VII.6 Relative Homology and Excision

정의 1 $A \underset{i}{\hookrightarrow} X$, a subspace of a topological space $X \Rightarrow S_p(A) \underset{i_\sharp}{\hookrightarrow} S_p(X)$

$$\Rightarrow \quad \begin{array}{ccccccccc} 0 & \longrightarrow & S_p(A) & \xrightarrow{i} & S_p(X) & \xrightarrow{q} & S_p(X)/S_p(A) & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \bar{\partial} & & \\ 0 & \rightarrow & S_{p-1}(A) & \rightarrow & S_{p-1}(X) & \rightarrow & S_{p-1}(X)/S_{p-1}(A) & \rightarrow & 0 \end{array}$$

,where $\bar{\partial}$ is an induced boundary homomorphism s.t. $\bar{\partial}^2 = \bar{\partial}^2 = 0$.
$\Rightarrow \{S_p(X, A), "\partial" = \bar{\partial}\} = S(X, A)$: relative chain complex or singular chain complex for a pair (X, A).
$\Rightarrow H_p(S(X, A)) =: H_p(X, A)$: singular homology for a pair (X, A)(or relative homology X mod A).

Remark 1. This is a general construction for $\forall$ chain complexes $\mathcal{C}' \subset \mathcal{C}$, and hence can talk about $H_p(K, A)$ for a simplicial pair $A < K$.

Remark 2. (geometric interpretation)
$S_p(X, A)$ can be viewed as a free group generated by p-simplices not contained in A.

$\widetilde{Z_p}(X, A) := q^{-1}(ker\bar{\partial}) = \partial^{-1}(S_{p-1}(A)) = \{c \in S_p(X) | \partial c \in S_{p-1}(A)\}$.
$\widetilde{B_p}(X, A) := q^{-1}(im\bar{\partial}) = \partial S_{p+1}(X) + S_p(A)$
$= \{c \in S_p(X) : \exists a \in S_{p+1}(X) \text{ s.t. } c \equiv \partial a \bmod S_p(A)\}$.

Then $H_p(X, A) \underset{\overleftarrow{q_*}}{\cong} \widetilde{Z_p}(X, A)/\widetilde{B_p}(X, A)$.
(by an isomorphism theorem, $M/p/N/p \cong M/N$)

1. Functorial Property

$f : (X, A) \to (Y, B)$

$$\Rightarrow \quad \begin{array}{ccccccccc} 0 & \longrightarrow & S_p(A) & \longrightarrow & S_p(X) & \longrightarrow & S_p(X)/S_p(A) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f_\sharp|} & & \big\downarrow{\scriptstyle f_\sharp} & & \vdots\,{\scriptstyle \bar{f}_\sharp = \text{"}f_\sharp\text{"}} & & \\ 0 & \longrightarrow & S_p(B) & \longrightarrow & S_p(Y) & \longrightarrow & S_p(Y)/S_p(B) & \longrightarrow & 0 \end{array} \qquad \text{and "}f_\sharp\text{" is a chain map.}$$

$\Rightarrow f_* : H_p(X, A) \to H_p(Y, B).$
$\quad\quad\quad \{c\} \mapsto \{f \circ c\}$

(i) $id : (X, A) \circlearrowleft \Rightarrow id_* = id.$
(ii) $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C) \Rightarrow (g \circ f)_* = g_* \circ f_*$

Example.
(i) $\{X_\alpha\}$: path-components of X and $A_\alpha = A \cap X_\alpha \Rightarrow H_p(X, A) = \bigoplus_\alpha H_p(X_\alpha, A_\alpha)$
(ii) X : path-connected and $A \neq \phi \Rightarrow H_0(X, A) = 0.$

2. Homotopy Invariance

$f \underset{F}{\simeq} g : (X, A) \to (Y, B) \quad (F(A \times I) \subseteq B) \Rightarrow f_* = g_*.$

