AFFINE DIFFERENTIAL GEOMETRY AND FLAT PROJECTIVE STRUCTURES

JOHN C. LOFTIN

In these lectures I will explore several aspects of the differential geometry of flat projective structures, with an emphasis on affine differential geometry and the rich structure which hyperbolic affine spheres provide for properly convex $\mathbb{RP}^n$ manifolds.

Here is a list of topics. The topics toward the end are somewhat tentative and we will cover them as time permits. (Each topic may not correspond to a single hour lecture.)

1. Projective equivalence of connection on the tangent bundle. Projectively flat connections. Lifting to a "normal" connection on a larger bundle. The Ricci and Weyl tensors. Relations to hypersurface (centroaffine) theory. Constructing $\mathbb{RP}^n$ coordinates. The tautological bundle, convexity and lifting to $\mathbb{S}^n$ structures.
2. An introduction to affine differential geometry: The affine normal, connection, metric, shape operator, cubic form. Formulas in local coordinates. Quadrics. Affine spheres. The hyperboloid. Calabi's example $\Pi x_i = c$. Cones.
3. Duality in affine differential geometry. Conormal maps. Conjugate connection. The affine conormal vector. Legendre transform. Dual projective structures. The Calabi/Loewner-Nirenberg equation $\det(u_{ij}) = (-\frac{1}{u})^{n+2}$.
4. Cheng-Yau's results on hyperbolic affine spheres. Consequences for $\mathbb{RP}^n$ manifolds. Canonical metrics and connections.
5. Toward a proof of this proposition (which directly follows from Cheng-Yau): A compact $\mathbb{S}^n$ manifold (w/o boundary) is properly convex if and only if there is a negative section u of the dual of the tautological bundle satisfying $\det(u_{ij}) = (-\frac{1}{u})^{n+2}$. A complete proof of $\Leftarrow$: Gradient estimates, length with respect to the affine metric.
6. (if time permits) Some basics about the real Monge-Ampère equation. Idea of proof of $\Rightarrow$.
7. (if time permits) C.P. Wang's work on two-dimensional hyperbolic affine spheres. A complex structure on the deformation space of convex $\mathbb{RP}^2$ surfaces of genus ≥ 2.

8. (if time permits) More on the real Monge-Ampère equation. A proof of $\Rightarrow$.

References: For much of the first part, I will follow the exposition in my thesis (available at www.math.columbia.edu/˜loftin), while filling in much of the details.

Other references: for affine differential geometry:

- Nomizu and Sasaki, *Affine Differential Geometry*, Cambridge UP 1994.
- Cheng and Yau, Complete affine hyperspheres, Part I, *Comm. Pure Appl. Math.*, 30:41–68, 1986.
- Li, Simon and Zhang *Global Affine Differetial Geometry of Hypersurfaces*, de Gruyter 1993.

These last two include the elements of Cheng-Yau's proof. Also for elements of the proof, see:

- A.M. Li, Calabi conjecture on hyperbolic affine hyperspheres, *Math Z.*, 203:408–491, 1990; Calabi conjecture on hyperbolic affine hyperspheres (2), *Math. Ann.*, 293:485–493, 1992.
- Gigena, On a conjecture of E. Calabi, *Geom. Ded.*, 11:387–396, 1981; Integral invariants of convex cones, *JDG*, 13:191–222, 1978.

These papers of Gigena give another approach to the duality theory. For the real Monge-Ampère equation, Cheng-Yau's solution to $\det(u_{ij}) = (-\frac{1}{u})^{n+2}$ with zero Dirichlet boundary condition on a bounded convex domain is in

- Cheng and Yau, On the regularity of the Monge-Ampère equation $\det((\partial^2 u/\partial x^i \partial x^j)) = F(x,u)$. *Comm. Pure Appl. Math.*, 30:31–68, 1977.

A (by now) more standard approach to real Monge-Ampère equations is given in

- Caffarelli, Nirenberg and Spruck, The Dirichlet problem for nonlinear second order elliptic equations, I. Monge-Ampère equations. *Comm. Pure Appl. Math.*, 37:369–402, 1983.
- Gilbarg and Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, 1983.

Wang's work on 2-dimensional affine spheres is

- C.P. Wang, Some examples of complete hyperbolic affine 2-spheres in $\mathbb{R}^3$, in *Global Differential Geometry and Global Analysis*, Springer *Lect. Notes in Math.* 1481:272–280, 1991.