VII.3 Homotopy invariance of homology

Goal. $f \simeq g : X \to Y \Rightarrow f_* = g_* : H_p(X) \to H_p(Y)$.

정의 1 (Chain homotopy) Let $f, g : \mathcal{C} \to \mathcal{C}'$ be chain maps i.e.,
$f = \{f_p : C_p \to C'_p \mid f_{p-1}\partial_p = \partial_p{}' f_p\}$, $g = \{g_p : C_p \to C'_p \mid g_{p-1}\partial_p = \partial_p{}' g_p\}$.
A ***Chain homotopy*** D between f and g ($D : f \simeq g$) is a collection of homomorphisms $\{D_p : C_p \to C_{p+1}{}'\}$ such that $\partial_{p+1}{}' D_p + D_{p-1}\partial_p = f_p - g_p$.
(simply denote by $\partial D + D\partial = f - g$)

정리 1 *If $f \simeq g : \mathcal{C} \to \mathcal{C}'$, then $f_* = g_* : H_p(\mathcal{C}) \to H_p(\mathcal{C}')$.*

증명 $z \in Z_p(C) \Rightarrow \partial Dz = f(z) - g(z) \Rightarrow f(z) \sim g(z) \Rightarrow f_*[z] = g_*[z]$ □

Special case : $\mathcal{C} = \mathcal{C}',\ f = id,\ g = 0$
Then a chain homotopy D between id and 0 is called a chain contraction and if $\mathcal{C}$ has a chain contraction, then $\mathcal{C}$ is said to be ***chain contractible***. In this case, $\partial D + D\partial = 1$.

정리 2 *If $\mathcal{C}$ is a free chain complex, i.e., each C_p is a free abelian group, then $\mathcal{C}$ is acyclic $\Leftrightarrow$ $\mathcal{C}$ is chain contractible.*

증명 $\Leftarrow$) by theorem 1, $id_* = 0 \Rightarrow H_p(C) = 0\ \forall p$
$\Rightarrow$) Define D_p inductively.

$$
\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "id"] & C_1 \arrow[r] \arrow[d, "id"] \arrow[dl, dotted, "\exists D_1"] & C_0 \arrow[r] \arrow[d, "id"] \arrow[dl, dotted, "\exists D_0"] & 0 \arrow[dl, dotted, "0"] & :\ \text{free} \\
\cdots \arrow[r] & C_2 \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & 0 & :\ \text{acyclic}
\end{tikzcd}
$$

그림에서와 같이 0에서 C_0로 가는 zero homomorphism을 정의하고 귀납적으로 D를 정의한다.
C_p가 free이므로, D가 존재함을 보이기 위해서는 C_p의 generator에 대해 조건을 만족하면서 C_{p+1}로 가는 함수가 존재함을 보이면 된다. 즉, 각 generator g에 대하여 $\partial D(g) = g - D\partial(g)$를 만족하는 C_{p+1}의 원소 $D(g)$를 대응시키면 된다. (Induction의 가정에서 우변의 D는 이미 정의된 것이고 좌변의 D를 정의하여야 한다.)
위 diagram에서 아래의 sequence가 exact이므로, $\partial(g - D\partial(g)) = 0$이면 $\partial(h) = g - D\partial(g)$인 $h \in C_{p+1}$이 존재하고 $D(g) = h$로 정의한다.
따라서, $\partial(1 - D\partial) = 0$임을 확인하면 된다. 그런데,

$$\partial(1 - D\partial) = \partial - \partial D\partial = \partial - (1 - D\partial)\partial = \partial - \partial + D\partial^2 = 0$$

이므로 증명이 끝났다. □

Note. Comparison theorem in homological algebra
(1) Existence of chain map lifting a given $k : \mathbb{Z} \to \mathbb{Z}$.

