in Lemma 5.1 gives an equivariant map k: $N \to (S^3 - S_1)$ such that $k(m) = S_2$.

For a point $(z_1, z_2) \in S^3 - L_0 = S^3 - S_1 \cup S_2$, we set $(z_1, z_2) = (\sqrt{t} \exp i\theta, \sqrt{1-t} \exp i\psi)$, where $0 < t < 1$, $0 \leqq \theta < 2\pi$ and $0 \leqq \psi < 2\pi$. In this parametrization, ι is represented by $\iota(\theta, \psi, t) = (-\theta, \psi + \pi, t)$.

Each torus section T of the cusp of N has a meridean-longitude pair of closed geodesic curves on it with respect to its euclidean structure. By construction of k in the proof of Lemma 5.1, we can make the following assumption on k,

(**) In a small neighborhood of the end of N, k maps each flat torus section T to a torus $T_c = \{(\theta, \psi, t) | t = c\}$, and k maps each euclidean closed geodesic curve on T parallel to the meridean (resp. the longitude) on it to a curve $\{\theta = \text{const}\}$ (resp. $\{\psi = \text{const}\}$) on T_c.

For each $u \in U$, if we deform the hyperbolic structure on N to N_u, each flat torus section of the cusp of N becomes a δ-torus section of the end of N_u for some $\delta > 0$. Hence k has the same property (**) in a neighborhood of the end of N_u.

Let $\mu(t)$ be a smooth monotone increasing function defined on $[0, 1]$ such that $\mu(t) = 0$ $(0 \leqq t \leqq 1/3)$ and $\mu(t) = 1$ $(2/3 \leqq t \leqq 1)$. Let $\mathscr{F}_0$ be the framing on $S^3 - L_0$ defined by, for $x = (\theta, \psi, t)$,

$$\mathscr{F}_0(x) = \left(\frac{1}{\sqrt{1-t}} \partial/\partial\psi, -\frac{1}{\sqrt{t}} \partial/\partial\theta, 2\sqrt{t(1-t)}\, \partial/\partial t\right) \cdot \begin{pmatrix} \cos \pi\mu(t) & 0 & -\sin \pi\mu(t) \\ 0 & 1 & 0 \\ \sin \pi\mu(t) & 0 & \cos \pi\mu(t) \end{pmatrix}.$$

Then

$$\mathscr{F}_0 = \left(-\frac{1}{\sqrt{1-t}} \partial/\partial\psi, -\frac{1}{\sqrt{t}} \partial/\partial\theta, -2\sqrt{t(1-t)}\, \partial/\partial t\right) \text{ near } S_1$$

$$\mathscr{F}_0 = \left(\frac{1}{\sqrt{1-t}} \partial/\partial\psi, -\frac{1}{\sqrt{t}} \partial/\partial\theta, 2\sqrt{t(1-t)}\, \partial/\partial t\right) \text{ near } S_2.$$

Lemma 5.4. *There is an orthonormal framing $\mathscr{F} = (e_1, e_2, e_3)$ on $N - m$ such that*

(i) $\mathscr{F}$ is a simple framing $\mathscr{F}(\varepsilon)$ on a neighborhood of the cusp and F has a special singularity at m,

(ii) $\int_{s(N-m)} Q = 0$, where s: $N - m \to F(N)$ is the section defined by $\mathscr{F}$, and

(iii) the bundle map Tk in Lemma 5.2 can be deformed by fibre homotopy so that it may satisfy $Tk(\mathscr{F}) = \mathscr{F}_0$.

Proof. Let Tk: $TN \to T(S^3 - S_1)$ be the restriction of the bundle map in Lemma 5.2. There is a framing $\mathscr{F}'=(e_1', e_2', e_3')$ on $N-m$ such that $Tk(\mathscr{F}'(x)) = \mathscr{F}_0(k(x))$ for $x \in N-m$. For each $x \in N-m$, applying the Schmidt orthonormalization to $\mathscr{F}'(x)$, we obtain an orthonormal framing $\mathscr{F}(x)=(e_1(x), e_2(x), e_3(x))$. By (**) and the construction of k, we may assume that, near the cusp, the e_3-vectors are transversal to each torus section of the cusp and $\mathscr{F}$ is ι_N-equivariantly homotopic to a simple framing. Near m, the e_1-vectors are directed along m and $\mathscr{F}$ is ι_N-equivariantly homotopic to a framing which has a special singularity at m. Hence we may deform $\mathscr{F}$, ι_N-equivariantly near m, so that it may satisfy (i). Since $\iota_* Tk = Tk\, \iota_{N*}$ and ι_* maps the vector fields $\{\partial/\partial\psi\}$, $\{\partial/\partial\theta\}$ and $\{\partial/\partial t\}$ to $\{\partial/\partial\psi\}$, $\{-\partial/\partial\theta\}$ and $\{\partial/\partial t\}$ respectively, ι_{N*} maps the frame field (e_1', e_2', e_3') to $(e_1', -e_2', e_3')$ and hence (e_1, e_2, e_3) to $(e_1, -e_2, e_3)$. From this, it follows that $\iota_N^* s^* Q = s^* Q$ by definition of the Chern-Simons form Q. As ι_N reverses the orientation of N, $\int_{N-m} s^* Q = \int_{N-m} \iota_N^* s^* Q = -\int_{N-m} s^* Q$ and it must be zero. This proves (ii). By construction, we may deform Tk by fibrewise homotopy so that it may satisfy (iii). q.e.d.

Using the orthonormal framing $\mathscr{F}$ on N in Lemma 5.4, by Proposition 3.2 and its proof, we obtain a family of orthonormal framings $\{\mathscr{F}_u\}_{u\in U}$ on the family of hyperbolic manifolds $\{N_u\}_{u\in U}$ such that each $\mathscr{F}_u$ has the properties mentioned in Proposition 3.2 and $\mathscr{F}_{u^0}=\mathscr{F}$, where u^0 corresponds to the original complete hyperbolic structure on N.

