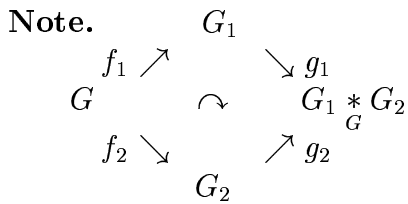


III.1 Amalgamated product

정의 1 G, G_1, G_2 : groups with $f_i : G \rightarrow G_i$, homomorphism for $i = 1, 2$.
 $F :=$ the free group generated by $G_1 \amalg G_2$. Denote by $x \cdot y$ the product in F .
 $R :=$ the normal subgroup of F generated by the words $(xy) \cdot y^{-1} \cdot x^{-1}$,
 where both $x, y \in G_i$, $i = 1, 2$ and $f_1(z) \cdot f_2(z)^{-1}$ for $z \in G$.
 The amalgamated product of G_1 and G_2 over G , $G_1 *_G G_2 := F/R$

Remark. $G_1 * G_2 := G_1 *_{{\{1\}}} G_2$



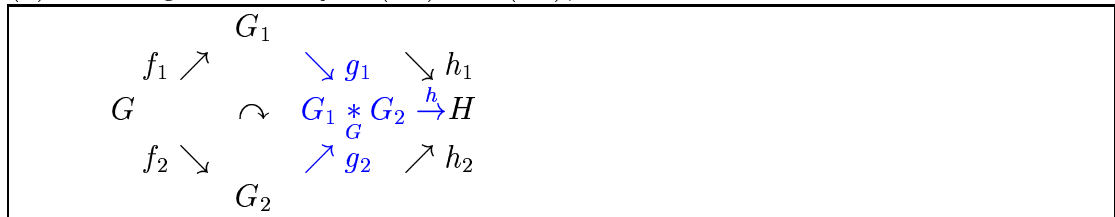
where the canonical map $g_i : G_i \xrightarrow{i} F \xrightarrow{p} F/R = G_1 *_G G_2$ is a homomorphism.

Then $g_1 f_1(z) = g_2 f_2(z)$ by the definition of F/R .

2. universal property of amalgamation

(a) Suppose $h_i : G_i \rightarrow H$ is a homomorphism with $h_1 f_1 = h_2 f_2$.
 Then $\exists!$ homomorphism $h : G_1 *_G G_2 \rightarrow H$ such that $h g_i = h_i$, $i = 1, 2$.

(b) If H is generated by $h_1(G_1) \cup h_2(G_2)$, then h is onto.



증명

(a) Define $h' : F \rightarrow H$ as the unique homomorphism determined by the condition $h'|_{G_i} = h_i$.

$R \subset \ker(h')$:

R 의 generator $(xy) \cdot y^{-1} \cdot x^{-1}, f_1(z) \cdot f_2(z)^{-1}$ 에 대해서만 check하면 된다.

$$h_1(xy)h_1(y^{-1})h_1(x^{-1}) = 1 \text{ for } x, y \in G_1,$$

$$h_2(xy)h_2(y^{-1})h_2(x^{-1}) = 1 \text{ for } x, y \in G_2 \text{ and}$$

$$h_1(f_1(z))h_2(f_2(z)^{-1}) = h_1f_1(z)(h_2f_2(z))^{-1} = 1 \quad (\because h_1f_1 = h_2f_2)$$

$\therefore \exists h : F/R \rightarrow H$.

G_i 상에서 $h' = h_i \Rightarrow h_i = h'i = hpi = hg_i \Rightarrow hg_i = h_i$

$$\begin{array}{ccc} G_i & \xrightarrow{i} F & \xrightarrow{h'} H \\ g_i \searrow \curvearrowright \downarrow p \curvearrowleft \nearrow h & & \\ & G_1 *_{G} G_2 & \\ & F/R & \end{array} \quad \text{diagram commutes from definitions of } g_i \text{ and } h.$$

Uniqueness is obvious since h is already determined on $g_i(G_i)$ and $g_1(G_1) \cup g_2(G_2)$ generates $G_1 *_{G} G_2$.