$$\begin{tikzcd}
\mathcal{C} & \cdots \arrow[r] & C_2 \arrow[r, "\partial"] \arrow[d, dotted, "\exists k_2"] & C_1 \arrow[r, "\partial"] \arrow[d, dotted, "\exists k_1"] & C_0 \arrow[r, "\partial"] \arrow[d, dotted, "\exists k_0"] & \mathbb{Z} \arrow[r, "\partial"] \arrow[d, "k"] & 0 & : \text{ free} \\
\mathcal{C}' & \cdots \arrow[r] & C_2' \arrow[r, "\partial"] & C_1' \arrow[r, "\partial"] & C_0' \arrow[r, "\partial"] & \mathbb{Z} \arrow[r, "\partial"] & 0 & : \text{ acyclic}
\end{tikzcd}$$

$\partial k_{p+1} = k_p\partial$인 k_p들이 존재함을 보이려면 정리 2에서와 마찬가지로 $\partial k_p \partial = 0$임을 확인하면 된다. $\partial k_p \partial = k_{p-1}\partial^2 = 0$이므로 귀납적으로 k_p를 정의해 나갈 수 있다.

(2) Existence of chain homotopy between two liftings.

$$\begin{tikzcd}
\mathcal{C} & \cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "{k_2, k_2'}"] & C_1 \arrow[r] \arrow[d, "{k_1, k_1'}"] \arrow[dl, dotted, "\exists D_1"] & C_0 \arrow[r] \arrow[d, "{k_0, k_0'}"] \arrow[dl, dotted, "\exists D_0"] & \mathbb{Z} \arrow[r] \arrow[d, "k"] \arrow[dl, dotted, "0"] & 0 & : \text{ free} \\
\mathcal{C}' & \cdots \arrow[r] & C_2 \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 & : \text{ acyclic}
\end{tikzcd}$$

먼저 $D : \mathbb{Z} \to C_0$를 $D = 0$으로 주고, $\partial D = k_p - k_p' - D\partial$가 되도록 D를 귀납적으로 정의해나간다. 따라서 앞서와 마찬가지로 $\partial(k_p - k_p' - D\partial) = 0$임을 확인하면 된다.
그런데, $\partial(k_p - k_p' - D\partial) = \partial k_p - \partial k_p' - \partial D\partial = k_p\partial - k_p'\partial - \partial D \partial = (k_p - k_p' - \partial D)\partial = D\partial\partial = 0$이므로 증명이 끝났다.

보조정리 3 *X is a star shaped subset of $\mathbb{R}^n$* $\Rightarrow \widetilde{H}(X) = 0$.

증명 We may assume X is star shaped from the origin. Construct a chain contraction D:

$$\begin{tikzcd}
\cdots \arrow[r] & S_2 \arrow[r, "\partial"] \arrow[d, "id"] & S_1 \arrow[r, "\partial"] \arrow[d, "id"] \arrow[dl, dotted, "D_1"] & S_0 \arrow[r, "\epsilon"] \arrow[d, "id"] \arrow[dl, dotted, "D_0"] & \mathbb{Z} \arrow[r] \arrow[d, "id"] \arrow[dl, dotted, "D_{-1}"] & 0 \\
\cdots \arrow[r] & S_2 \arrow[r] & S_1 \arrow[r] & S_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Define $D_{-1}(1)$ =constant map to the origin.
Given $\sigma : \triangle^p \to X$, define $D(\sigma) \in S_{p+1}(X)$ by $D(\sigma)(se_0 + (1-s)t) = (1-s)"\sigma"(t)$, where $s \in I$ and $t \in \langle e_1, \cdots, e_{p+1}\rangle$. (다시말해 $D(\sigma)(e_0) = 0$ and $D(\sigma) \circ f^0_{p+1} = \sigma$.

$p=0$의 경우에는 $D(\sigma)^{(0)} = \sigma$, $D(\sigma)^{(1)} = 0$이므로, $\partial D(\sigma) = \sigma - D_{-1}(1) = \sigma - D_{-1}\epsilon(\sigma)$이다. 따라서, $\partial D + D\epsilon = 1$이 성립한다.