We have chosen the meridean curve m in N lying on a flat torus section of the cusp. We choose an orthonormal framing $\kappa=(f_1, f_2, f_3)$ defined on m as follows: at each $y \in m$, $f_1(y)$ is the unit tangent vector at y of m having the same direction as the first vectors of $\mathscr{F}$ near y, $f_2(y)$ is tangent to the torus section on which m lies, and $f_3(y)$ is normal to it. Note that this framing κ induces a product structure $D^2 \times m$ on a tubular neighborhood of m such that $1 \times m$ is homotopic to zero in $S^3 - m$.

Lemma 5.5. *Let s: $m \to F(N)$ be the section defined by κ. Then* $\int_{s(m)} \theta_{23} = 0$.

Proof. By definition of κ, it follows that $s^* \theta_{23} = 0$. q.e.d.

For each $u \in U$, let κ_u be the orthonormal framing on m defined by the Schmidt orthonormalization of κ with respect to the hyperbolic metric of N_u.

We define the complex function $f(u)$ on U by, for $u \in U$,

$$f(u) = \int_{s(N_u - m)} C - \frac{1}{2\pi} \int_{s(m)} (\theta_1 - i\,\theta_{23}),$$

where s: $N_u - m \to F(N_u)$ and s: $m \to F(N_u)$ are the sections defined by $\mathscr{F}_u$ and κ_u respectively.

Proposition 5.1. *$f(u)$ is a complex analytic function on U.*

Proof. By the proof of Theorem 3.1, it follows that we may take as V in Theorem 3.1 any neighborhood of u^0 such that $V \cap P$ is a proper analytic subset in V, where P is defined as before. In this case $P=\{u^0\}$ and we may set $V = U$. q.e.d.

Theorem 5.1. *For $u=(z,w)\in U$, setting $f(z,w)=f(u)$, we have*

$$f(z,w)=-\frac{i}{\pi^2}\left(R(z)+R(w)-\frac{\pi^2}{6}\right),$$

where $R(x)$ is the function on the upper half plane defined by

$$R(x)=\tfrac{1}{2}\log x\log(1-x)-\int_0^x \log(1-t)\,d\log t.$$

Proof. At first we calculate the real part of $f(z,w)$. Let $u=(z,w)$ be a point of $U-u^0$. Let $E_\delta(\varepsilon)$ be the δ-neighborhood of the end ε of N_u (see Sect. 3, for definition) and let $T_\delta(\varepsilon)=\partial E_\delta(\varepsilon)$ be the δ-torus section of ε, where $\delta>0$ is sufficiently small. We may assume that $F_u=F(\varepsilon)$ is a simple framing on $E_\delta(\varepsilon)$ by (ii) of Proposition 3.2. Let X_δ be the closure of $s(N_u-E_\delta(\varepsilon)-m)$ in $F(N_u)$, where s: $N_u-m\to F(N_u)$ is the section defined by $\mathscr{F}_u$. Then X_δ is a 3-manifold in $F(N_u)$ with boundary $\partial X=s(T_\delta(\varepsilon))\cup R$, where R is mapped onto m by the bundle projection $F(N_u)\to N_u$. We have

$$\begin{aligned}\operatorname{Re} f(z,w)&=\int_{s(N_u-m)}\operatorname{Re} C-\frac{1}{2\pi}\int_{s(m)}\theta_1\\ &=\lim_{\delta\to 0}\int_{X_\delta}\operatorname{Re} C-\frac{1}{2\pi}\int_{s(m)}\theta_1.\end{aligned}\tag{1}$$

Now

$$\operatorname{Re} C=\frac{1}{\pi^2}\theta_1\wedge\theta_2\wedge\theta_3-\frac{1}{4\pi^2}d\Theta,$$

where $\Theta=\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12}$. Hence

$$\begin{aligned}\lim_{\delta\to 0}\int_{X_\delta}\operatorname{Re} C&=\lim_{\delta\to 0}\int_{X_\delta}\frac{1}{\pi^2}\theta_1\wedge\theta_2\wedge\theta_3-\lim_{\delta\to 0}\int_{X_\delta}\frac{1}{4\pi^2}d\Theta\\ &=\frac{1}{\pi^2}\operatorname{vol}(N_u)-\lim_{\delta\to 0}\int_{s(T_\delta(\varepsilon))}\frac{1}{4\pi^2}\Theta-\int_R\frac{1}{4\pi^2}\Theta,\end{aligned}$$

by Stokes's theorem.

Let ψ: $S^1\times m\to R$ be the diffeomorphism defined by

$$\begin{aligned}\psi(v,y)=(f_1(y),(\cos v)f_2(y)-(\sin v)f_3(y),\\ (\sin v)f_2(y)+(\cos v)f_3(y)),\end{aligned}$$

where $v\in S^1=\mathbb{R}/2\pi\mathbb{Z}$ and $y\in m$ and $\kappa_u(y)=(f_1(y),f_2(y),f_3(y))$. Then $\psi^*\theta_i(\partial/\partial v)=\psi^*\theta_{i1}(\partial/\partial v)=0$ $(i=1,2,3)$ and $\psi^*\theta_{23}(\partial/\partial v)=1$ as in the proof of Lemma 1.5. The orientation of $S^1\times m$ is given by $(f_1,-\partial/\partial v)$. Hence by calculations similar to the ones in the proof of Lemma 1.5, we have

$$\begin{aligned}-\frac{1}{4\pi^2}\int_R\Theta&=-\frac{1}{4\pi^2}\int_{S^1\times m}\psi^*\theta_1\wedge\psi^*\theta_{23}.\\ &=\frac{1}{2\pi}\int_{s(m)}\theta_1,\end{aligned}$$

where s: $m\to F(N_u)$ is the section defined by κ_u.

