Convex hyperbolic domains and Clans

Yuncherl Choi

November 2, 1999

Contents

1. Convex hyperbolic cones **1**

2. Convex hyperbolic domains **3**

3. Algebra of a convex hyperbolic homogeneous domain **5**

4. Decomposition of clan **9**

5. Examples of clans **13**

1. Convex hyperbolic cones

Let V be a finite dimensional real vector space and V^* be the dual vector space of V.

Definition. (1) A open subset $\Omega \subset V$ is called a <u>*cone*</u> if for all $x \in \Omega$, tx is contained in Ω where t is a positive real number.

(2) A subset $\Omega \subset V$ is called <u>*convex*</u> if for any two points $x, y \in \Omega$ the line segment $tx + (1-t)y,\ 0 \leq t \leq 1$ contained in Ω.

(3) A subset $\Omega \subset V$ is called <u>*hyperbolic*</u> if it doesn't any straight full line.

(4) For any cone Ω, the <u>*dual cone*</u> Ω^* is the set of elements $y \in V^*$ such that $y(x) > 0$ for all $x \in \bar{\Omega},\ x \neq 0$.

Proposition 1 *Let Ω be a cone in a vector space V. Then we have followings:*

(1) *For any nonempty cone Ω, if $\Omega^* \neq \emptyset$, then $\Omega^{**} = \Omega$ under the natural identification of V^{**} and V.*

(2) *Ω is hyperbolic if and only if $\Omega \cap -\Omega = \emptyset$.*

(3) *Ω is hyperbolic if and only if $\Omega^* \neq \emptyset$.*

Definition. (1) A cone Ω is called <u>*self dual*</u> if $\Omega = \Omega^*$.

(2) *Aut*$(\Omega) = \{g \in GL(V) | g\Omega \subset \Omega\}$ is called the <u>*automorphism group*</u> of Ω.

(3) A cone Ω is called <u>*homogeneous*</u> if *Aut*(Ω) acts on Ω transitively.

(4) A cone Ω is called <u>*symmetric*</u> if it is homogeneous and self dual.

Proposition 2 (1) *If Ω is a convex hyperbolic cone, then Aut$(\Omega^*) =$ Aut$(\Omega)^*$.*

(2) *For any $x \in \Omega$, Aut$(\Omega)_x$ is compact. Furthermore, if H is a compact subset of Aut(Ω), then there exists an element $x \in \Omega$ such that $H \subset$ Aut$(\Omega)_x$.*

Definition. Let Ω be a convex hyperbolic cone.

(1) The <u>*characteristic function*</u> ϕ of Ω is defined for $x \in \Omega$ by

$$\phi(x) = \int_{\Omega^*} e^{-y(x)} dy,$$

where dy denotes the Euclidian measure on V.

(2) For any point $x \in \Omega$, define <u>*dual point*</u> x^* as a differential of $-\log \phi$ at x, i.e.

$$x^* = -D \log \phi(x).$$

Proposition 3 *Let Ω be a convex hyperbolic cone. Then*

(1) *The characteristic function ϕ is uniformly convergent for x in any compact subset of Ω.*

(2) $\phi(gx) = \dfrac{\phi(x)}{|\deg(g)|}$

(3) *For any sequence $\{x_m\} \subset \Omega$, if $\{x_m\}$ converges to x_0 as $m \to \infty$ where $x_0 \in \partial\Omega$, then we have:*

$$\lim_{m\to\infty} \phi(x_m) = \infty.$$

(4) *$D^2 \log \phi(x)$ is positive definite at each point $x \in \Omega$.*

(5) *The map $x \mapsto x^*$ is a bijection of Ω onto Ω^*.*

(6) *$x^*(x) = n$ for all $x \in \Omega$ where $n = \dim V$.*

(7) *The map $x \mapsto x^*$ has a unique fixed point. Hence we have $\Omega \cap \Omega^* \neq \emptyset$.*

(8) *For all $x \in \Omega$ and $g \in$ Aut(Ω), $(gx)^* = (g^*)^{-1} x^*$.*

(9) *If Ω is homogeneous, then $\phi(x)\phi(x^*) =$ const and $(x^*)^* = x$.*

Proposition 4 *Let Ω be a symmetric cone. Then*

(1) *Aut$(\Omega) =$ Aut$(\Omega)^*$.*

(2) *If x_0 be a unique point in Ω such that $x_0^* = x_0$, then*

$$Aut(\Omega) \cap O(V) = Aut(\Omega)_{x_0},$$

where $O(V)$ is the orthogonal group of V.

