Rigidity theorem

Survey on the work of A. Nijenhuis and R.W. Richardsen Jr.

Yuncherl Choi

December 2, 1999

Definition ([4]) A *graded Lie algebra* over a field $\mathcal{K}$ of characteristic $\neq 2$ is a graded vector space $E = \oplus_\alpha E^\alpha$ over $\mathcal{K}$ together with a bilinear map $(a, b) \mapsto [a, b]$ of $E \times E$ into E which satisfies the following conditions :

GLA1 : $[E^\alpha, E^\beta] \subset E^{\alpha+\beta}$

GLA2 : If $a \in E^\alpha$ and $b \in E^\beta$, then $[a, b] = -(-1)^{\alpha\beta}[b, a]$.

GLA3 : If $a \in E^\alpha$, $b \in E^\beta$ and $c \in E^\gamma$, then

$$(-1)^{\alpha\gamma}[[a, b], c] + (-1)^{\beta\alpha}[[b, c], a] + (-1)^{\gamma\beta}[[c, a], b] = 0.$$

In the GLA3, the Jacobi identity has some equivalent forms :

$$[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]]$$
$$[a, [b, c]] - (-1)^{\alpha\beta}[b, [a, c]] = [[a, b], c]$$
$$(-1)^{\alpha\gamma}[a, [b, c]] + (-1)^{\beta\alpha}[b, [c, a]] + (-1)^{\gamma\beta}[c, [a, b]] = 0.$$

Example ([4]) Let V be a finite dimensional vector space over $\mathcal{K}$. And let

$$C_a^* = \oplus_n C_a^n,\ C^* = \oplus_n C^n, \text{ and } C_l^* = \bigoplus_n C_l^n$$

be the cochain complexes where C_a^n, C^n and C_l^n are the sets of n-multilinear, left skew symmetric n-multilinear, skew symmetric n-multilinear maps from V into V respectively. The composition products $\circ$ on these complexes are defined as followings

(a) $\underline{\text{On } C_a^*}$: $f \in C_a^{m+1},\ g \in C_a^{n+1} \Rightarrow f \circ g \in C_a^{m+n+1}$

$$\begin{aligned}&(f \circ g)(x_1, \cdots, x_{m+n+1})\\ &= \sum_{i=1}^{m+1} (-1)^{n(i-1)} f(x_1, \cdots, x_{i-1}, g(x_i, \cdots, x_{i+n}), x_{i+n+1}, \cdots, x_{m+n+1}).\end{aligned}$$

(b) <u>On C^*</u> : $f \in C^{m+1},\ g \in C^{n+1} \Rightarrow f \circ g \in C^{m+n+1}$

$$\begin{aligned}(f \circ g)&(x_1, \cdots, x_{m+n+1})\\ &= \sum_1 \mathrm{sgn}\sigma f(g(x_{\sigma(1)}, \cdots, x_{\sigma(n+1)}), x_{\sigma(n+2)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1})\\ &+ \sum_2 \mathrm{sgn}\sigma f(x_{\sigma(1)}, \cdots, x_{\sigma(m)}, g(x_{\sigma(m+1)}, \cdots, x_{\sigma(m+n)}, x_{m+n+1}))\end{aligned}$$

where the $\sum_1$ means that $\sigma(1) < \cdots < \sigma(n)$ and $\sigma(n+2) < \cdots < \sigma(m+n)$, and the $\sum_2$ means that $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$.

(c) <u>On C_l^*</u> : $f \in C_l^{m+1},\ g \in C_l^{n+1} \Rightarrow f \circ g \in C_l^{m+n+1}$

$$\begin{aligned}(f \circ g)&(x_1, \cdots, x_{m+n+1})\\ &= \sum \mathrm{sgn}\sigma f(g(x_{\sigma(1)}, \cdots, x_{\sigma(n+1)}), x_{\sigma(n+2)}, \cdots, x_{\sigma(m+n+1)})\end{aligned}$$

where the $\sum$ means that $\sigma(1) < \cdots < \sigma(n+1)$ and $\sigma(n+2) < \cdots < \sigma(m+n+1)$.