증명 Let $i_0, i_1 : (X, A) \to (X \times I, A \times I)$ be level maps as before.
Need a chain homotopy $D_{X,A} : S_p(X, A) \to S_{p+1}(X \times I, A \times I)$
between $i_{0\sharp}$ and $i_{1\sharp}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_p(A) & \longrightarrow & S_p(X) & \longrightarrow & S_p(X, A) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle D_A = D_X|} & & \big\downarrow{\scriptstyle D_X} & & \vdots\,{\scriptstyle \bar{D}_X = D_{X,A}} & & \\ 0 & \longrightarrow & S_{p+1}(A \times I) & \longrightarrow & S_{p+1}(X \times I) & \longrightarrow & S_{p+1}(X \times I, A \times I) & \longrightarrow & 0 \end{array}$$

Check $\bar{\partial}\bar{D}\bar{\sigma} + \bar{D}\bar{\partial}\bar{\sigma} = \bar{i}_{0\sharp}\bar{\sigma} - \bar{i}_{1\sharp}\bar{\sigma}$: trivial.

$(X, A) \overset{i_0}{\underset{i_1}{\rightrightarrows}} (X \times I, A \times I) \xrightarrow{F} (Y, B)$ s.t. $f = F \circ i_0$ and $g = F \circ i_1$.
$\bar{D} : i_{0\sharp} \cong i_{1\sharp} \Rightarrow i_{0*} = i_{1*} : H(X, A) \to H(X \times I, A \times I)$
$\therefore f_* = (F \circ i_0)_* = F_* \circ i_{0*} = F_* \circ i_{1*} = (F \circ i_1)_* = g_*$ $\square$

3. $0 \to S(A) \to S(X) \to S(X,A) \to 0$
$\overset{\text{snake}}{\Rightarrow}$ $\exists$ a functorial long exact sequence

$$\cdots \longrightarrow H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{q_*} H_p(X,A) \xrightarrow{\partial_*} H_{p-1}(A) \longrightarrow \cdots$$

$$\{c\} \mapsto \{\partial c\}$$

Remark. If $A \neq \phi$, define $\widetilde{H}(X,A) = H(X,A)$. Then 3 holds for reduced case also.

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_0(A) & \to & S_0(X) & \to & S_0(X,A) & \to & 0 \\
 & & \downarrow \epsilon & & \downarrow \epsilon & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Example. $H_p(B^n, \partial B^n)$

$$\cdots \to \widetilde{H_p}(\partial B^n) \to \widetilde{H_p}(B^n) \to \widetilde{H_p}(B^n, \partial B^n) \overset{\partial}{\to} \widetilde{H}_{p-1}(\partial B^{n-1}) \to \cdots$$

$\widetilde{H_p}(B^n) = 0 \ \ ,\forall p \ \Rightarrow \ \partial :\cong \Rightarrow$
$H_p(B^n, \partial B^n) = \widetilde{H}_p(B^n, \partial B^n) \cong \widetilde{H}_{p-1}(S^{n-1}) = \mathbb{Z} \quad p = n$
$= 0 \quad p \neq n$

4. Long Exact Sequence for Triple

Let $A \subset B \subset X$. Then we have a functorial long exact sequence

$$\cdots \to H_p(B,A) \to H_p(X,A) \to H_p(X,B) \overset{\partial}{\to} H_{p-1}(B,A) \to \cdots$$

증명

$$0 \to S(B)/S(A) \to S(X)/S(A) \to S(X)/S(B) \to 0 \qquad : \text{s.e.s.}$$

and apply snake lemma. $\square$

5. Five Lemma

$$\begin{array}{ccccccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 & & \text{:exact} \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 & & \text{all} \curvearrowright \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 & & \text{:exact} \end{array}$$

f_1: onto , $f_5 : 1-1$ and $f_2, f_4 :\cong \;\Rightarrow\; f_3 :\cong$.
(In particular, $f_i :\cong \;\; i = 1,2,4,5. \;\Rightarrow\; f_3 :\cong$.)