(3) *The map $x \mapsto x^*$ is an isometry of Riemannian manifold Ω, i.e. Ω is a Riemannian symmetric space.*

2. Convex hyperbolic domains

Let U be a finite dimensional affine space.

Definition. (1) Any open subset $\Omega \subset U$ is called a *domain*.

(2) *Aut*$(\Omega) = \{g \in$ *Aff*$(U) | g\Omega \subset \Omega\}$ is called the *automorphism group* of the affine domain Ω.

(3) A domain Ω is called *homogeneous* if *Aut*(Ω) acts on Ω transitively.

Proposition 5 *Let Ω be a convex hyperbolic homegeneous domain in U and $G \subset$ Aut(Ω) be a group that acts transitively on Ω. If there exists an affine subspace $U_1 \subset U$ such that it is invariant under the linear parts of all $g \in G$, then $(x_0 + U_1) \cap \Omega$ is a convex hyperbolic homogeneous domain in $x_0 + U_1$ for all $x_0 \in \Omega$.*

Proof. Let G_1 be the subgroup of G formed by the element $g \in G$ that leave the affine subspace $x_0 + U_1$ invariant. Since the subspace U_1 is invariant under the linear part of all $g \in G$, it follows that every transformation $g \in G$ for which $gx_0 \in x_0 + U_1$ belongs to G_1. It is now obvious that G_1 acts transitively on $(x_0 + U_1) \cap \Omega$. $\square$

Definition. Given a convex hyperbolic domain Ω in an n-dimensional affine space U, imbed the space U in the $(n+1)$-dimensional vector space V as a hyperplane not passing through 0. Then we can define the cone $\tilde{\Omega}$ as following:

$$\tilde{\Omega} = \{tx \in V | x \in \Omega,\ t > 0\}.$$

In this case, the cone $\tilde{\Omega}$ is said to be *fitted* onto Ω.

Remark. (1) The cone $\tilde{\Omega}$ is clearly hyperbolic and convex.

(2) If Ω is homogeneous, then $\tilde{\Omega}$ is also.

(3) The map $\phi = \phi_{\tilde{\Omega}}|_{\Omega}$ is a characteristic fuction of Ω.

(4) Every nonempty cross section of a convex hyperbolic cone by a hyperplane not passing through 0 is a convex hyperbolic domain.

Proposition 6 *Let Ω be a convex hyperbolic homegeneous domain in U and G be a Lie subgroup of Aut(Ω) that acts transitively on Ω. Then*

(1) *G_{x_0} is a maximal compact subgroup of G for all $x_0 \in \Omega$.*

(2) *The number of connected components of the group G is finite.*

(3) *The identity component Aut$_0(\Omega)$ of Aut(Ω) coincides with the idnetity component of its normalizer in the complete affine group Aff(U). Moreove it coincides with the idnetity component of an affine algebraic group.*

(4) *Let K be the stability subgroup of some point $x_0 \in \Omega$ and T be a maximal connected triangular subgroup of* (Ω). *Then we have*

$$Aut(\Omega) = K \cdot T,$$

where $K \cap T = \{id\}$. The subgroup T acts simply transitively in Ω.

(5) *For every convex hyperbolic homogeneous cone $\tilde{\Omega}$, there exists a convex hyperbolic homogeneous domain Ω of dimension one less such that $\tilde{\Omega}$ is isomorphic to the cone fitted onto Ω.*

Proof. (1) Let U_{x_0} be the vector space such that x_0 is zero point. Then $G_{x_0} \subset GL(U_{x_0})$, and G_{x_0} leave the bounded subset $(\Omega - x_0) \cap (x_0 - \Omega) \subset U_{x_0}$. Hence we conclude that G_{x_0} is compact.

Suppose now that K is a compact subgroup of G. We take a bounded open set $M \subset \Omega$ and consider the set

$$K \cdot M = \{gx | g \in K, \ x \in M\},$$

which is invariant under all the transformations in K. Since K is compact, the set $K \cdot M$ is bounded. Its center of gravity x_1 lies in Ω and is a fixed point for K. Thus $K \subset G_{x_1}$ and therefore is conjugate to a subgroup K' of G_{x_0}. $\square$

Question. Let Ω be a convex hyperbolic homegeneous domain in U and G be a Lie subgroup of *Aut*(Ω) that acts transitively on Ω. Then

(1) Is the identity component G_{x_0} of G is algebraic?

(2) Does G have a decomposition as proposition 6(4)?

3. Algebra of a convex hyperbolic homogeneous domain

Let Ω be a convex hyperbolic homogeneous domain in an affine space U. By proposition 6(4), there exists a triangular affine subgroup G of *Aut*(Ω) such that acts simply transitvely on Ω.