$p>0$의 경우에는 $D(\sigma)^{(0)} = \sigma$이고, $D(\sigma)^{(i+1)} = D(\sigma^{(i)})$가 성립한다.
$(\because D(\sigma)^{(i+1)}(e_1,\cdots,e_p) = D\sigma(e_1,\cdots,\hat{e}_{i+1},\cdots,e_{p+1})$
$D(\sigma^{(i)})(e_1,\cdots,e_p) = \sigma^{(i)}(e_0,\cdots,e_{p-1})$
$= \sigma(e_0,\cdots,\hat{e}_i,\cdots,e_p) = D\sigma(e_1,\cdots,\hat{e}_{i+1},\cdots,e_{p+1}))$
따라서

$$\partial D\sigma = \sum_{i=0}^{p}(-1)^{i+1}(D\sigma)^{(i+1)} + D(\sigma)^{(0)} = \sum_{i=0}^{p}(-1)^{i+1}D(\sigma^{(i)}) + \sigma = -D(\partial\sigma) + \sigma$$

이므로, $\partial D + D\partial = 1$이 성립한다. □

보조정리 4 *Let* $i_0 : \begin{array}{l} X \to X \times I \\ x \mapsto (x,0) \end{array}$ *and* $i_1 : \begin{array}{l} X \to X \times I \\ x \mapsto (x,1) \end{array}$.
Then for any space X, $\forall p, \exists D_X : S_p(X) \to S_{p+1}(X \times I)$ *such that*
(1) $\partial D + D\partial = i_{1\sharp} - i_{0\sharp}$
(2) D_X *is natural, i.e.,* $\forall f : X \to Y$,

$$\begin{array}{ccc} S_p(X) & \xrightarrow{D_X} & S_{p+1}(X \times I) \\ \downarrow f_\sharp & & \downarrow (f\times id)_\sharp \\ S_p(Y) & \xrightarrow{D_Y} & S_{p+1}(Y \times I) \end{array} \quad \textit{commutes.}$$

만약 이 보조정리를 증명하였다면 다음 정리가 성립한다.

정리 5 $f \simeq g : X \to Y \Rightarrow f_* = g_* : H_*(X) \to H_*(Y)$

증명 There exist $F : X \times I \to Y$ such that $f = F \circ i_0$ and $g = F \circ i_1$.
By the above lemma, there exists $D : i_{0\sharp} \simeq i_{1\sharp}$. Hence, $i_{0*} = i_{1*} : H(X) \to H(X \times I)$. Therefore $f_* = (F \circ i_0)_* = F_* \circ i_{0*} = F_* \circ i_{1*} = (F \circ i_1)_* = g_*$. □

따름정리 6 $X \simeq Y \Rightarrow H_*(X) \cong H_*(Y)$

따라서 보조정리만 증명하면 된다.

보조정리의 증명 (Induction on p)
idea. 먼저 D를 "model space" $\triangle^p$에 대해서 정의하고, D의 naturality를 이용하여 임의의 X에 대하여 D_X를 정의한다.

먼저 $p = 0$인 경우 $D : S_0(\triangle^0) \to S_1(\triangle^0 \times I)$를 정의하자. $\triangle^0$은 한 point이므로, $S_0(\triangle^0)$는 constant map σ로 generate되는 free abelian group이다. 여기서, $D\sigma$를 obvious한 1-chain으로 정의한다. 즉, $D\sigma : \triangle^1 \to \triangle^0 \times I$에서 $\triangle^1$와 $\triangle^0 \times I$는 본질적으로 같은 space이므로 $D\sigma$는 identity map으로 정의한다. 이때, $\partial D\sigma = i_1 \circ \sigma - i_0 \circ \sigma = i_{1\sharp}(\sigma) - i_{0\sharp}(\sigma)$이 된다.
$p = 0$인 경우에는 같은 방법으로 임의의 space X에 대하여 $D_X : S_0(X) \to S_1(X \times I)$를 바로 정의할 수 있는데, $\sigma = x$(constant map), $x \in X$에 대하여 $D\sigma$를 $\triangle^1$에서 $\{x\} \times I$로 가는 "identity map"으로 정의하면 위의 조건 (1),(2)를 만족함을 쉽게 확인 할 수 있다.

