Lobachevsky function and dilogarithm function

Jinseok Cho

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1 The Lobachevsky functon

The Lobachevsky function $\Lambda(\theta)$ is defined by the formula

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin t| dt.$$

This function has a lot of names and definitions. For example, the well-known Clausen function $Cl_2(\theta)$ is defined by the similar formula

$$Cl_2(\theta) = -\int_0^\theta \log\left|2\sin\frac{s}{2}\right| ds,$$

and it is easy to show $Cl_2(2\theta) = 2\Lambda(\theta)$.

Theorem 1.1 *The function $\Lambda(\theta)$ is well defined, continuous for all θ, π-periodic, and odd. Moreover, for each positive integer n, $\Lambda(\theta)$ satisfies the identity*

$$\Lambda(n\theta) = n\sum_{j=0}^{n-1} \Lambda(\theta + j\pi/n).$$

Proof. See §10.4 of [1]. □

As an application of Theorem 1.1, we obtain the useful formula

$$\frac{1}{2}\Lambda(2\theta) = \Lambda(\theta) + \Lambda(\theta + \frac{\pi}{2}) = \Lambda(\theta) - \Lambda(\frac{\pi}{2} - \theta). \tag{1}$$

Substituting $\theta = \frac{\pi}{6}$ in (1), we have

$$\Lambda(\frac{\pi}{3}) = \frac{2}{3}\Lambda(\frac{\pi}{6}) = 0.3383138...$$

Now we will find the maximum and minimum value of $\Lambda(\theta)$. By the fundamental theorem of calculus, we have

$$\begin{aligned}\frac{d\Lambda(\theta)}{d\theta} &= -\log|2\sin\theta| \\ \frac{d^2\Lambda(\theta)}{d\theta^2} &= -\cot\theta.\end{aligned}$$

Consequently, $\Lambda(\theta)$ attains its maximum value $\Lambda(\frac{\pi}{6}) = 0.5074708...$ at $\theta = \frac{\pi}{6}$ and its minimum value $\Lambda(\frac{5\pi}{6}) = -\Lambda(\frac{\pi}{6}) = -0.5074708...$ at $\theta = \frac{5\pi}{6}$.

The following is a fundamental theorem of the hyperbolic volume theory.

Theorem 1.2 *Let $T_{\alpha,\beta,\gamma}$ be an ideal tetrahedron in $\mathbb{H}^3$ with the three dihedral angles α, β, γ of edges incident to a vertex. Then the volume of $T_{\alpha,\beta,\gamma}$ is given by*

$$Vol(T_{\alpha,\beta,\gamma}) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

Moreover, a tetrahedron of maximum volume in $\mathbb{H}^3$ is a regular ideal tetrahedron with the volume $3\Lambda(\frac{\pi}{3})$.

Proof. See §10.4 of [1]. □

2 The dilogarithm function

Traditionally the dilogarithm function $\mathrm{Li}_2(z)$is defined by the Taylor series espansion

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \tag{2}$$

for a complex number z with $|z| \leq 1$. The domain of Li_2 can be extended analytically to the whole complex plane by the following way :

Definition 2.1 *For a complex number z, let $\log z$ have the principal branch cut, i.e. $|\arg z| < \pi$. It defines the following function*

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt$$

*in a unique way with $0 < \arg(z-1) < 2\pi$. (We assume the integrand path from 0 to t is a straight line.) This well-defined function is called **the (Euler) dilogarithm function**.*

Note that **Definition 2.1** is an analytic continuation of (2).

The following identities can be obtained easily by the definition and differentials of each terms. (see [2])

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \frac{1}{2}\mathrm{Li}_2(z^2) \tag{3}$$

$$\mathrm{Li}_2(-z) + \mathrm{Li}_2(-\frac{1}{z}) = 2\mathrm{Li}_2(-1) - \frac{1}{2}\log^2 z \tag{4}$$

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \mathrm{Li}_2(1) - \log z \log(1-z) \tag{5}$$

From these equations, one obtains

$$\mathrm{Li}_2(1) = \frac{\pi^2}{6},\ \mathrm{Li}_2(-1) = -\frac{\pi^2}{12}.$$

One can find more formulas in pages 2808-2809 of [3].

Remark 2.2 *Roger's dilogarithm function $L(z)$, which in general has better properties than $\mathrm{Li}_2(z)$, is defined by the formula*

$$L(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log z \log(1-z) = -\frac{1}{2}\int_0^z \left(\frac{\log(1-t)}{t} + \frac{\log t}{1-t}\right) dt.$$

3 The Lobachevsky function and the dilogarithm function

The Lobachevsky function and the imaginary part of the dilogarithm function play an important role in hyperbolic volume theory. In fact, the Lobachevsky function can be considered as an imaginary part of the dilogarithm function.

Theorem 3.1 *For $0 < \theta < \pi$, we have*

$$\mathrm{Li}_2(e^{2i\theta}) = \mathrm{Li}_2(1) + \theta(\theta - \pi) + 2i\Lambda(\theta).$$

Proof. See §10.4 of [1]. □

The following theorem (due to E. Kummer(1840)) shows more general relation between these two functions.