Choose a point x_0 in Ω such that $\phi(x_0) = 1$. The map ev_{x_0} which is defined by $ev_{x_0}(g) = g \cdot x_0$ is an affine diffeomorphism from G onto Ω. With this evaluation map, we define the affine flat connection ∇ on G by

$$\nabla = ev_{x_0}^* D,$$

where D is the usual connection on Ω induced from the directional derivative on the affine space U. Since the connection ∇ is left invariant, it induces a multiplication on the Lie algebra $\mathfrak{g} = Lie(G)$:

$$X \cdot Y = \nabla_X Y.$$

This multiplication is compatible with Lie multiplication and left symmetric from the flat condition of connection.

Definition. The LSA $\mathcal{A} = (\mathfrak{g}, \cdot)$ is called the *algebra of the convex hyperbolic homogeneous domain* Ω with respact to $x_0 \in \Omega$.

The characteristic function $\tilde{\phi}$ on G is indeced from the characteristic function ϕ on Ω by the evaluation map: for any $g \in G$,

$$\tilde{\phi}(g) = (ev_{x_0}^* \phi)(g) = \phi(g \cdot x_0) = \frac{1}{\det(g)} \phi(x_0) = \frac{1}{\det(g)}.$$

Hence we have for any $X \in \mathfrak{g}$,

$$\log \tilde{\phi}(\exp X) = \log \frac{1}{\det(\exp X)} = -\mathrm{tr} X (= -\mathrm{tr} \lambda_X).$$

Now define the 1-form $s(X) = \mathrm{tr} X$. Clearly it is a Lie algebra homomorphism, hence we have that for $X, Y \in \mathfrak{g}$

$$s(XY) = s(YX).$$

By the way, $\exp X = id + X + \frac{1}{2!} X^2 + \frac{1}{3!} X^3 + \cdots$. Hence we have the following by the Taylor expansion:

$\log \tilde{\phi}(\exp X)$

$$
\begin{aligned}
= \quad & \nabla_X \log \tilde{\phi}(id) + \frac{1}{2!}[\nabla_{X^2} \log \tilde{\phi}(id) + \nabla_X^2 \log \tilde{\phi}(id)] \\
& +[\frac{1}{3!}\nabla_{X^3} \log \tilde{\phi}(id) + \frac{1}{2!}\nabla_X \nabla_{X^2} \log \tilde{\phi}(id) + \frac{1}{2!}\nabla_{X^2}\nabla_X \log \tilde{\phi}(id) + \frac{1}{3!}\nabla_X^3 \log \tilde{\phi}(id)] \\
& + \cdots .
\end{aligned}
$$

From this, we obtain

$$
\begin{aligned}
s(X) &= -\nabla_X \log \tilde{\phi}(id) \\
s(X^2) &= \nabla_X^2 \log \tilde{\phi}(id) \\
s(XY) &= \nabla_X \nabla_Y \log \tilde{\phi}(id) \\
s(X^3) &= -2\nabla_X^3 \log \tilde{\phi}(id) \\
\vdots & \qquad \vdots \qquad .
\end{aligned}
$$

Definition. A LSA $\mathcal{A}$ is called a <u>*clan*</u> if the operator of left multiplication have only real eigenvalue and there exists a Lie algebra homomorphism $s : \mathcal{A}^- \to \mathbb{R}$ such that the induced bilinear form $<,>$ defined by $< x, y >= s(x \cdot y)$ is positive definite.

Proposition 7 (1) *The algebra of the convex hyperbolic homogeneous domain is a clan.*

(2) *The algebra of the convex hyperbolic homogeneous cone is a clan with identity.*

(3) *The algebra of the cone $\tilde{\Omega}$ fitted onto the convex hyperbolic homogeneous domain Ω is obtained from the algebra of the domain Ω by adjunction of an identity element.*

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r, "-s"] \arrow[d, "\exp"'] & \Re \arrow[d, shift left, "\exp"] \\
G \arrow[r, "\tilde{\phi}"] \arrow[d, "ev_{x_0}"'] & \Re_+ \arrow[u, shift left, "\log"] \\
\Omega \arrow[ur, "\phi"] &
\end{tikzcd}
$$

Conversely, from the clan $\mathcal{A}$ we obtain a convex hyperbolic homogeneous domain by evauating the Lie group G which is simply connected and $\mathcal{A}^- = Lie(G)$ with a point in the affine space U whose dimension is same as $\mathcal{A}$.