We compute $-\int_{s(T_\delta(\varepsilon))} \frac{1}{4\pi^2}\Theta$. Let $\tilde{N}_u$ be the universal cover of N_u and let $\widetilde{E_\delta(\varepsilon)}$ be a connected component of the inverse image of $E_\delta(\varepsilon)$ of the covering projection. Let d_u: $\tilde{N}_u \to H^3$ be a developing map of N_u. By conjugation by an element of $PSL_2(\mathbb{C})$, if necessary, we may assume that $\widetilde{E_\delta(\varepsilon)}$ is mapped by d_u into the cylinder around the t-axis,

$$E_\delta = \left\{(r, \beta, \phi) \in H^3 \,\middle|\, \log\cot\left(\frac{\pi}{4} - \frac{\beta}{2}\right) \leqq \delta\right\},$$

where (r, β, ϕ) is the polar coordinate of H^3 defined in Sect. 3. Set $T_\delta = \partial E_\delta$. Let $\tilde{T}_\delta$ be the universal cover of T_δ. On $\tilde{T}_\delta$, we can put the complex coordinate $z = \log r + i\phi$ for $z \in \tilde{T}_\delta$, where (r, ϕ) is the part of the polar coordinate of the image of z in T_δ by the covering projection. Then $T_\delta(\varepsilon)$ is identified with the quotient space of $\tilde{T}_\delta$ by the $\mathbb{Z} \times \mathbb{Z}$-action generated by the translations by two complex numbers

$$\{\log w(1-z), 2\log z(1-z)\}$$

by (III) at the beginning of this section [14]. Let I be the parallelogram spanned by these two complex numbers in the complex plane. Then I is a fundamental domain of this $\mathbb{Z} \times \mathbb{Z}$-action. From the equations in the proof of Lemma 3.2, we have, on $\tilde{T}_\delta$,

$$-s^*\Theta = \left(\frac{1+\sin^2\beta}{\cos^2\beta}\right)(d\log r) \wedge (d\phi),$$

where s: $\tilde{T}_\delta \to PSL_2(\mathbb{C})$ is defined by the simple framing $\mathscr{F}(t\text{-axis})$. Since $\mathscr{F}_u = \mathscr{F}(\varepsilon)$ on $E_\delta(\varepsilon)$, we have

$$\begin{aligned}
-\int_{s(T_\delta(\varepsilon))} \frac{1}{4\pi^2}\Theta &= \frac{1}{4\pi^2}\left(\frac{1+\sin^2\beta}{\cos^2\beta}\right)\int_I d\log r \wedge d\phi \\
&= \frac{1}{4\pi^2}\left(\frac{1+\sin^2\beta}{\cos^2\beta}\right) \\
&\quad \cdot 2[\arg z(1-z)\log|w(1-z)| - \arg w(1-z)\log|z(1-z)|].
\end{aligned}$$

As $\delta \to 0$, $\beta \to 0$ and from (1) above, we obtain

$$\begin{aligned}
\operatorname{Re} f(z, w) &= \frac{1}{\pi^2}\operatorname{vol}(N_u) \\
&\quad + \frac{1}{2\pi^2}[\arg z(1-z)\log|w(1-z)| - \arg w(1-z)\log|z(1-z)|].
\end{aligned} \tag{2}$$

Now $N_u = \Delta(z) \cup \Delta(w)$ and $\operatorname{vol}(N_u) = \operatorname{vol}(\Delta(z)) + \operatorname{vol}(\Delta(w))$. There is a well-known formula for the volume of an ideal simplex in H^3 (see [6, 13]), and we have

$$\begin{aligned}
\frac{1}{\pi^2}\operatorname{vol}(N_u) &= \frac{1}{\pi^2}\left[\arg(1-z)\log|z| - \operatorname{Im}\int_0^z \log(1-t)\, d\log t\right] \\
&\quad + \frac{1}{\pi^2}\left[\arg(1-w)\log|w| - \operatorname{Im}\int_0^w \log(1-t)\, d\log t\right].
\end{aligned}$$

Using the equation (I) in Introduction, the second term of the right hand side of (2) can be written as

$$\frac{1}{2\pi^2}[\arg z \log|1-z| - \arg(1-z)\log|z|]$$
$$+\frac{1}{2\pi^2}[\arg w \log|1-w| - \arg(1-w)\log|w|].$$

Summing up, we have

$$\operatorname{Re} f(z,w) = \frac{1}{\pi^2}\operatorname{Im}\left[\tfrac{1}{2}\log z \log(1-z) - \int_0^z \log(1-t)\, d\log t\right]$$
$$+\frac{1}{\pi^2}\operatorname{Im}\left[\tfrac{1}{2}\log w \log(1-w) - \int_0^w \log(1-t)\, d\log t\right]$$
$$= -\operatorname{Re}\left(\frac{i}{\pi^2}(R(z)+R(w))\right).$$

Both sides of this equation are smooth functions on U, and hence the equality holds at $u^0=(e^{\pi i/3}, e^{\pi i/3})$. By Proposition 5.1, $f(u)$ is an analytic function on U. Since two complex analytic functions with the same real parts differ from each other only by an imaginary constant, it follows that

$$f(z,w) = -\frac{i}{\pi^2}(R(z)+R(w)) + \mathrm{ic},$$

for some real constant c. By Lemmas 5.4 and 5.5, we have $\operatorname{Im} f(u^0)=0$. We may calculate the value $\operatorname{Re} R(e^{\pi i/3})=\pi^2/12$, and we have $c=1/6$. This proves the theorem. q.e.d.

Here we insert the following subsection.

Subsection: Lens space.