$p > 0$인 경우, $p - 1$까지 (1),(2)를 만족하는 D_X가 존재한다고 가정하자.
$i = id : \triangle^p \to \triangle^p$에 대하여 $\partial Di + D\partial i = i_{1\sharp}(i) - i_{2\sharp}(i)$를 만족하는 $Di \in S_{p+1}(\triangle^p \times I)$를 정의하자.
$c = i_{1\sharp}(i) - i_{0\sharp}(i) - D\partial i \in S_p(\triangle^p \times I)$로 두고, $c = \partial Di$가 되도록 Di를 정의해야 한다. $\triangle^p \times I$가 star shaped이므로 앞의 보조정리에 의하여 $\widetilde{H}(\triangle^p \times I) = 0$이 되어 $S_p(\triangle^p \times I)$의 augmented chain이 acyclic이다. 따라서, 정리 2의 증명에서와 같이 $\partial c = 0$만 확인하면 된다.
그런데, induction의 가정을 이용하면

$$\partial c = i_{1\sharp}(\partial i) - i_{0\sharp}(\partial i) - (i_{1\sharp}(\partial i) - i_{0\sharp}(\partial i) - D\partial(\partial i)) = 0$$

이므로, Di가 잘 정의된다.

이제 Di를 이용하여 임의의 space X에 대하여 $D_X : S_p(X) \to S_{p+1}(X \times I)$를 정의하자.
임의의 $\sigma \in S_p(X)$는 $\sigma : \triangle^p \to X$인 map이므로, $\sigma_\sharp$은 $\begin{array}{l} \sigma_\sharp : S_p(\triangle^p) \to S_p(X) \\ \qquad\qquad\quad i \mapsto \sigma \end{array}$ 인 map이다. 따라서, $D_X\sigma = (\sigma \times id)_\sharp Di$로 정의한다.

$S_p(X)$는 σ들로 generate되는 free abelian group이므로, D_X는 well-defined이고, 이제 조건 (1),(2)만 확인하면 증명이 끝난다.

Check (1):

$$
\begin{tikzcd}
 & S_{p+1}(\triangle^p) \arrow[rr, "\partial"] \arrow[dl, "\sigma_\sharp"'] \arrow[dd, "i_{0\sharp}"', "i_{1\sharp}"] & & S_p(\triangle^p) \arrow[rr, "\partial"] \arrow[dl] \arrow[dd] \arrow[ddll, bend right, "D"'] & & S_{p-1}(\triangle^p) \arrow[dl] \arrow[dd] \arrow[ddll, bend right, "D"'] \\
S_{p+1}(X) \arrow[rr] \arrow[dd, "{i_{0\sharp},i_{1\sharp}}"'] & & S_p(X) \arrow[rr] \arrow[dd] \arrow[ddll, bend right, "D_X"'] & & S_{p-1}(X) \arrow[dd] \arrow[ddll, bend right, "D_X"'] & \\
 & S_{p+1}(\triangle^p \times I) \arrow[rr] \arrow[dl, "(\sigma\times id)_\sharp"] & & S_p(\triangle^p \times I) \arrow[rr] \arrow[dl] & & S_{p-1}(\triangle^p \times I) \arrow[dl] \\
S_{p+1}(X \times I) \arrow[rr] & & S_p(X \times I) \arrow[rr] & & S_{p-1}(X \times I) &
\end{tikzcd}
$$

위,아래, 앞, 뒤의 사각형들은 ∂와 $\sharp$이 commute한다는 사실로부터 commute하고, 가운데(세로로 보이는) 사각형은 space들 간의 map이 commute하므로 functorial property로부터 commute한다. ($i_{0\sharp}, i_{1\sharp}$의 naturality.) 오른쪽의 비스듬한 사각형은 induction의 가정으로부터 commute한다. (D의 naturality.) 따라서,

$$\partial D_X\sigma = \partial(\sigma \times id)_\sharp Di = (\sigma \times id)_\sharp \partial Di = (\sigma \times id)_\sharp(i_{1\sharp}(i) - i_{0\sharp}(i) - D\partial)$$

이 때, $i_{1\sharp}$의 naturality로부터 ($i_{0\sharp}$의 경우도 마찬가지)

$$(\sigma \times id)_\sharp i_{1\sharp}(i) = i_{1\sharp}\sigma_\sharp(i) = i_{1\sharp}(\sigma)$$

또한, D의 naturality로부터

$$(\sigma \times id)_\sharp D(\partial i) = D_X\sigma_\sharp(\partial i) = D_X\partial\sigma_\sharp(i) = D_X\partial\sigma$$

따라서, $\partial D_X = i_{1\sharp} - i_{0\sharp} - D_X\partial$임을 확인할 수 있다.