Proposition 8 [2] *A connected linear Lie group G(*resp. *linear Lie algebra $\mathfrak{g}$) is triangular if and only if every element of G(*resp. $\mathfrak{g}$*) has only real eigenvalues.*

Proof. It is enough to show that the connected linear Lie group G whose elements have only real eigenvalues is solvable. If this were not the case, G would contain a nontrivial semisimple subgroup. Since every nontrivial semisimple linear group contains a nontrivial compact subgroup, G contains a nontrivial compact subgroup, which is impossible. $\square$

At first, let $\mathcal{A}$ be a clan with an identity e and G be a simply connected Lie group such that its Lie algebra is $\mathcal{A}^-(=\mathfrak{g})$. We consider the subset $\mathcal{A}_0$ of $\mathcal{A}$ defined by

$$\mathcal{A}_0 = \{x \in \mathcal{A} | s(x) = 0\}.$$

Then $\mathcal{A}_0^-$ is an ideal of $\mathcal{A}^-$ since $s([x, y]) = 0$ for all $x, y \in \mathcal{A}^-$. Hence the Lie algebra $\mathcal{A}^-$ is the direct sum of the Lie ideals, i.e.

$$A^- = \mathcal{A}_0^- \oplus \{te\}.$$

From this, we say that the Lie group G is a direct product of the normal subgroups:

$$G = G_0 \times \{t \cdot id\}_{t>0},$$

where the Lie algebra of G_0 is $\mathcal{A}_0^-$.

Let $\Omega = G \cdot e$ be the orbit of G in $\mathcal{A}$ as a vector space and $S = G_0 \cdot e$ be the orbit of the normal subgroup G_0 at e. Then we see that

$$\Omega = \bigcup_{t>0} tS.$$

We consider the tangent space T_eS, then

$$T_eS = \{e + a | s(a) = 0\} = \{x | s(x) = s(e)\}.$$

If $a \in \mathcal{A}_0$, then

$$(\exp a) \cdot e = e + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \cdots.$$

From this we obtain the inequality: for surficiently small $a \in \mathcal{A}_0$,

$$s((\exp a)e) \geq s(e).$$

This says that S is convex at e and therefore it is convex at all points since S is homogeneous. Hence we conclude that the cone Ω is convex.

From the convecity of S, $s(x) \geq s(e)$ for all $x \in S$ and $s(tx) \geq ts(e) > 0$. This shows that for all $x \in \Omega$

$$s(x) > 0.$$

We define the map from G to $\mathcal{A}^*$ the dual vector space of $\mathcal{A}$ as the dual evaluation map at $s \in \mathcal{A}^*$: for all $x \in \mathcal{A}$ and $g \in G$

$$(g^* \cdot s)(x) = s(g \cdot x).$$

Then the map is regular at identity and the differential of this map is an isomorphism from $\mathcal{A}^-$ onto $\mathcal{A}^*$. Hence $G^* \cdot s$ fill a neiborhood of s and g^*s is positive on Ω for all $g \in G$. From this we conclude that Ω is hyperbolic.

Now we consider the clan $\mathcal{A}$ which does not have an identity. By adjunction of an identity e, we obtain an algebra $\tilde{\mathcal{A}}$ from $\mathcal{A}$:

$$\tilde{\mathcal{A}} = \mathcal{A} \oplus \{te\}.$$

Then $\tilde{\mathcal{A}}$ is a clan:

(1) $\tilde{\mathcal{A}}$ is an LSA.

(2) The operator of left multiplication have only real eigenvalues.

Let s be a linear form in $\mathcal{A}$ such that it is symmetric and the induced 2-form $< x, y >= s(xy)$ is positive definite. Let α be the maximum value of s on the unit sphere $s(a^2) = 1$. Then for all $a \in \mathcal{A}$, we have

$$(s(a))^2 \leq \alpha^2 s(a^2).$$

Now extend the form s to $\tilde{\mathcal{A}}$ by putting

$$s(e) = 1 + \alpha^2.$$

(3) s is symmetric in $\tilde{\mathcal{A}}$.

(4) The induced 2-form is positive definite in $\tilde{\mathcal{A}}$.

The simply connected Lie group $\tilde{G}$ whose Lie algebra $\tilde{\mathcal{A}}^-$ is decomposed of normal subgroups:

$$\tilde{G} = G \times \{t\ id\}_{t>0}$$

where G is the simply connected Lie group and the Lie algebra of G is $\mathcal{A}^-$. Then since the orbit of $\tilde{G}$ is a convex hyperbolic cone $\tilde{\Omega}$, the orbit of G, $e + \mathcal{A} \cap \tilde{\Omega}$, is a convex hyperbolic domain in $e + \mathcal{A}$. Hence we conclude that the orbit of G at 0, $G \cdot 0$, is a convex hyperbolic domain in $\mathcal{A}$.