증명 by diagram chasing. □

Example. Show $j : (B^n, S^{n-1}) \to (B^n, B^n \setminus 0)$ induces an isomorphism
$j_* : H(B^n, S^{n-1}) \to H(B^n, B^n \setminus 0)$.

$$\begin{array}{ccccccccccc} \cdots \to H_p(S^{n-1}) & \to & H_p(B^n) & \to & H_p(B^n, S^{n-1}) & \to & H_{p-1}(S^{n-1}) & \to & H_{p-1}(B^n) & \to \cdots \\ \downarrow j_* :\cong & & \downarrow j_* := & & \downarrow \text{five lem.에 의해} j_* :\cong & & \downarrow j_* := & & \downarrow j_* := & \\ \cdots \to H_p(B^n \setminus 0) & \to & H_p(B^n) & \to & H_p(B^n, B^n \setminus 0) & \to & H_{p-1}(B^n \setminus 0) & \to & H_{p-1}(B^n) & \to \cdots \end{array}$$

6. Split Short Exact Sequence

Given short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$, TFAE :
(i) $\exists A \xleftarrow{j} B$ s.t. $j \circ i = id$.
(ii) $\exists B \xleftarrow{s} C$ s.t. $p \circ s = id$.
In this case, $B = A \oplus C$ and the short exact sequence is said to be split.

증명 (i)⇒(ii):
$0 \to A \underset{i}{\overset{j}{\leftrightarrows}} B \underset{p}{\to} C \to 0 \Rightarrow B = i(A) \oplus ker(j) : b = ij(b) + (b - ij(b))$
($\because j(b) - jij(b) = 0 \Rightarrow b - ij(b) \in ker(j)$ and $ji(a) = 0 \Rightarrow a = 0 \Rightarrow i(a) = 0$.)

$p|_{ker(j)} : ker(j) \to C :\cong$ since $ker(p) = i(A) \Rightarrow \exists s = (p|)^{-1}$ s.t. $p \circ s = id$.

(ii)⇒(i) : exactly same. □

Example. $A \underset{r}{\overset{i}{\rightleftarrows}} X$, a retract $\Rightarrow H_p(X) = H_p(A) \oplus H_p(X, A)$.

Excision

정리 1 (X, A) : *a pair of spaces and let* $U \subset X$ *be s.t.* $\bar{U} \subset \mathring{A}$.
Then the inclusion $i : (X - U, A - U) \hookrightarrow (X, A)$ *induces an isomorphism* $i_* : H_*(X - U, A - U) \xrightarrow{\cong} H_*(X, A)$. *(i.e., U may be excised without altering relative homology.)*

증명 Let $\mathcal{U} = \{X - U, A\}$ so that $\{(X - U)^\circ, \mathring{A}\}$ covers X. (See the footnote.[1])
Then $i : S^{\mathcal{U}}(X) \hookrightarrow S(X)$ induces an isomorphism on homology.

$$\begin{array}{ccccccccc} 0 & \to & S(A) & \to & S^{\mathcal{U}}(X) & \to & S^{\mathcal{U}}(X)/S(A) & \to & 0 \\ & & \downarrow{\scriptstyle i=i|} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \\ 0 & \to & S(A) & \to & S(X) & \to & S(X)/S(A) & \to & 0 \end{array}$$

where j is induced as quotient map and becomes a chain map as before.

$$\Rightarrow \quad \begin{array}{ccccccccccc} \cdots & \to & H_p(A) & \to & H_p(S^{\mathcal{U}}(X)) & \to & H_p(S^{\mathcal{U}}(X)/S(A)) & \to & H_{p-1}(A) & \xrightarrow{\partial_*} & \cdots \\ & & \downarrow{\scriptstyle i_*=id} & & \downarrow{\scriptstyle i_*\cong} & & \downarrow{\scriptstyle j_*:\cong \text{by five lem.}} & & \downarrow{\scriptstyle =} & & \\ \cdots & \to & H_p(A) & \to & H_p(X) & \to & H_p(X, A) & \to & H_{p-1}(A) & \xrightarrow{\partial_*} & \cdots \end{array}$$