With these composition products, $C_a^*,\ C^*,$ and C_l^* become the graded right symmetric algebras by reducing the degree 1, i.e., for $f \in C^{m+1},\ g \in C^{n+1},\ h \in C^*$,

$$(h \circ f) \circ g - (-1)^{mn}(h \circ g) \circ f = h \circ (f \circ g) - (-1)^{mn} h \circ (g \circ f).$$

Now define the bracket by for $f \in C^{m+1},\ g \in C^{n+1}$

$$[f, g] = -(-1)^{mn} f \circ g + g \circ f.$$

Then $C_a^*,\ C^*$ and C_l^* become the graded Lie algebras.

Definition ([4]) Let $E = \oplus_\alpha E^\alpha$ be a GLA(graded Lie algebra) over $\mathbb{R}$ or $\mathbb{C}$ and let G be a Lie group whose Lie algebra is E^0 and $\tau : G \to GL(E)$ be a continuous homomorphism(i.e. a representation of G on E). Then (E, G, τ) is *analytic graded Lie algebra* if it satisfies the following conditions

(i) For each α, $\dim E^\alpha < \infty$.

(ii) $d\tau = \mathrm{ad}_E : E^0 \to gl(E)$.

(iii) For each $a \in G$, $\tau(a) \in Aut(E)$.

Remark If we assume the base field of the vector space V is $\mathbb{R}$ or $\mathbb{C}$, then $C_a^*,\ C^*$ and C_l^* are the analytic GLA with $G = GL(V)$ and the representation τ which is defined by

$$(\tau(a)f)(x_1, \cdots, x_n) = a^{-1} f(ax_1, \cdots, ax_n),$$

for $f \in C_a^n$(or C^n, C_l^n and $a \in GL(V)$.

(1) $\tau(ab)f = \tau(b)\tau(a)f$ i.e., the action occur on the right side.

(2) $d\tau(a)(f) = [a, f]$

$$\begin{aligned} d\tau(a)(f) &= \frac{d}{dt}\Big|_{t=0} \exp(-ta)f(\exp(ta)x_1, \cdots, \exp(ta)x_n) \\ &= -af(x_1, \cdots, x_n) \\ &\quad +f(ax_1, x_2, \cdots, x_n) + \cdots + f(x_1, \cdots, x_{n-1}, ax_n) \\ &= (-a \circ f)(x_1, \cdots, x_n) + (f \circ a)(x_1, \cdots, x_n) \\ &= [a, f](x_1, \cdots, x_n) \end{aligned}$$

Let's denote $\mathcal{M}$, the deformation space, as the set of elements of E^1 satisfying $[a, a] = 0$, i.e.,

$$\mathcal{M} = \{a \in E^1 | [a, a] = 0\}.$$

For each $\mu \in \mathcal{M}$, we can define the derivation D_μ of E,

$$D_\mu(f) = [\mu, f]$$

from the Jacobi identity, i.e., for $f \in E^\alpha,\ g \in E$

$$\begin{aligned} D_\mu([f, g]) &= [\mu, [f, g]] \\ &= [[\mu, f], g] + (-1)^\alpha [f, [\mu, g]] \\ &= [D_\mu(f), g] + (-1)^\alpha [f, D_\mu(g)] \end{aligned}$$

Moreover, we have $D_\mu \circ D_\mu = 0$. Hence we obtain the graded supspace $Z(\mu),\ B(\mu)$ and $H(\mu)$ of E defined by

$$Z(\mu) = \ker D_\mu,\ B(\mu) = \mathrm{im} D_\mu,\ H(\mu) = Z/B.$$

In this case, $Z(\mu)$ is the subalgebra of E and $B(\mu)$ is the ideal of $Z(\mu)$.