Proposition 9 *There is a one to one corespondence between the set of clans and the set of convex hyperbolic homogeneous domains. In particular, there is also a one to one corespondence between the set of clans with identity and the set of convex hyperbolic homogeneous cones.*

Remark. { Clan's }/$\sim \overset{1-1}{\Longleftrightarrow}$ { Convex hyperbolic homogeneous domain's}/ $\sim$

4. Decomposition of clan

Definition. A LSA $\mathcal{A}$ is called a <u>*clan*</u> if the operator of left multiplication have only real eigenvalue and there exists a Lie algebra homomorphism $s : \mathcal{A}^- \to \mathbb{R}$ such that the induced bilinear form $<,>$ defined by $< x, y >= s(x \cdot y)$ is positive definite.

In this definition, since s is Lie algebra homomorphism, the bilinear form $<,>$ is symmetric.

Let $\{e_1, \cdots, e_n\}$ be an orthornormal basis with respect to $<,>$ and put $u = \sum_{i=1}^{n} s(e_i)e_i$, then we have

$$s(ux) = s(x) = s(xu), \tag{1}$$

for all $x \in \mathcal{A}$. Furthermore, u is an unique element of $\mathcal{A}$ satisfying (1) because the bilinear form $<,>$ is nondegenerate.

Proposition 10 (1) *ρ_u is self-adjoint with respect to $<,>$.*

(2) *u is an idempotent, i.e. $u \cdot u = u$. In this case $u \in \mathcal{A}$ is called principal idempotent.*

(3) $\rho_u^2 = \rho_u$.

Proof. (1) For any $x, y \in \mathcal{A}$,

$$\begin{aligned} < x, \rho_u y > - < y, \rho_u x > &= s(x(yu)) - s(y(xu)) \\ &= s([\lambda_x, \lambda_y](u)) \\ &= s(\lambda_{[x,y]}(u)) \\ &= s([x, y]u) = s([x, y]) = 0. \end{aligned}$$

This proves.

(2) For any $x \in \mathcal{A}$, we have

$$< uu, x >=< \rho_u(u), a >=< u, \rho_u(x) >= s(u(xu)) = s(xu) =< u, x > .$$

Since the bilinear form $<,>$ is nondegenerate, we prove.

(3) Let $P := \rho_u - \rho_u^2 = \rho_{uu} - \rho_u^2 = [\lambda_u, \rho_u]$. Since ρ_u is self-adjoint, P is also. Moreover, P is commute with ρ_u. Now,

$$\begin{aligned} \mathrm{tr}P^2 &= \mathrm{tr}[\lambda_u, \rho_u]P = \mathrm{tr}(\lambda_u\rho_u P - \rho_u\lambda_u P) \\ &= \mathrm{tr}\lambda_u\rho_u P - \mathrm{tr}\rho_u\lambda_u P = \mathrm{tr}\lambda_u\rho_u P - \mathrm{tr}\lambda_u P\rho_u \\ &= \mathrm{tr}(\lambda_u\rho_u P - \lambda_u P\rho_u) = \mathrm{tr}\lambda_u[\rho_u, P] \\ &= 0. \end{aligned}$$

Thereofre we conclude that $P = 0$. $\square$

Proposition 10(3) says that ρ_u is an orthogonal projector onto a subspace $\mathcal{A}_0 \subset \mathcal{A}$, hence we have an orthogonal decomposition

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{K},$$

where $\mathcal{K}$ is the kernel of ρ_u. And it is called *principal decomposition* of $\mathcal{A}$.

Remark. If $\mathcal{A}$ has an identity e, then by the uniqueness of pricipal idempotent, we have $u = e$. Hence we conclude that $\mathcal{A}_0 = \mathcal{A}$.

Proposition 11 (1) $\lambda_u + \lambda_u^* = \rho_u + id$.
(2) $\lambda_u = \frac{1}{2}(\rho_u + id)$, *i.e.* $\lambda_u = id$ *on* $\mathcal{A}_0$ *and* $\lambda_u = \frac{1}{2}id$ *on* $\mathcal{K}$.