Let (p,q) be a coprime pair of integers. Let r and s be the integers such that $0 \leqq r < |p|$ and $ps+qr=1$. The Lens space $L(p,q)$ is the quotient space of S^3 by $\mathbb{Z}_p$ action generated by $\zeta(z_1,z_2)=(\zeta z_1, \zeta^r z_2)$ for (z_1,z_2), where $\zeta=\exp 2\pi i/p$. $L(p,q)$ is naturally oriented and has the standard metric of constant sectional curvature 1 as the quotient space of S^3. We denote the image of $(z_1,z_2)\in S^3$ by the canonical projection by $[z_1,z_2]$. Thus $[\zeta z_1, \zeta^r z_2]=[z_1,z_2]$.

Let ϕ_1 and ϕ_2 be the embeddings $D^2\times S^1 \to L(p,q)$ defined by

$$\phi_1(a,b)=\left[\frac{b^{1/p}}{\sqrt{|a|^2+1}}, \frac{a\, b^{r/p}}{\sqrt{|a|^2+1}}\right]$$
$$\phi_2(a,b)=\left[\frac{a\, b^{q/p}}{\sqrt{|a|^2+1}}, \frac{b^{1/p}}{\sqrt{|a|^2+1}}\right],$$

where a and b represent complex numbers such that $0\leqq|a|\leqq 1$ and $|b|=1$. Then $L(p,q)=D^2\times S^1 \cup_g D^2\times S^1$, where $g=\phi_2^{-1}\phi_1$: $\partial D^2\times S^1 \to \partial D^2 \times S^1$ is given by $g(a,b)=(a^{-q}b^s, a^p b^r)$, for $(a,b)\in\partial D^2\times S^1$.

Let $S_1=\{(z_1,0)||z_1|=1\}$, $S_2=\{(0,z_2)||z_2|=1\}$ and $L_0=S_1\cup S_2$ as before. The action of ζ on S^3-L_0 is written as $\zeta(\theta,\psi,t)=\left(\theta+\frac{2}{p},\ \psi+\frac{r}{p}2\pi,\ t\right)$. We denote the image of the point (θ,ψ,t) in $L(p,q)$ by $[\theta,\psi,t]$. Let $\bar{L}_0=\bar{S}_1\cup\bar{S}_2$ be the image of L_0 in $L(p,q)$. Define the map

$$h\colon S^3-L_0\to L(p,q)-\bar{L}_0$$

by $h(\theta,\psi,t)=\left[\frac{1}{p}\theta,\psi+\frac{r}{p}\theta,t\right]$. Then h is an orientation-preserving diffeomorphism, and it extends naturally to a diffeomorphism $\tilde{h}\colon S^3-S_1\to L(p,q)-\bar{S}_1$. We identify $L(p,q)-\bar{L}_0$ with S^3-L_0 and $L(p,q)-\bar{S}_1$ with S^3-S_1 by h and $\tilde{h}$ respectively. With this identification, the metric on $L(p,q)$ is written on S^3-L_0 as

$$ds^2=\left(\sqrt{t}\frac{1}{p}d\theta\right)^2+\left(\sqrt{1-t}\left(d\psi+\frac{r}{p}d\theta\right)\right)^2+\left(\frac{1}{2\sqrt{t(1-t)}}dt\right)^2.$$

Define the framing $\mathscr{F}_0(p,q)$ on S^3-L_0 by, for $x=(\theta,\psi,t)$,

$$\mathscr{F}_0(p,q)(x)=\left(\frac{1}{\sqrt{1-t}}\partial/\partial\psi,\ -\frac{1}{\sqrt{t}}(p\,\partial/\partial\theta-r\,\partial/\partial\psi),\ 2\sqrt{t(1-t)}\,\partial/\partial t\right)$$
$$\cdot\begin{pmatrix}\cos\pi\mu(t) & 0 & -\sin\pi\mu(t)\\ 0 & 1 & 0\\ \sin\pi\mu(t) & 0 & \cos\pi\mu(t)\end{pmatrix}$$

where $\mu\colon[0,1]\to[0,1]$ is the smooth function defined before the definition of the framing $\mathscr{F}_0$. Then $\mathscr{F}_0(p,q)$ is orthonormal with respect to the above Lens space metric. Actually $\mathscr{F}_0(p,q)$ is the orthonormalization of $\mathscr{F}_0$ with respect to that metric. Near $\bar{S}_1\subset L(p,q)$,

$$\mathscr{F}_0(p,q)=\left(-\frac{1}{\sqrt{1-t}}\partial/\partial\psi,\ -\frac{1}{\sqrt{t}}(p\,\partial/\partial\theta-r\,\partial/\partial\psi),\ -2\sqrt{t(1-t)}\,\partial/\partial t\right)$$

and near $\bar{S}_2\subset L(p,q)$

$$\mathscr{F}_0(p,q)=\left(\frac{1}{\sqrt{1-t}}\partial/\partial\psi,\ -\frac{1}{\sqrt{t}}(p\,\partial/\partial\theta-r\,\partial/\partial\psi),2\sqrt{t(1-t)}\,\partial/\partial t\right).$$

From this we see that $\mathscr{F}_0(p,q)$ has a special singularity at $\bar{S}_2$ but it does not at $\bar{S}_1$. We deform it slightly near $\bar{S}_1$ and define the framing $\mathscr{F}(p,q)$ on S^3-L_0 as follows: for $x=(\theta,\psi,t)$, setting $\mathscr{F}(p,q)(x)=(e_1(x),e_2(x),e_3(x))$ and $\mathscr{F}_0(p,q)(x)=(e_1'(x),e_2'(x),e_3'(x))$, we define

$$e_1(x)=(\cos v(t))\,e_1'(x)+(\sin v(t))\,e_2'(x)$$
$$e_2(x)=-(\sin v(t))\,e_1'(x)+(\cos v(t))\,e_2'(x)$$
$$e_3(x)=e_3'(x)$$

where v: $(0,1)\to\mathbb{R}/2\pi\mathbb{Z}$ is a smooth map such that $v(t)=0$ for $0<t<1-\varepsilon(\varepsilon>0$ is sufficiently small), $v(t)=v_0$ (constant) for $1-(\varepsilon/2)<t<1$ and in this range $e_1(x)=(\cos v_0)e_1'(x)+(\sin v_0)e_2'(x)$ has the direction along $\bar{S}_1$.