Now consider $S^{\mathcal{U}}(X)/S(A) : S_p^{\mathcal{U}} = S_p(X - U) + S_p(A)$

$$\Rightarrow S_p^{\mathcal{U}}/S_p(A) = \frac{S_p(X-U)+S_p(A)}{S_p(A)} \underset{\overleftarrow{i}}{\cong} S_p(X - U)/(S_p(X - U) \cap S_p(A))$$
$$= S_p(X - U)/S_p(A - U) = S_p(X - U, A - U)$$

$$\Rightarrow H_p(X - U, A - U) \underset{\overrightarrow{i_*}}{\cong} H_p(S^{\mathcal{U}}(X)/S(A)) \underset{\overrightarrow{j_*}}{\cong} H_p(X, A)$$ □

Remark. Recall $X_1, X_2 \subset X, X = X_1 \cup X_2$.
$\{X_1, X_2\}$: an excisive couple if $S(X_1) + S(X_2) \hookrightarrow S(X_1 \cup X_2 = X)$ induces an isomorphism on homology.

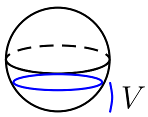
[1] $(X - U)^\circ = (U^C)^\circ = \bar{U}^C$

숙제 24. $\{X_1, X_2\}$ is an excisive couple. $\Leftrightarrow (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2)$ is an excision map, i.e., induces an isomorphism on homology.
증명 Hint: Consider the following commutative diagram.

$$\begin{array}{ccc} S(X_1)/S(X_1 \cap X_2) & \xrightarrow{\quad j \quad} & S(X_1 \cup X_2)/S(X_2) \\ & {}_{i}\searrow \quad \nearrow_{k} & \\ & S(X_1) + S(X_2)/S(X_2) & \end{array}$$

□

Example.



$X = S^n, E^n_+$: 북반구, $E^n_- = A$: 남반구, $U = \mathring{A}$라 두면 U는 excision theorem을 이용하여 X에서 바로 excise할 수 없다. ($\because \bar{U} \not\subseteq \mathring{A}$) 그렇지만 V를 그림과 같이 잡아서 V를 excision theorem 에 의해 excise한 다음 deformation retract를 이용하여 U를 "excise"할 수 있다.

$H_p(S^n, E^n_-) \underset{\overleftarrow{i_*}}{\cong} H_p(S^n - V, E^n - V)$

Note $j : (E^n_+, S^{n-1}) \underset{\overleftarrow{\exists r}}{\hookrightarrow} (S^n - V, E^n_- - V)$ where r is a deformation retraction (i.e., $rj = id, jr \simeq id$) $\Rightarrow j_* :\cong \Rightarrow H_p(S^n, E^n_-) \underset{\overleftarrow{i_*}}{\cong} H_p(E^+_n, S^{n-1})$

$\therefore \{E^n_+, E^n_-\}$: excisive couple and hence can apply Mayer-Vietori sequence directly.

In general, $V \subset U \subset A$ and suppose that V can be excised. If $(X - U, A - U)$ is a deformation retract of $(X - V, A - V)$, then U can be excised by the exactly same argument as above. Hence in this case, $\{X - U, A\}$ is an excisive couple.

Equivalence of simplicial and singular homology

Let K be a simplicial complex. Recall that we have 3 homology theories.
(1) $C(K) \rightsquigarrow H(K)$; simplicial homology
(2) $\triangle(K) \rightsquigarrow H^{\triangle}(K)$; ordered simplicial homology
(3) $S(|K|) \rightsquigarrow H(|K|)$; singular homology

And we showed $\mu : \triangle(K) \to C(K)$ is a natural chain equivalence,
$$(v_0, \cdots v_p) \mapsto [v_0, \cdots, v_p]$$
so that $\mu_* : H^{\triangle}(K) \to H(K)$ is a natural isomorphism.

이제 (2)와 (3)을 비교해보자.
Let $\theta : \triangle(K) \to S(|K|)$
$(v_0, \cdots v_p) \mapsto l$
where $l : \triangle^p \to |K|$ is an affine map determined by $l(e_i) = v_i, i = 0, 1, \cdots, p$.

$\Rightarrow \theta$ is a natural(augmentation-preserving) chain map so that $\theta_* : H^{\triangle}(K) \to H(|K|)$ is a natural transformation.