Proof. (1) For any $x, y \in \mathcal{A}$, we have

$$\begin{aligned} < (\lambda_u + \lambda_u^*)(x), y > &= < \lambda_u x + \lambda_u^* x, y > = < ux, y > + < x, uy > \\ &= s((ux)y + x(uy)) = s(u(xy) + (xu)y) \\ &= s(xy) + s((xu)y) = < x, y > + < \rho_u x, y > \\ &= < (\rho_u + id)(x), y > . \end{aligned}$$

Hence by the nondegneracy of $<, >$, we prove.
(2) Put $h = \frac{1}{2}(\rho_u + id) - \lambda_u$, then h is skew-symmetric from the proposition 10. It means that h has only pure imaginary eigenvalues. However, we have $[\rho_u, \lambda_u] = 0$ from proposition 10, hence $[\rho_u + id, \lambda_u] = 0$, and $\rho_u + id,\ \lambda_u$ have real eigen values. By the Lie Theorem, h has real eigen values. Therefore we conclude that $h = 0$. $\square$

Lemma 1 *Let $\mathcal{A}$ be a LSA and suppose that*

$$\lambda_z x = t_x x,\ \lambda_z y = t_y y,\ \rho_z x = s_x x,$$

where $x, y, z \in \mathcal{A}$ and $t_x, t_y, s_x \in \mathbb{R}$. Then we have

$$\lambda_z(xy) = (t_x + t_y - s_x)xy.$$

Proof.

$$\begin{aligned} \lambda_z(xy) &= z(xy) = (zx)y + x(zy) - (xz)y \\ &= t_x xy + t_y xy - s_x xy = (t_x + t_y - s_x)xy \end{aligned}$$

□

By applying the lemma, we have the followings:

$$\lambda_u(xy) = \begin{cases} xy & \text{if } x, y \in \mathcal{A}_0 \\ \frac{1}{2}xy & \text{if } x \in \mathcal{A}_0, y \in \mathcal{K} \\ \frac{3}{2}xy & \text{if } x \in \mathcal{K}, y \in \mathcal{A}_0 \\ xy & \text{if } x, y \in \mathcal{K} \end{cases}$$

Hence the multiplcation table is given as followings:

$\cdot$	$\mathcal{A}_0$	$\mathcal{K}$
$\mathcal{A}_0$	$\mathcal{A}_0$	$\mathcal{K}$
$\mathcal{K}$	0	$\mathcal{A}_0$

$[,]$	$\mathcal{A}_0^-$	$\mathcal{K}^-$
$\mathcal{A}_0^-$	$\mathcal{A}_0^-$	$\mathcal{K}^-$
$\mathcal{K}^-$	$\mathcal{K}^-$	0

In fact, the multiplication restricted to $\mathcal{K}$ is commutative since for any $x, y \in \mathcal{K}$ we have

$$xy = \rho_u(xy) = (xy)u = x(yu) + (yx)u - y(xu) = (yx)u = \rho_u(yx) = yx.$$

Hence we conculude that the associated Lie algebra $\mathcal{K}^-$ of $\mathcal{K}$ is abelian.

Corollary *Let $\mathcal{A}$ be a clan and assume its associated Lie algebra $\mathcal{A}^-$ is unimodula, then the principal idempotent u is an identity of $\mathcal{A}$.*

Proof. From the principal decomposition, we have

$$\frac{1}{2}\dim \mathcal{K} = \mathrm{tr}\lambda_u|_{\mathcal{K}} = \mathrm{tr}\rho_u|_{\mathcal{K}} = 0.$$

Hence $\dim \mathcal{K} = 0$, i.e. $\rho_u = \lambda_u = id$. □

Proposition 12 *Let $\mathcal{A}$ be a clan with the identity e. Then there exists a subalgebra $\mathcal{A}_1$ and an idempotent $v \in \mathcal{A}$ such that*

$$\mathcal{A} = \mathcal{A}_1 + \{tv\},$$

and $u = e - v$ is the principal idempotent of the clan $\mathcal{A}_1$.

Proof. From the proposition 6(5), the cone $\tilde{\Omega}$ obtained from $\mathcal{A}$ is isomorphic to the cone fitted onto a convex hyperbolic domain Ω. If $\mathcal{A}_1$ is the algebra of Ω, then $\mathcal{A}$ is obtained from $\mathcal{A}_1$ by adjunction of an identity. Let u be the principal idempotent of the clan $\mathcal{A}_1$. We put $v = e - u$. Then the decomposition follows. □

Lemma 2 *Let L be a unimodular Lie algebr and L_0 be an ideal of L. Then L_0 is also unimodular.*

Proof. For any $x \in L_0$, $\mathrm{tr}ad_x|_{L_0} = \mathrm{tr}ad_x$. $\square$