Lemma 5.6. *The framing $\mathscr{F}(p,q)$ is an orthonormal framing on $S^3-L_0=L(p,q)-\bar{L}_0$ having a special singularity at $\bar{L}_0$. Moreover $s^*Q=0$, where Q is the Chern-Simons form on $F(L(p,q))$ (the $SO(3)$ frame bundle of $L(p,q)$) and s: $L(p,q)-\bar{L}_0\to F(L(p,q))$ is the section defined by $\mathscr{F}(p,q)$.*

Proof. By construction, $\mathscr{F}(p,q)$ has a special singularity at $\bar{L}_0$. Since $L(p,q)$ has the metric of constant sectional curvature 1, $\Omega_{ij}=\theta_i\wedge\theta_j$, where Ω_{ij} and θ_i are the curvature forms and the fundamental forms of $L(p,q)$ respectively $(i,j=1,2,3)$. It is easy to prove that the Chern-Simons form vanishes for the framing

$$\left(\frac{1}{\sqrt{1-t}}\partial/\partial\psi,\ -\frac{1}{\sqrt{t}}(p\,\partial/\partial\theta-r\,\partial/\partial\psi),\ 2\sqrt{t(1-t)}\,\partial/\partial t\right).$$

From this it follows that

$$\begin{aligned} s^*Q= & -d(\pi\mu(t))\wedge d\left(-\sqrt{t}\left(d\psi+\frac{r}{p}d\theta\right)\right)\\ & -d(v(t))\wedge d\left((\sin\pi\mu(t))\sqrt{1-t}\,\frac{1}{p}\,d\theta\right)\\ = & \,0. \quad \text{q.e.d.}\end{aligned}$$

The tangent space at $[z_1,z_2]$ of $L(p,q)$ is identified with the tangent space at (z_1,z_2) of S^3, and each vector in $\mathbb{C}^2$ which is orthogonal to the vector $(z_1,z_2)\in S^3$ is considered as a tangent vector at $[z_1,z_2]$ of $L(p,q)$.

Lemma 5.7. *There is an orthonormal framing $\alpha(p,q)$ on $L(p,q)$ such that at each $[z_1,0]\in\bar{S}_1$,*

$$\alpha(p,q)=((\mathrm{i}z_1,0),(0,z_1^{r+\lambda p}),(0,\mathrm{i}z_1^{r+\lambda p}))$$

and at each $[0,z_2]\in\bar{S}_2$

$$\alpha(p,q)=((0,\mathrm{i}z_2),(z_2^{q+\nu p},0).\ (\mathrm{i}z_2^{q+\nu p},0))$$

where $\lambda=\dfrac{1+(-1)^r}{2}$ *and* $\nu=\dfrac{1+(-1)^q}{2}$.

Proof. Let α_1 and α_2 be orthonormal framings of $L(p,q)$ defined on $\bar{S}_1$ and $\bar{S}_2$ respectively. The only obstruction to extending $\alpha_1\cup\alpha_2$ to an orthonormal framing on $L(p,q)$ lies in the group $H^2(L(p,q),\ \bar{S}_1\cup\bar{S}_2,\ \pi_1(SO(3)))$. From the cohomology exact sequence of the pair $(L(p,q),\bar{S}_1\cup\bar{S}_2)$ with $\mathbb{Z}_2$-coefficients, we see that this group is isomorphic to $\mathbb{Z}_2+\mathbb{Z}_2$. If $\alpha_1=((\mathrm{i}z_1,0),(0,z_1^m),(0,\mathrm{i}z_1^m))$ at $[z_1,0]\in\bar{S}_1$ and $\alpha_2=((0,\mathrm{i}z_2),(z_2^n,0),(\mathrm{i}z_2^n,0))$ at $[0,z_2]\in\bar{S}_2$ for some integers m and n respectively, then the obstruction is equal to $((m+1)\bmod 2,(n+1)\bmod 2)$. This can be seen by considering the restriction of the obstruction to the Moore spaces in $L(p,q)$, $M_1=\{z_2=\text{real}>0\}$ and $M_2=\{z_1=\text{real}>0\}$. If q or r is even, then p must be odd by $qr=1\bmod p$, and the lemma follows. q.e.d.

Lemma 5.8. *Let $\alpha(p,q)$ be the framing on $L(p,q)$ in Lemma 5.7. Then*

$$\tau(\bar{S}_1, \alpha(p,q)) = \left(\frac{r}{p} + \lambda\right) 2\pi$$

and

$$\tau(\bar{S}_2, \alpha(p,q)) = \left(\frac{q}{p} + \nu\right) 2\pi,$$

where λ and ν are as in Lemma 5.7.

Proof. By definition, $\tau(\bar{S}_1, \alpha(p,q)) = -\int_{s(\bar{S}_1)} \theta_{23}$, where $s\colon \bar{S}_1 \to F(L(p,q))$ is the section defined by $\alpha(p,q)$. Parametrizing as $\bar{S}_1 = [0, \exp it]$ $(0 \leqq t < 2\pi/p)$, we have $s^*\theta_{23} = -(r+\lambda p)\,dt$. Hence $\tau(\bar{S}_1, \alpha(p,q)) = \int_0^{2\pi/p} (r+\lambda p)\,dt = \left(\frac{r}{p}+\lambda\right) 2\pi$. For $\tau(\bar{S}_2, \alpha(p,q))$, the proof is similar. q.e.d.