Remark In the case $C_a^*,\ C^*$ and C_l^*, $\mathcal{M}$ is really the deformation space of associative, left symmetric and Lie algebras respectively. And D_μ is the coboundary map of the algebra defined by μ on V.

Now fix $\mu \in \mathcal{M}$ and put $D = D_\mu$. And some notations are given :

Notation

Z : kernel of D

B : image of D

C : supplementaty subspace to Z in E

H : supplementaty subspace to B in Z

$$Z^\alpha = Z \cap E^\alpha,\ B^\alpha = B \cap E^\alpha,\ C^\alpha = C \cap E^\alpha,\ H^\alpha = H \cap E^\alpha$$

$N(V)$: the neighborhood of the origin of the vecto space of V

With the above notations, we have a decomposition of E

$$E = B \oplus H \oplus C,$$

which is called *Hodge decomposition* of E. Of cource the decomposition in each degree is the same, i.e.,

$$E^\alpha = B^\alpha \oplus H^\alpha \oplus C^\alpha,$$

Lemma ([4]) *There exist $N(\mu, E^1)$, $N(C^0)$, $N(H^1)$, $N(C^1)$ such that $F : C^0 \times H^1 \times C^1 \to E^1$ which is defined by*

$$F(a, h, b) = \exp(a) \cdot (\mu + h + b)$$

is anaylitic and its restriction to $N(C^0) \times N(H^1) \times N(C^1)$ is homeomorphic onto $N(\mu, E^1)$.

Proof.

$$\begin{aligned} dF_{(0,0,0)}(a, h, b) &= \left.\frac{d}{dt}\right|_{t=0} F(ta, th, tb) \\ &= \left.\frac{d}{dt}\right|_{t=0} \exp(ta) \cdot (\mu + th + tb) \\ &= [a, \mu] + h + b \\ &= -D(a) + h + b \end{aligned}$$

Since D is an isomorphism of C^0 onto B^1 and $E^1 = B^1 \oplus H^1 \oplus C^1$, $dF_{(0,0,0)}$ is an isomorphism. By the inverse function theorem, F is locally diffeomorphic at $(0, 0, 0)$. □

Theorem (Rigidity theorem [4]) *Let E be the analytic GLA and μ be an element of $\mathcal{M}$ such that $H^1(E, D_\mu) = \{0\}$, then μ is rigid, i.e., there exist $N(\mu, \mathcal{M})$ and $N(C^0)$ such that $\mu' \in N(\mu, \mathcal{M})$ implies that $\mu' = \exp(a) \cdot \mu$ for some $a \in N(C^0)$*

Proof. From the lemma, there exist $N(\mu, E^1)$, $N(C^0)$, $N_1(C^1)$ such that

$$\text{if } \mu' \in N(\mu, \mathcal{M}), \ \text{ then } \mu' = \exp(a) \cdot (\mu + b)$$

where $a \in N(C^0),\ b \in N_1(C^1)$. Since $\mathcal{M}$ is invariant under G, we have

$$\mu' \in \mathcal{M} \text{ if and only if } \mu + b \in \mathcal{M}.$$

Now define

$$P : C^1 \to E^2$$

by $P(b) = [\mu + b, \mu + b]$. Then we have

(1) $\mu + b \in \mathcal{M}$ if and only if $P(b) = 0$

(2) $dP_0(b) = 2D(b)$

$\because dP_0(b) = \frac{d}{dt}\Big|_{t=0} P(tb) = \frac{d}{dt}\Big|_{t=0} [\mu + tb, \mu + tb] = 2[\mu, b] = 2D(b)$

Therefore dP_0 is injective, hence there exist $N_2(C^1)$ such that $P|_{N_2(C^1)}$ is injective.