Corollary *Let $\mathcal{A}$ be a clan and assume its associated Lie algebra $\mathcal{A}^-$ is unimodula, then there exist the set of idempotents $\{e_1, \cdots, e_n\}$ where n is the dimension of $\mathcal{A}$ and satisfies*

$$\mathcal{A} = \{t_1 e_1\} + \cdots + \{t_n e_n\}.$$

Proposition 13 *Let $\mathcal{A}$ be a clan with an identity. Then*

(1) *$\mathcal{A}$ has a normal decomposition, i.e.*

$$\mathcal{A} = \sum_{i \leq j} \mathcal{A}_{ij} \quad (i, j = 1, \cdots, m)$$

where (i) *$\mathcal{A}_{ii}$ is 1-dimensional and generated by idempotent e_i for all $i = 1, \cdots, m$.*

(ii) *the operaters $\lambda_i = \lambda_{e_i}$, $\rho_i = \rho_{e_i}$ leave the space $\mathcal{A}_{jk}$ invariant with the following table*

	$\mathcal{A}_{ii}$	$\mathcal{A}_{ji}(j \neq i)$	$\mathcal{A}_{ik}(k \neq i)$	$\mathcal{A}_{jk}(j, k \neq i)$
λ_i	1	$\frac{1}{2}$	$\frac{1}{2}$	0
ρ_i	1	1	0	0

(2) *The multiplication law satisfies the followings:*

$$\begin{aligned} \mathcal{A}_{ij}\mathcal{A}_{kl} &= 0 \ \text{ if } j \neq k, l \\ \mathcal{A}_{ij}\mathcal{A}_{jk} &\subset \mathcal{A}_{ik} \\ \mathcal{A}_{ij}\mathcal{A}_{kj} &\subset \mathcal{A}_{ik} \text{ or } \mathcal{A}_{ki}. \end{aligned}$$

(3) *The any idempotent v of $\mathcal{A}$ can be expressed in the form*

$$v = e_{i_1} + \cdots + e_{i_k},$$

where $i_1 < \cdots < i_k$.

5. Examples of clans

< 2-dimensional Case >

(1) $\dim A_0 = 1,\ A_0 =< e_1 >,\ u = e_1$

	e_1	e_2
e_1	e_1	$\frac{1}{2}e_2$
e_2	0	te_1

$[,]$	e_1	e_2
e_1	0	$\frac{1}{2}e_2$
e_2	$-\frac{1}{2}e_2$	0

For $x = ae_1 + be_2$, $\lambda_x = \begin{pmatrix} a & tb \\ 0 & \frac{1}{2}a \end{pmatrix}$, $\rho_x = \begin{pmatrix} a & tb \\ \frac{1}{2}b & 0 \end{pmatrix}$, $p(x) = 1 + a - \frac{1}{2}tb^2$

For $x = a_1e_1 + b_1e_2,\ y = a_2e_1 + b_2e_2,\ \ xy = (a_1a_2 + tb_1b_2)e_1 + \frac{1}{2}a_1b_2e_2$
$s_\lambda(xy) = \frac{3}{2}(a_1a_2 + tb_1b_2)$, $s_\rho(xy) = a_1a_2 + tb_1b_2$

(2) $\dim A_0 = 2,\ A_0 = A,\ u = e_1 + e_2,\ A_{11} =< e_1 >,\ A_{22} =< e_2 >$

	e_1	e_2
e_1	e_1	0
e_2	0	e_2

, abelian

For $x = ae_1 + be_2$

$$\lambda_x = \rho_x = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad p(x) = (1+a)(1+b)$$

For $x = a_1e_1 + b_1e_2,\ y = a_2e_1 + b_2e_2,\ xy = a_1a_2e_1 + b_1b_2e_2$
$s_\lambda(xy) = s_\rho(xy) = a_1a_2 + b_1b_2$

< 3-dimensional Case >

(1) $\dim A_0 = 1,\ A_0 =< e_1 >,\ u = e_1$

	e_1	e_2	e_3
e_1	e_1	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$
e_2	0	re_1	se_1
e_3	0	se_1	te_1

$[,]$	e_1	e_2	e_3
e_1	0	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$
e_2	$-\frac{1}{2}e_2$	0	0
e_3	$-\frac{1}{2}e_3$	0	0

For $x = ae_1 + be_2 + ce_3$,

$$\lambda_x = \begin{pmatrix} a & rb+sc & sb+tc \\ 0 & \frac{1}{2}a & 0 \\ 0 & 0 & \frac{1}{2}a \end{pmatrix},\ \rho_x = \begin{pmatrix} a & rb+sc & sb+tc \\ \frac{1}{2}b & 0 & 0 \\ \frac{1}{2}c & 0 & 0 \end{pmatrix},$$