Theorem 3.2 *For $0 < \theta < 2\pi$, we have*

$$\mathrm{ImLi}_2(re^{i\theta}) = \omega \log r + \Lambda(\theta) + \Lambda(\omega) - \Lambda(\theta + \omega),$$

where ω is defined by the following identities :

$$\tan\omega = \frac{r\sin\theta}{1 - r\cos\theta} \quad \text{or} \quad r = \frac{\sin\omega}{\sin(\omega+\theta)}.$$

Proof. See page 15 of [2]. □

From the proof of the theorem, we can easily find the following relation

$$\mathrm{ReLi}_2(re^{i\theta}) = -\frac{1}{2}\int_0^r \frac{\log(1 - 2y\cos\theta + y^2)}{y} dy$$

for $0 < \theta < 2\pi$. (See **Proposition A** of [4])

4 The general form of the dilogarithm function

We have thus far considered only the principal branch of the dilogarithm. But if we permit any branch, the dilogarithm function has the following general form

$$\mathrm{Li}_2(z) = \mathrm{Li}_2^*(z) + 2m\pi i \log z + 4k\pi^2$$

where $\mathrm{Li}_2^*(z)$ is the dilogarithm function with the principal branch and m, k are integers. (See page 2810 of [3]) The integers m and k depend on the integrand path, specifically the number of encircling the branch points 1 and 0.

Note that the imaginary part of the general dilogarithm function does not depend on the branch point 0, but only on the branch point 1.

5 Bloch-Wigner function and the hyperbolic volume of an ideal tetrahedron

The following function plays an important role in hyperbolic volume and K-theory.

Definition 5.1 *We define* ***the Bloch-Wigner function*** $D(z)$ *by the formula*

$$D(z) = \mathrm{Im}\mathrm{Li}_2(z) + \arg(1-z)\log|z|$$

for $z \in \mathbb{C}\backslash\mathbb{R}$*, and* $D(z) = 0$ *for* $z \in \mathbb{R} \cup \{\infty\}$.

Note that, for $\theta \in \mathbb{R}$,

$$D(e^{i\theta}) = \mathrm{Im}\mathrm{Li}_2(e^{i\theta}) = 2\Lambda(\frac{\theta}{2}). \tag{6}$$

The usefulness of this equation will be shown later.

Definition 5.2 *A sequence (a,b,c,d,e) of five complex numbers is called a* ***5-cycle*** *if*

$$a = 1 - cd, \quad b = 1 - de, \quad c = 1 - ea, \quad d = 1 - ab, \quad e = 1 - bc.$$

We say that a 5-cycle is ***nontrivial*** *if it consists of nonzero elements.*

Remark 5.3 1. *The dihedral group* D_5 *acts on the set of all 5-cycles.*

2. *If* $d \neq 0$*, then* (a, b, c, d, e) *is 5-cycle if and only if*

$$(a, b, c, d, e) = \left(a, b, \frac{1-a}{1-ab}, 1-ab, \frac{1-b}{1-ab}\right).$$

Lemma 5.4 *For* $z \in \mathbb{C} \cup \{\infty\}$*, we have*

1. $D(z) + D(1-z) = 0$,
2. $D(z) + D(\frac{1}{z}) = 0$,
3. $D(z) + D(\overline{z}) = 0$,
4. *If* (a, b, c, d, e) *is a 5-cycle, then*

$$D(a) + D(b) + D(c) + D(d) + D(e) = 0.$$

Proof. 1-3 are easy. From these properties, we know

$$D(1-ab) = -D(ab), \; D(\frac{1-a}{1-ab}) = D(1-\frac{1}{\frac{1-a}{1-ab}}) = D(-a\frac{1-b}{1-a}),$$

$$D(\frac{1-b}{1-ab}) = D(1-\frac{1}{\frac{1-b}{1-ab}}) = D(-b\frac{1-a}{1-b}).$$

Using the well-known equality (see [2])

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(y)+\mathrm{Li}_2(-x\frac{1-y}{1-x})+\mathrm{Li}_2(-y\frac{1-x}{1-y})-\mathrm{Li}_2(xy) = -\frac{1}{2}\log^2\left[\frac{1-x}{1-y}\right]$$

and the **Remark 5.3**, we have

$$\begin{aligned} D(a)+D(b)+D(c)+D(d)+D(e) &= D(a)+D(b)+D(\frac{1-a}{1-ab})-D(ab)+D(\frac{1-b}{1-ab}) \\ &= \mathrm{ImLi}_2\left[-\frac{1}{2}\log^2\left(\frac{1-a}{1-b}\right)\right]+\arg(1-a)\log|a|+\arg(1-b)\log|b| \\ &+\arg\frac{1-ab}{1-a}\log\left|a\frac{1-b}{1-a}\right|+\arg\frac{1-ab}{1-b}\log\left|b\frac{1-a}{1-b}\right|-\arg(1-ab)\log|ab| = 0. \end{aligned}$$

□

Theorem 5.5 *For $z \in \mathbb{C}\backslash\mathbb{R}$, we have*

$$D(z) = \Lambda(\theta_1)+\Lambda(\theta_2)-\Lambda(\theta_1+\theta_2)$$

where $\theta_1 = \arg z$, $\theta_2 = \arg(1-\overline{z})$.

Proof. See pages 245-246 of [2]. □

The following corollary follows directly from **Theorem 1.2** and **Theorem 5.5**.

Corollary 5.6 *For a complex number z with $0 < \arg z < \pi$,*

$$D(z) = \Lambda(\arg z)+\Lambda(\arg(1-\frac{1}{z}))+\Lambda(\arg\frac{1}{1-z}) = Vol(T_z)$$

where T_z is the hyperbolic ideal tetrahedron parametrized by the complex number z.

References

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