Using the framing $\mathscr{F}(p,q)$ on $L(p,q) - \bar{L}_0$ and the framing $\alpha(p,q)$ on $L(p,q)$, we apply Theorem 1 in Introduction to the η-invariant of $L(p,q)$. Then by Lemmas 5.6 and 5.8, we have

$$\eta(L(p,q)) = -\frac{1}{3}\left(\frac{q}{p} + \frac{r}{p} + \lambda + \nu\right) + \tfrac{2}{3} d(\mathscr{F}(p,q), \alpha(p,q)) + \delta(L(p,q), \alpha(p,q)).$$

In [2], the following has been proved,

$$\eta(L(p,q)) = -\frac{1}{p} \sum_{k=1}^{p-1} \cot\frac{k}{p}\pi \cot\frac{k}{p} q\pi = \frac{1}{p} \operatorname{def}(p; q, 1).$$

Thus we obtain

Proposition 5.2. *Let $\alpha(p,q)$ and $\mathscr{F}(p,q)$ be as above. Then*

$$\tfrac{2}{3} d(\mathscr{F}(p,q), \alpha(p,q)) + \delta(L(p,q), \alpha(p,q)) = \frac{1}{3}\left(\frac{q}{p} + \frac{r}{p} + \lambda + \nu\right) + \frac{1}{p} \operatorname{def}(p; q, 1).$$

This finishes Subsection.

Let (p,q) be a coprime pair of integers such that $|p| \geqq 5$ if $|q| = 1$. Let (r,s) be the pair of integers such that $0 \leqq r < |p|$ and $ps + qr = 1$. Let $u = (z, w)$ be the point of U satisfying the equation (II). Then as before mentioned N_u can be completed to a closed hyperbolic manifold $M_{p,q}$ by adjoining a closed geodesic γ to the end of N_u. Let E_1 be a small tubular neighborhood of K such that $E_1 \subset E$ and $E_1 \cap m = \phi$, where E is defined in Lemma 5.1. Let (m_1, l_1) be a meridean-longitude pair in $T_1 = \partial E_1$. Then $M_{p,q}$ can be written as $M_{p,q} = D^2 \times S^1 \bigcup_f (S^3 - \mathring{E}_1)$, where $f\colon \partial D^2 \times S^1 \to T_1 = \partial(S^3 - \mathring{E}_1)$ is a diffeomorphism such that $f(\partial D^2 \times 1)$ and $f(1 \times S^1)$ represent the homology classes $p[m_1]+q[l_1]$ and $r[m_1]-s[l_1]$ in $H_1(T_1)$ respectively. The geodesic loop γ corresponds to the core curve $0 \times S^1$ of the solid torus $D^2 \times S^1$. On the other hand, we have the diffeomorphism $\phi=\phi_1 \cup \phi_2$: $D^2 \times S^1 \bigcup_g D^2 \times S^1 \to L(p,q)$ as in Subsection. Let k: $S^3 - \mathring{E}_1 \to S^3 - k(\mathring{E}_1)$ be the restriction of the map k in Lemma 5.1. Note that, since $E \supset E_1$, k maps E_1 diffeomorphically onto a tubular neighborhood $k(E_1)$ of S_1 and the pair $(k(m_1), k(l_1))$ forms a meridean-longitude pair of the trivial knot $S_1 \subset S^3$. Identifying $S^3 - k(\mathring{E}_1)$ with $D^2 \times S^1$ so that S_2 may be identified with $0 \times S^1$ and the pair $(1 \times S^1,\ \partial D^2 \times 1)$ in $\partial D^2 \times S^1$ may form a meridean-longitude pair of the trivial knot $S_1 \subset S^3$, we may write the map k as a map k: $S^3 - \mathring{E}_1 \to D^2 \times S^1$ such that $k(m_1)=1 \times S^1$, $k(l_1)=\partial D^2 \times 1$ and $k(m)=0 \times S^1$. Then the composition $(k|\partial)f$: $\partial D^2 \times S^1 \to \partial D^2 \times S^1$ is isotopic to $g=\phi_2^{-1}\phi_1|\partial$, and we may assume that $(k|\partial)f=g$. Define the map

$$H': M_{p,q}=D^2 \times S^1 \bigcup_f (S^3 - \mathring{E}_1) \to D^2 \times S^1 \bigcup_g D^2 \times S^1$$

by $H'|D^2 \times S^1 =$ the identity map and $H'|S^3 - \mathring{E}_1 = k$. By composing with ϕ, we obtain a map H: $M_{p,q} \to L(p,q)$ such that H maps a neighborhood of $\gamma \cup m$ diffeomorphically onto a neighborhood of $\bar{S}_1 \cup \bar{S}_2$ and $H(\gamma)=\bar{S}_1$ and $H(m)=\bar{S}_2$.

Similarly the bundle map Tk of Lemma 5.2 gives a bundle map TH: $TM_{p,q} \to TL(p,q)$ which covers H. $TM_{p,q}$ and $TL(p,q)$ have the orthogonal bundle structures induced from the Riemannian metric on them. We may assume that TH is a bundle map between these orthogonal bundles.