Put $N(C^1) = N_1(C^1) \cap N_2(C^1)$, then we have that for $b \in N(C^1)$, $\mu + b \in \mathcal{M}$ if and only if $b = 0$. This proves. □

Fact Let S be an irreducible algebraic set in $\mathbb{C}^n$ (resp. $\mathbb{R}^n$) and S_0 be the set of simple points of S. Then

(a) S_0 is a Zariski dense open subset of S (see [1] p.40)

(b) For any $x \in S_0$, there exist a neighborhood $N(x, \mathbb{C}^n)$(resp. $N(x, \mathbb{R}^n)$) such that

$$S \cap N(x, \mathbb{C}^n) \ (\text{resp.} S \cap N(x, \mathbb{R}^n))$$

is a complex submanifold of $N(x, \mathbb{C}^n)$(resp. $N(x, \mathbb{R}^n)$).

(c) For any $x \in S_0$, $T_x S_0 \cong T_x S$, hence $\dim S_0 = \dim S$.

(d) For any Zariski open subset $U \subset S$, U is connected in the topology induced on U from the usual topology of $\mathbb{C}^n$.(In the case of $\mathbb{R}^n$, U has finite number of components)

Lemma ([5] p.104) *If an algebraic group G acts on a reducible quasiprojective variety S, then the elements of G^0, the identity component, transform each irreducible component of S into itself.*

Proposition ([4]) *Let S be an algebraic set in $\mathbb{C}^n$*(resp. $\mathbb{R}^n$) *and G be a Lie subgroup of $GL(\mathbb{C}^n)$*(resp. $GL(\mathbb{R}^n)$) *such that $G \cdot S \subset S$. Suppose there exists a point $s_0 \in S$ whose orbit Gx_0 is an open subset of S. Then Gx_0 is a Zariski open subset of S*(resp. *one component of a Zariski open subset of S*).

Proof. May assume that G is connected.

Each irreducible component of S is stable under the action of G.

($\because$ Let $\tilde{G}$ be an algebraic hull of G(Zariski closure of G), then $G \subset \tilde{G}^0$.

Now using the lemma above.)
Now may assume S is irreducible.
Let k be the dimension of S as an algebraic set, and $\mathfrak{g}$ be the Lie algebra of G.
For any $x \in \mathbb{C}^n$, the orbit Gx is a complex submanifold of $\mathbb{C}^n$.
Then $T_x Gx = \mathfrak{g}(x) := \{T(x) | T \in \mathfrak{g}\}$
If $x \in S$, then $\dim \mathfrak{g}(x) \leq k$.
Put $U = \{x \in S | \dim \mathfrak{g}(x) = k\}$, then

(1) U is a Zariski dense open supset of S(see [1] p.40)
(2) U is nonempty ($\because$ given point x_0)
(3) $U \subset S_0$ where S_0 is the set of simple points

For $x \in U$, Gx is an open subset of U.
Hence, the orbit of G divide U into disjoint open subsets.
Since U is connected, there exist unique orbit of G on U. $\square$

Theorem ([4]) *Let E be the analytic GLA with the base field* $\mathbb{C}$(resp. $\mathbb{R}$) *and let $\mu \in \mathcal{M}$ be such that $H^1(E, D_\mu) = \{0\}$. Then the orbit $G\mu$ is a Zarisk open subset of $\mathcal{M}$*(resp. *is one component of a Zariski open subset of $\mathcal{M}$*).

References

[1] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.

[2] Albert Nijenhuis. On a class of common properties of some different types of algebras. *Nieuw Arch. Wisk. (3)*, 17:17–46, 1969.

[3] Albert Nijenhuis. On a class of common properties of some different types of algebras. II. *Nieuw Arch. Wisk. (3)*, 17:87–108, 1969.

[4] Albert Nijenhuis and Jr. Richardson, R. W. Cohomology and deformations in graded Lie algebras. *Bull. Amer. Math. Soc.*, 72:1–29, 1966.

[5] A. L. Onishchik and È. B. Vinberg. *Lie groups and algebraic groups*. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.