Check (2):
임의의 map $f : X \to Y$에 대하여 다음 diagram을 생각해 보자.

$$
\begin{tikzcd}
 & S_p(X) \arrow[rr, "D_X"] \arrow[dd, "f_\sharp"'] & & S_{p+1}(X \times I) \arrow[dd, "(f\times id)_\sharp"] \\
S_p(\triangle^p) \arrow[ur, "\sigma_\sharp"] \arrow[dr, "(f\circ\sigma)_\sharp"'] \arrow[rr, "D"] & & S_{p+1}(\triangle^p \times I) \arrow[ur, "(\sigma\times id)_\sharp"] \arrow[dr, "(f\circ\sigma\times id)_\sharp"'] & \\
 & S_p(Y) \arrow[rr] & & S_{p+1}(Y \times I)
\end{tikzcd}
$$

양쪽의 삼각형은 functorial property에 의하여 commute하고, $i \in S_p(\triangle^p)$에 대해 D_X의 정의에 의하여 위의 사각형이, D_Y의 정의에 의하여 아래의 사각형

형이 commute한다. 따라서, 뒤쪽 사각형이 commute한다는 것을 확인 할 수 있다. □

Review of the proof of 2.
Need
(i) $\{S(\triangle^p \times I), \partial, \epsilon\}$ is acyclic.(to define D for model space, $\triangle^p$ and $i \in S_p(\triangle^p)$)
(ii) $\{\sigma_\sharp i | \sigma : \triangle^p \to X\}$ form a basis for $S_p(X)$.(to define D_X for $\forall X$)
(iii) $i_{0\sharp}$ and $i_{1\sharp}$ are natural. (to check (1))
$\Rightarrow$with functorial properties,
$\exists$chain homotopy $D : i_0 \simeq i_1$ s.t. naturality of D holds.

Note.
$\mathcal{T}$= the category of topological spaces.
$\mathcal{S}$= the category of chain complexes.(with augmentation)
$S : \mathcal{T} \longrightarrow \mathcal{S}$ is a functor.
$X \mapsto S(X)$
$S' : \mathcal{T} \longrightarrow \mathcal{S}$ is a functor.
$X \mapsto S(X \times I)$
$i_{0\sharp}$ and $i_{1\sharp}$ are natural transformations : $S \to S'$, i.e.,

$$\begin{array}{lccc} S : S(X) & \xrightarrow{f_\sharp} & S(Y) \\ i_{0\sharp} \downarrow \quad \downarrow i_{0\sharp}^X & \curvearrowright & \downarrow i_{0\sharp}^Y \\ S' : S(X \times I) & \xrightarrow{(f \times id)_\sharp} & S(Y \times I) \end{array}$$

정의 2 $\mathcal{T}$= the category of chain complexes , $\mathcal{S}$= the category of (augmented) chain complexes and chain maps. Let $\mathcal{M} \subset Ob(\mathcal{T})$ (called models).
Let $S : \mathcal{T} \to \mathcal{S}$ be a functor from $\mathcal{T}$ to $\mathcal{S}$.
(1) S is acyclic relative to $\mathcal{M}$ if $S(M)$ is acyclic $\forall M \in \mathcal{M}$.
(2) S is free relative to $\mathcal{M}$ if $\forall p \geq 0, \exists J_p$ and $\exists$ an indexed family of $\{M_\alpha\}_{\alpha \in J_p}, M_\alpha \in \mathcal{M}$ and $\exists$ an indexed family of $\{i_\alpha\}_{\alpha \in J_p}, i_\alpha \in S_p(M_\alpha)$ s.t. $\{S(\sigma) i_\alpha | \alpha \in J_p, \sigma \in hom(M_\alpha, X)\}$ is a basis for $S_p(X)$.