If $K_0 < K$, a subcomplex, then

$$\begin{array}{ccccccccc} 0 & \to & \triangle(K_0) & \xrightarrow{i} & \triangle(K) & \to & \triangle(K)/\triangle(K_0) & \to & 0 \\ & & \downarrow \theta| & & \downarrow \theta & & \downarrow \bar{\theta} = "\theta" & & \\ 0 & \to & S(|K_0|) & \to & S(|K|) & \to & S(|K|)/S(|K_0|) & \to & 0 \end{array}$$

$\Rightarrow \theta_* : H^{\triangle}(K, K_0) \to H(|K|, |K_0|)$ is naturally well-defined.

> $\theta_* : \widetilde{H^{\triangle}}(K) \to \widetilde{H}(|K|)$ is a natural isomorphism. (Hence so is $\theta_* : H^{\triangle}(K) \to H(|K|)$ and $H_* : H^{\triangle}(K, K_0) \to H(|K|, |K_0|)$ if $K_0 < K$ by 5-lemma.)

step 1 Assume that K is finite.
Prove by induction on n(the number of simplices in K).

n=1 $\quad K = \{v\} \Rightarrow \widetilde{H^{\triangle}}(K) = \widetilde{H}(|K|) = 0.$

$n > 1$
Let σ be a simplex of K of maximal dimension so that $K_1 := K - \{\sigma\}$ is a subcomplex of K.
$\underline{\sigma} := \sigma$ as a subcomplex of K

$bd\underline{\sigma}$:= subcomplex of $\underline{\sigma}$ consisting of proper faces of σ

$$\begin{array}{ccc} \widetilde{H_p^{\triangle}}(K) & \xrightarrow{\theta_*} & \widetilde{H_p}(|K|) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ H_p^{\triangle}(K,\underline{\sigma}) & \xrightarrow{\theta_*} & H_p(|K|,\sigma) \\ {\scriptstyle j_*}\uparrow & & {\scriptstyle j_*}\uparrow \\ H_p^{\triangle}(K_1, bd\underline{\sigma}) & \xrightarrow{\theta_*} & H_p(|K_1|, bd\sigma) \end{array}$$

위의 diagram에서 j_*는 excision theorem에 의해서 isomorphism이고, 가장 아래의 θ_*는 induction hypothesis에 의해 isomorphism이다. 마지막으로 i_*는 각각 $(|K|,\sigma), (K,\underline{\sigma})$ pair에 대한 long exact sequence에 의해서 isomorphism이다.($\because \sigma$에 대한 homology는 항상 0 이므로) 따라서,

$$\widetilde{H_p^{\triangle}}(K) \cong \widetilde{H_p}(|K|)$$

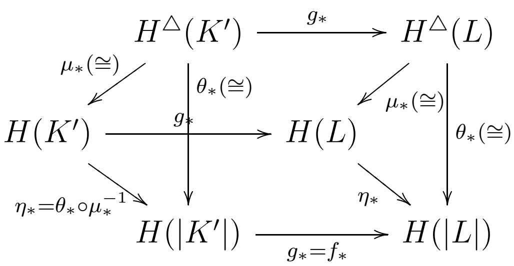
step2 Assume that K is infinite.
(1) θ_* is onto.
Given $\{z\} \in \widetilde{H_p}(|K|), |z|$:= support of z. Then $|z|$ is compact, and hence is contained in $|L|$, a finite subcomplex of K.

$$\begin{array}{cccc} \widetilde{H_p^{\triangle}}(L) & \overset{\theta_*}{\underset{\cong}{\to}} & \widetilde{H_p}(|L|) \ni & \{z\} \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & \downarrow \\ \widetilde{H_p^{\triangle}}(K) & \overset{\theta_*}{\to} & \widetilde{H_p}(|K|) \ni & \{z\} \end{array}$$

step 1 으로 부터 L에 대한 θ_*가 $\cong$이고 diagram이 commute하므로 K에 대한 θ_*는 onto이다.