Now recall the construction of the framing $\mathscr{F}_u$ on $N_u - m = M_{p,q} - \gamma \cup m$ which has a special singularity at $\gamma \cup m$. $\mathscr{F}_u$ was constructed as follows. At first we pull back by Tk the framing $\mathscr{F}_0$ on $S^3 - L_0$ to $N_u - m$. Next we orthonormalize it with respect to the metric of N_u, and finally in a neighborhood of the end of N_u, we rotate about its e_3-vectors so that it may become a simple framing there. Also recall the construction of the framing $\mathscr{F}(p,q)$ on $L(p,q)-\bar{L}_0$. It was defined by rotating the framing $\mathscr{F}_0(p,q)$ about its e_3-vectors in a small neighborhood of $\bar{S}_1$ so that it may have a special singularity at $\bar{S}_1$. As noted before, $\mathscr{F}_0(p,q)$ is the orthonormalization of the framing $\mathscr{F}_0$ with respect to the metric of $L(p,q)$. The map k in Lemma 5.1 maps the end of N_u diffeomorphically onto a neighborhood of $\bar{S}_1$ ($\bar{S}_1$ is deleted) in a nice way (assumption (**)) and the two rotations about e_3-vectors in the above two constructions can be carried in the same way. That is, we may assume that (deforming fibre homotopically if necessary) TH maps $\mathscr{F}_u$ to $\mathscr{F}(p,q)$, $TH(\mathscr{F}_u(x))=\mathscr{F}(p,q)(H(x))$ for each $x \in M_{p,q} - \gamma \cup m$.

We define the orthonormal framing α_u on $M_{p,q}$ as the pull-back of the orthonormal framing $\alpha(p,q)$ on $L(p,q)$ by TH, $TH(\alpha_u(x))=\alpha(p,q)(H(x))$ for each $x \in M_{p,q}$. This is possible, for TH preserves the orthogonal structures on the fibres.

Using the framings α_u, $\mathscr{F}_u$ and κ_u, we apply Theorem 3.2 to the η-invariant of $M_{p,q}$. We must calculate the terms in the right hand side of the equation in Theorem 3.2. The term $\operatorname{Im} f(u)$ is given by Theorem 5.1.

Lemma 5.9. $d(\mathscr{F}_u, \alpha_u)=d(\mathscr{F}(p,q), \alpha(p,q))$.

Proof. Let $\bar{L}(p,q)$ be the closure of $s(L(p,q)-\bar{L}_0)$ in $F(L(p,q))$, where $s\colon L(p,q)-\bar{L}_0\to F(L(p,q))$ is the section defined by $\mathscr{F}(p,q)$. Let $\bar{M}_{p,q}$ be the closure of $s(M_{p,q}-\gamma\cup m)$ in $F(M_{p,q})$, where $s\colon M_{p,q}-\gamma\cup m\to F(M_{p,q})$ is the section defined by $\mathscr{F}_u$. Then H induces the map $\bar{H}\colon \bar{M}_{p,q}\to\bar{L}(p,q)$. If $f\colon \bar{L}(p,q)\to SO(3)$ is the difference map of $\mathscr{F}(p,q)$ and $\alpha(p,q)$ as in Sect. 1, then $f\cdot H$ gives that of $\mathscr{F}_u$ and α_u. Since $\bar{H}$ is of degree 1, the lemma follows. q.e.d.

Lemma 5.10. $\delta(M_{p,q},\alpha_u)=\delta(L(p,q),\,\alpha(p,q))$.

Proof. There is a decomposition

$$\begin{pmatrix} p & r\\ -q & s\end{pmatrix}=\begin{pmatrix} -d_n & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} -d_{n-1} & 1\\ -1 & 0\end{pmatrix}\cdots\begin{pmatrix} -d_1 & 1\\ -1 & 0\end{pmatrix}$$

where $\{d_j\}_{j=1,\ldots,n}$ are integers. For $j=1,\ldots,n$, let A_j be the oriented D^2-bundle over the oriented 2-sphere S_j^2 whose euler class is d_j times the orientation class of S_j^2. Let W be the 4-manifold obtained by plumbing of $A_1,\ldots,A_n$ according to the following weighted tree (see [8], § 8),

$$(d_1)\!-\!(d_2)-\cdots-(d_n)$$

where each (d_j) represents A_j. Let D_1^2 (resp. D_n^2) be a smooth 2-disc in S_1^2 (resp. S_n^2) which does not intersect with A_2 (resp. A_{n-1}) in the above plumbing. Let $D_1^2\times D^2$ (resp. $D_n^2\times D^2$) be the sub D^2-bundle of A_1 (resp. A_n) over D_1^2 (resp. D_n^2). Then the boundary of W, ∂W, is diffeomorphic to the manifold $D_1^2\times S^1\bigcup_g D_n^2\times S^1$, where $g\colon \partial D_1^2\times S^1\to\partial D_n^2\times S^1$ is the map defined by $g(a,b)=(a^{-q}b^s,a^p b^r)$ for $(a,b)\in\partial D_1^2\times S^1$ [8], here we consider the 2-discs as the unit 2-disc in the complex plane. We identify the curve $\partial D_n^2\times 0\subset\partial(D_n^2\times D^2)=S^3$ with the curve $S_1=\{(z_1,0)\,|\,|z_1|=1\}$ in S^3. Then the map k in Lemma 5.1 gives a diffeomorphism $k_1\colon E_1\to\partial D_n^2\times D^2$, where E_1 is the tubular neighborhood of the figure eight knot K in S^3 and k_1 maps a longitude curve in ∂E_1 to $\partial D_n^2\times 1$. Let W' be the 4-manifold obtained from the disjoint union $(W-\mathring{D}_n^2\times D^2)\cup D^4$ by identifying each point $x\in E_1\subset\partial D^4$ with $k_1(x)\in\partial D_n^2\times D^2\subset\partial(W-\mathring{D}_n^2\times D^2)$. Then $\partial W'=M_{p,q}$. Using the maps $\tilde{k}$ and $T\tilde{k}$ of Lemma 5.3, we obtain a map $G\colon W\to W'$ and a bundle map $TG\colon TW'\to TW$ which covers G such that $G|D^4=\tilde{k}$, $G|W-\mathring{D}_n^2\times D^2=$ the identity map, $TG|TD^4=T\tilde{k}$ and $TG|T(W-\mathring{D}_n^2\times D^2)=$ the identity map. Then $G|\partial W'=H$ and $TG|T\partial W'=TH$. Now,

$$\delta(L(p,q),\alpha(p,q))=\tfrac{1}{3}P_1[W]-\mathrm{Sign}(W)$$

and

$$\delta(M_{p,q},\alpha_u)=\tfrac{1}{3}P_1[W']-\mathrm{Sign}(W').$$

Clearly $\mathrm{Sign}(W)=\mathrm{Sign}(W')$. Since the Pontrjagin class is functorial with respect to the bundle map, $P_1[W]=P_1[W']$. q.e.d.