정리 7 ***Acyclic Model Theorem***
Let $S, S' : \mathcal{T} \to \mathcal{S}$ be functors and $\mathcal{M} \subset Ob(\mathcal{T})$.
S : free relative to $\mathcal{M}$ and S' : acyclic relative to $\mathcal{M}$.
$\Rightarrow$(a) $\exists$ a natural transformation : $\tau : S \to S'$ as a lifting of a given $\tau_0 : S_0 \to S'_0$.(or $k : \mathbb{Z} \to \mathbb{Z}$ for augmented case)
(b) Given two such natural transformations $\tau_0, \tau_1 : S \to S', \exists$ a natural chain homotopy $D : \tau_0 \simeq \tau_1$.

증명(a) Use induction.
1st, define τ_p for $i_\alpha \in S_p(M_\alpha)$:

$$\begin{tikzcd}
\cdots \arrow[r] & S_p(M_\alpha) \arrow[r, "\partial"] \arrow[d, dashed, "\tau_p"] & S_{p-1}(M_\alpha) \arrow[r, "\partial"] \arrow[d, "\tau_{p-1}"] & \cdots \\
\cdots \arrow[r] & S'_p(M_\alpha) \arrow[r, "\partial"] & S'_{p-1}(M_\alpha) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

S'이 acyclic이므로 τ_p가 존재함을 보이려면 $\partial\tau_{p-1}\partial$=0임을 보이면 된다. 그런데 induction 가정에 의해 $\partial\tau_{p-1}\partial = \tau_{p-2}\partial\partial = 0$.

2nd on X : Use Naturality.

$$\begin{tikzcd}
 & i_\alpha \in S_p(M_\alpha) \arrow[rr, "\partial"] \arrow[dl, "S_p(\sigma)"'] \arrow[dd] & & S_{p-1}(M_\alpha) \arrow[dl] \arrow[dd, "\tau_{p-1}"] \\
S(\sigma)i_\alpha \in S_p(X) \arrow[rr] \arrow[dd, dashed, "\tau_p"'] & & S_{p-1}(X) \arrow[dd] & \\
 & S'_p(M_\alpha) \arrow[rr] \arrow[dl] & & S'_{p-1}(M_\alpha) \arrow[dl] \\
S'_p(X) \arrow[rr] & & S'_{p-1}(X) &
\end{tikzcd}$$

$S(\sigma)i_\alpha$들이 $S_p(X)$의 basis를 이루므로
Define τ_p for $S(\sigma)i_\alpha$: 위 diagram box에서 i_α에 대해서는 이미 τ_p가 정의되어 있으므로 왼쪽 면이 commute하도록 τ_p를 정의하면 앞면을 제외하고 모든 면이 commute하므로 앞면도 commute.
$\therefore \tau_p$ is a chain map, $\forall X$.

τ_p is natural: $f : X \to Y$일 때,

$$\begin{tikzcd}
& S_p(M_\alpha) \arrow[dl, "S_p(\sigma)"'] \arrow[dr, "S_p(f\circ\sigma)"] \arrow[dd, "\tau"] & \\
S_p(X) \arrow[rr, "S_p(f)"] \arrow[dd, "\tau_X"] & & S_p(Y) \arrow[dd, "\tau_Y"] \\
& S'_p(M_\alpha) \arrow[dl] \arrow[dr] & \\
S'_p(X) \arrow[rr] & & S'_p(Y)
\end{tikzcd}$$

이 세로 기둥에서 i_α에 대해 위에서 τ_p의 정의와 S_p의 functorial property에 의해 앞면만 빼고 다 commute.
$\therefore S(\sigma)i_\alpha$에 대해 앞면도 commute.