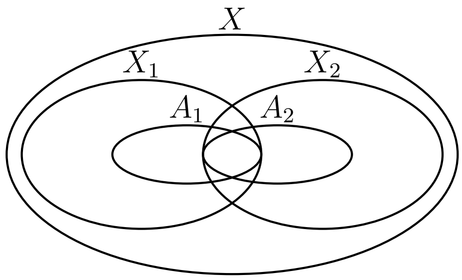
(2) θ_* is 1-1.
Suppose $\{z\} \in \widetilde{H_p^{\triangle}}(K)$ and $\theta_*\{z\} = \{\theta(z)\} = 0$. Then $\exists c \in \widetilde{S_{p+1}}(|K|)$ such that $\partial c = \theta(z)$. $|c|$ is compact, hence contained in L, a finite subcomplex of K. By above diagram, θ_* is 1-1.

Note Given a continuous map $f : |K| \to |L|$, let $g : K' \to L$ be a simplicial approximation of f. 그러면



위의 그림에서 모든 diagram이 commute하고, μ_*, θ_*가 natural isomorphism이므로 η_*도 natural isomorphism이 된다.

Relative Mayer-Vietoris sequence



위와 같은 경우 아래와 같은 sequence들을 생각할 수 있다.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S(A_1 \cap A_2) & \xrightarrow{\phi|} & S(A_1) \bigoplus S(A_2) & \xrightarrow{\psi|} & S(A_1) + S(A_2) & \to & 0 \quad \text{s.e.s} \\
 & & \downarrow i & & \downarrow i_1 \oplus i_2 & & \downarrow j & & \\
0 & \to & S(X_1 \cap X_2) & \xrightarrow{\phi} & S(X_1) \bigoplus S(X_2) & \xrightarrow{\psi} & S(X_1) + S(X_2) & \to & 0 \quad \text{s.e.s} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{S(X_1 \cap X_2)}{S(A_1 \cap A_2)} & \overset{\bar{\phi}}{\dashrightarrow} & \frac{S_(X_1)}{S(A_1)} \bigoplus \frac{S(X_2)}{S(A_2)} & \overset{\bar{\psi}}{\dashrightarrow} & \frac{S(X_1)+S(X_2)}{S(A_1)+S(A_2)} & \longrightarrow & 0 \quad (\star) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

(1) $\bar{\phi}$ and $\bar{\psi}$ are chain maps. ($\because$ chain map의 quotient는 chain map)
(2) $(\star)$ is exact by 9-lemma.

Suppose $\{X_1, X_2\}$ is an excisive couple of $X_1 \cup X_2$ and $\{A_1, A_2\}$ is an excisive couple of $A_1 \cup A_2$. Then $(\star)$ induces an long exact sequence

$$(*) \cdots \to H_p(X_1 \cap X_2, A_1 \cap A_2) \to H_p(X_1, A_1) \bigoplus H_p(X_2, A_2) \to H_p(X_1 \cup X_2, A_1 \cup A_2) \to \cdots$$

여기서 $H_p(X_1 \cup X_2, A_1 \cup A_2)$이 되는 이유는
$S(X_1) + S(X_2) \overset{i}{\hookrightarrow} S(X_1 \cup X_2)$ 와 $S(A_1) + S(A_2) \overset{i}{\hookrightarrow} S(A_1 \cup A_2)$이 homology에서 isomorphism을 induce하고, 5-lemma 에 의해서 $\frac{S(X_1)+S(X_2)}{S(A_1)+S(A_2)} \hookrightarrow \frac{S(X_1 \cup X_2)}{S(A_1 \cup A_2)}$ 도 homology에서 isomorphism을 induce하기 때문이다.

$(*)$를 **Relative Mayer-Vietoris sequence** 라고 부른다.

Special cases
(1) $X = X_1 \cup X_2 \Rightarrow H_p(X_1 \cup X_2, A_1 \cup A_2) = H_p(X, A_1 \cup A_2)$
(2) Furthermore, if $X_1 = X_2 = X$, then

$$\cdots \to H_p(X, A_1 \cap A_2) \to H_p(X, A_1) \bigoplus H_p(X, A_2) \to H_p(X, A_1 \cup A_2) \to \cdots$$

(3) If $X = X_1 \cup X_2$ and $A_1 = A_2 = A$, then

$$\cdots \to H_p(X_1 \cap X_2, A) \to H_p(X_1, A) \bigoplus H_p(X_2, A) \to H_p(X, A) \to \cdots$$