Proposition 5.3

(*i*) $\tau(\gamma,\alpha_u)=\left(\dfrac{r}{p}+\lambda\right)2\pi-\dfrac{2}{p}\arg z(1-z)$

(*ii*) $\tau(m,\kappa_u)-\tau(m,\alpha_u)=-v\,2\pi.$

Proof. (i) We use the coordinate (x, y, t) of $D^2 \times S^1$ defined by $a = x + iy$ and $b = \exp it$, where $(x, y) \in \mathbb{R}^2$ with $x^2 + y^2 \leqq 1$, $0 \leqq t < 2\pi$, $a \in D^2$ and $b \in S^1$. Let α'_0 be the framing on $D^2 \times S^1$ defined by $\alpha'_0 = (\partial/\partial t, \partial/\partial x, \partial/\partial y)$. In the representation $M_{p,q} = D^2 \times S^1 \bigcup_f (S^3 - \mathring{E}_1)$, $0 \times S^1$ represents γ and we may consider $\tau(\gamma, \alpha_0)$, where α_0 is the orthonormalization of α'_0 with respect to the metric of $M_{p,q}$. Since the curve $f(1 \times S^1) \subset \partial E_1$ represents the curve $r m_1 - s l_1$, it follows from (III) at the beginning of this section that

$$\tau(\gamma, \alpha_0) = r \arg w(1-z) - s \arg z^2(1-z)^2.$$

Since $p \arg w(1-z) + q \arg z^2(1-z)^2 = 2\pi$ and $ps + qr = 1$, we have

$$\tau(\gamma, \alpha_0) = \frac{r}{p} 2\pi - \frac{2}{p} \arg z(1-z).$$

Let $\phi = \phi_1 \cup \phi_2$: $D^2 \times S^1 \bigcup_g D^2 \times S^1 \to L(p, q)$ be the diffeomorphism defined in Subsection. Consider the framing $(\phi^{-1}) * \alpha(p, q)$. On $\phi^{-1}(\bar{S}_1) = 0 \times S^1$, we may write as $((\phi^{-1})_* \alpha(p, q))(t) = \alpha'_0(t)\, v(t)$ for $t \in \phi^{-1}(\bar{S}_1)$, where v: $\phi^{-1}(\bar{S}) \to SO(2)$ $(\subset SO(3))$ is the difference map as in the proof of Lemma 1.1 in Sect. 1. Then from the definitions of ϕ_1 and $\alpha(p, q)$, it follows that the mapping degree of v is λ. The map H': $M_{p,q} = D^2 \times S^1 \bigcup_f (S^3 - \mathring{E}_1) \to D^2 \times S^1 \bigcup_g D^2 \times S^1$ is the identity map on $D^2 \times S^1$. Since α_u is defined as the pull back of $\alpha(p, q)$ by the bundle map TH, it follows that, on $\gamma = 0 \times S^1$, we may write as $\alpha_u(t) = \alpha_0(t)\, v(t)$ for $t \in \gamma$, where v is the above map. By the calculation in the proof of Lemma 1.1, we have

$$\begin{aligned} \tau(\gamma, \alpha_u) &= \tau(\gamma, \alpha_0) + \lambda 2\pi \\ &= \left(\frac{r}{p} + \lambda\right) 2\pi - \frac{2}{p} \arg z(1-z). \end{aligned}$$

(ii) On $\phi^{-1}(\bar{S}_2) = 0 \times S^1$, $((\phi^{-1})_* \alpha(p, q))(t) = \alpha'_0(t)\, w(t)$ for $t \in \phi^{-1}(\bar{S}_2)$, where w: $\phi^{-1}(\bar{S}_2) \to SO(2) (\subset SO(3))$ is the difference map. From the definitions of ϕ_2 and $\alpha(p, q)$, we see that the mapping degree of w is v. Since $H' | (S^3 - E_1) = k$ and $Tk = k_*$ near m, it follows from the definition of κ_u and the construction of k that $TH'(\kappa_u)$ is isotopic to α'_0 on $H(m) = 0 \times S^1 \subset D^2 \times S^1$. This shows that, on m, the mapping degree of the difference map of α_u and κ_u is v. By the calculation in the proof of Lemma 1.1, we see that $\tau(m, \alpha_u) = \tau(m, \kappa_u) + v 2\pi$. q.e.d.

By Theorem 5.1, Propositions 5.2 and 5.3, and Lemmas 5.9 and 5.10, we have

$$\tfrac{1}{3} \operatorname{Im} f(u) = -\frac{1}{3\pi^2} \operatorname{Re} \left(R(z) + R(w) - \frac{\pi^2}{6}\right),$$

$$-\frac{1}{6\pi} \tau(\gamma, \alpha_u) = -\tfrac{1}{3} \left(\frac{r}{p} + \lambda\right) + \frac{1}{3p\pi} \arg z(1-z),$$

$$\tfrac{2}{3} d(\mathscr{F}_u, \alpha_u) + \delta(M_{p,q}, \alpha_u) = \tfrac{1}{3} \left(\frac{q}{p} + \frac{r}{p} + \lambda + v\right) + \frac{1}{p} \operatorname{def}(p; q, 1),$$

$$\frac{1}{6\pi} (\tau(m, \kappa_u) - \tau(m, \alpha_u)) = -\tfrac{1}{3} v.$$

Summing up these equations, by Theorem 3.2, we obtain Theorem 3 in Introduction.

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