(b) exactly same as before. (**숙제 22.**) □

정의 3 A chain map $\tau : \mathcal{C} \to \mathcal{C}'$ is a chain homotopy equivalence if $\exists$ a chain map $\tau' : \mathcal{C}' \to \mathcal{C}$ s.t. $\tau\tau' \simeq id_{\mathcal{C}'}$ and $\tau'\tau \simeq id_{\mathcal{C}}$.

Note. $\tau : \mathcal{C} \to \mathcal{C}'$: chain homopoty equivalence. $\Rightarrow \tau_* : H_*(\mathcal{C}) \cong H_*(\mathcal{C}')$.

따름정리 8 *$S, S' : \mathcal{T} \to \mathcal{S}, \mathcal{T}$ with models $\mathcal{M}$ and S, S': both free and acyclic relative to $\mathcal{M}$. $\Rightarrow \exists$ a natural chain map $\tau : S \to S'$ and any natural chain map is a natural chain homotopy equivalence.*

증명 AMT. (a) $\Rightarrow$ natural chain map τ exists and also $\exists \tau' : S' \to S$. Now $\tau\tau' : S' \to S'$
AMT (b) $\Rightarrow \tau\tau' \simeq id_{S'}$, similarly $\tau'\tau \simeq id_S$ □

An application. (Equivalence of homologies)
If K is a simplicial complex, then we have
$\mathcal{C}(K) \rightsquigarrow H(K) =$ simplicial homology
$\triangle(K) \rightsquigarrow H_\triangle(K) =$ ordered simplicial homology
$S(|K|) \rightsquigarrow H(|K|)$= singular homology for $|K|$

보조정리 9 *K: a simplicial complex, w: a vertex (as a simplicial complex) not in K. $\Rightarrow \{C(w * K), \partial, \epsilon\}$ and $\{\triangle(w * K), \partial, \epsilon\}$ are acyclic.*

증명 Construct a chain contraction:

$$\begin{tikzcd} \cdots \arrow[r] & \triangle_1 \arrow[r] \arrow[d, "id"] \arrow[dl, "D_1"'] & \triangle_0 \arrow[r, "\epsilon"] \arrow[d, "id"] \arrow[dl, "D_0"'] & \mathbb{Z} \arrow[d, "id"] \arrow[dl, "D_{-1}"'] \\ \cdots \arrow[r] & \triangle_1 \arrow[r] & \triangle_0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Let $D_{-1}(1) := w$.Then $\epsilon(w) = 1$이므로 OK.

Define $D(v_0, \cdots, v_p) = (w, v_0, \cdots, v_p)$: join operator.
Then $(\partial D + D\partial)(v_0, \cdots, v_p)$
$= \partial(w, v_0, \cdots, v_p) + D(\Sigma(-1)^i(v_0, \cdots, \hat{v_i}, \cdots, v_p))$
$= \Sigma(-1)^{j+1}(w, \cdots, \hat{v_j}, \cdots, v_p) + (v_0, \cdots, v_p) + \Sigma(-1)^i(w, \cdots, \hat{v_i}, \cdots, v_p)$
$= (v_0, \cdots, v_p)$.

Similarly for $\{\mathcal{C}(w * K), \partial, \epsilon\}$. □

따름정리 10 *(1)* $\forall \sigma \in K, \quad \tilde{H}(\sigma) = \tilde{H}_\triangle(\sigma) = 0$
(i.e. $\{\mathcal{C}, \partial, \epsilon\}$ *and* $\{\triangle, \partial, \epsilon\}$ *are acyclic w.r.t.* σ*.)*
(2) The chain map $\mu : \triangle(K) \to \mathcal{C}(K)$ *is a natural homotopy equivalence.*
$$(v_0, \cdots, v_p) \mapsto [v_0, \cdots, v_p]$$

증명(1) 보조정리9로부터 clear.
(2) $\triangle, \mathcal{C} : \mathcal{C}(K) \to \mathcal{S}$ are free and acyclic functors relative to models $\mathcal{M} = \{\sigma | \sigma \in K\}$, where $\mathcal{C}(K)$ is a category of subcomplexes of K with inclusion as morphism. Since $\mu : \triangle \to C$ is a natural chain map, it it a chain homotopy equivalence by the previous cor.(따름정리 8) □