$p(x) = 1 + a - \frac{1}{2}(rb^2 + 2sbc + tc^2)$

For $x = a_1e_1 + b_1e_2 + c_1e_3,\ y = a_2e_1 + b_2e_2 + c_2e_3$

$$s_\lambda(xy) = 2(a_1a_2 + rb_1b_2 + sb_1c_2 + sc_1b_2 + tc_1c_2)$$
$$s_\rho(xy) = a_1a_2 + rb_1b_2 + sb_1c_2 + sc_1b_2 + tc_1c_2$$

$rt - s^2 \neq 0 \ \Rightarrow$ nondegenerate
$r > 0, rt - s^2 > 0 \ \Rightarrow$ positive definit

(2) $\dim A_0 = 2,\ A_0 =< e_1, e_2 >,\ u = e_1 + e_2$

	e_1	e_2	e_3
e_1	e_1	0	$\frac{1}{2}e_3$
e_2	0	e_2	0
e_3	0	0	se_1

$[,]$	e_1	e_2	e_3
e_1	0	0	$\frac{1}{2}e_3$
e_2	0	0	0
e_3	$-\frac{1}{2}e_3$	0	0

For $x = ae_1 + be_2 + ce_3$
$\lambda_x = \begin{pmatrix} a & 0 & sc \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{2}a \end{pmatrix}$, $\rho_x = \begin{pmatrix} a & 0 & sc \\ 0 & b & 0 \\ \frac{1}{2}c & 0 & 0 \end{pmatrix}$,
$p(x) = (b+1)(a+1-\frac{1}{2}sc^2)$

For $x = a_1e_1 + b_1e_2 + c_1e_3,\ y = a_2e_1 + b_2e_2 + c_2e_3$

$$s_\lambda(xy) = \frac{3}{2}(a_1a_2 + sc_1c_2) + b_1b_2$$
$$s_\rho(xy) = a_1a_2 + b_1b_2 + sc_1c_2$$

if $s = 0$, then the inner product is degenerate.

(3) $\dim A_0 = 3,\ A_0 = A,\ u = e_1 + e_2,\ A_{11} =< e_1 >,\ A_{22} =< e_2 >$

	e_1	e_2	e_3
e_1	e_1	0	$\frac{1}{2}e_3$
e_2	0	e_2	$\frac{1}{2}e_3$
e_3	0	e_3	te_1

$[,]$	e_1	e_2	e_3
e_1	0	0	$\frac{1}{2}e_3$
e_2	0	0	$-\frac{1}{2}e_3$
e_3	$-\frac{1}{2}e_3$	$\frac{1}{2}e_3$	0

For $x = ae_1 + be_2 + ce_3$,

$$\lambda_x = \begin{pmatrix} a & 0 & tc \\ 0 & b & 0 \\ 0 & c & \frac{1}{2}a + \frac{1}{2}b \end{pmatrix},\ \rho_x = \begin{pmatrix} a & 0 & tc \\ 0 & b & 0 \\ \frac{1}{2}c & \frac{1}{2}c & b \end{pmatrix}$$

$p(x) = (1+b)((1+b)(1+a) - \frac{1}{2}tc^2)$

For $x = a_1e_1 + b_1e_2 + c_1e_3,\ y = a_2e_1 + b_2e_2 + c_2e_3$

$$s_\lambda(xy) = \frac{3}{2}(a_1a_2 + b_1b_2 + tc_1c_2)$$
$$s_\rho(xy) = a_1a_2 + 2b_1b_2 + tc_1c_2$$

(4) $\dim A_0 = 3,\ A_0 = A,\ u = e_1 + e_2 + e_3,$
$A_{11} =< e_1 >,\ A_{22} =< e_2 >\ \ A_{33} =< e_3 >$

	e_1	e_2	e_3
e_1	e_1	0	0
e_2	0	e_2	0
e_3	0	0	e_3

abelian

For $x = ae_1 + be_2 + ce_3,$

$$\lambda_x = \rho_x = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad p(x) = (1+a)(1+b)(1+c)$$

For $x = a_1e_1 + b_1e_2 + c_1e_3,\ y = a_2e_1 + b_2e_2 + c_2e_3,$

$$s_\lambda(xy) = s_\rho(xy) = a_1a_2 + b_1b_2 + c_1c_2$$

References

[1] Jacques Faraut and Adam Korányi. *Analysis on symmetric cones*. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.

[2] E. B. Vinberg. The Morozov-Borel theorem for real Lie groups. *Soviet Math. Dokl.*, 2:1416–1419, 1961.

[3] E.B. Vinberg. The theory of convex homogeneous cones. *Trans. of the Moscow Math. Soc.*, 12:340–403, 1963.