(b) Suppose $\forall a \in H$ is $a = a_1a_2 \cdots a_k$.

$$a_j \in h_1(G_1) \cup h_2(G_2) \Rightarrow a_j = h_i(x_j) \text{ for } i = i(a_j) = 1 \text{ or } 2$$

$$\Rightarrow a = \prod a_j = \prod h_i(x_j) = \prod hg_i(x_j) = h(\prod g_i(x_j))$$

□

숙제 9.

$$(1) G_1 *_{G} G_2 \cong G_1 * G_2 / \Gamma,$$

where Γ is the normal subgroup generated by $f_1(z) \cdot f_2(z)^{-1}$.

$$(2) f_1 : \text{onto} \Rightarrow g_2 : \text{onto}$$

$$(3) f_1 : \cong \Rightarrow g_2 : \cong$$

3. Group presentation.

$$\begin{aligned} \text{정리 1 } G_1 &= \langle x_1, \dots, x_n \mid r_1, \dots, r_k \rangle \\ G_2 &= \langle y_1, \dots, y_m \mid s_1, \dots, s_l \rangle \end{aligned}$$

$$G = \langle z_1, \dots, z_p \mid \text{any} \rangle \text{ with } G \begin{array}{c} f_1 \nearrow G_1 \\ f_2 \searrow G_2 \end{array}$$

$$\Rightarrow G_1 *_G G_2 \cong \langle x_1, \dots, x_n, y_1, \dots, y_m \mid r_1, \dots, r_k, s_1, \dots, s_l \text{ and } f_1(z_i)f_2(z_i)^{-1} \text{ for } i = 1, \dots, p \rangle$$

$$\text{증명 Let } \overline{G} = \langle x_1, \dots, x_n, y_1, \dots, y_m \mid r_1, \dots, r_k, s_1, \dots, s_l \text{ and } f_1(z_i)f_2(z_i)^{-1} \text{ for } i = 1, \dots, p \rangle.$$

Define $\phi : \overline{G} \rightarrow G_1 *_G G_2$ by

$$\begin{aligned} x_i &\mapsto g_1("x_i") \quad ("x_i" \in G_1) \\ y_i &\mapsto g_2("y_i") \quad ("y_i" \in G_2) \end{aligned}$$

$$\begin{aligned} r_i = x_{i_1} \cdots x_{i_k} \text{ in } \overline{G} &\Rightarrow \phi(r_i) = \phi(x_{i_1} \cdots x_{i_k}) = g_1("x_{i_1}") \cdots g_1("x_{i_k}") \\ &= g_1("x_{i_1} \cdots x_{i_k}") = g_1("r_i") \\ &= g_1(1) = 1 \end{aligned}$$

Similarly $\phi(s_j) = 1$

$$\phi(f_1(z_i)f_2(z_i)^{-1}) = g_1(f_1(z_i))g_2(f_2(z_i))^{-1} = 1$$

$\Rightarrow \phi$ is a homomorphism.

For the converse, use universal property:

아래 diagram에서 $h_1(x_i) = x_i, h_2(y_j) = y_j$ 라 두면 \overline{G} 의 정의에 의해 $h_1f_1(z_i) = h_2f_2(z_i)$ 가 성립하고 따라서 universal property에 의해

$$\exists! h : G_1 *_G G_2 \rightarrow \overline{G} \text{ s.t. } \begin{array}{ccccc} & & G_1 & & \\ & f_1 \nearrow & & \searrow g_1 & \searrow h_1 \\ \exists! h : G_1 *_G G_2 \rightarrow \overline{G} \text{ s.t. } & G & \hookrightarrow & G_1 *_G G_2 & \xrightarrow{h} \overline{G} \\ & f_2 \searrow & & \nearrow g_2 & \nearrow h_2 \\ & & G_2 & & \end{array} \text{ commute.}$$

Now $h\phi(x_i) = hg_1(x_i) = h_1(x_i) = x_i$ and

$h\phi(y_j) = hg_2(y_j) = h_2(y_j) = y_j$

$\Rightarrow h\phi = \text{id}.$

$$\begin{aligned} \phi h(g_1(x_i)) &= \phi h_1(x_i) = \phi(x_i) = g_1(x_i) \text{ and} \\ \phi h(g_2(y_j)) &= \phi h_2(y_j) = \phi(y_j) = g_2(y_j) \\ \Rightarrow \phi h &= \text{id.} \end{aligned}$$

$\therefore \phi$ is an isomorphism with inverse h .

□

4. Special Case

$$G_1 = \{1\} \text{ (or } G_2 = \{1\}) \Rightarrow G_1 *_G G_2 = G_2 / \langle f_2(G) \rangle^1$$

¹ $\langle f_2(G) \rangle$ is the normal subgroup generated by $f_2(G)$