VIII.2 Further Application to Sphere

정의 1 $f : S^n \to S^n \Rightarrow f_* : \widetilde{H_*}(S^n) \to \widetilde{H_*}(S^n)$.
Now let α be a generator of $\widetilde{H_*}(S^n) \cong \mathbb{Z}$ and let $f_*(\alpha) = d\alpha$.
Then the integer d is independent of choice of a generator α, and is called the degree of f.

Remark. M^n : connected orientable closed n-manifold $\Rightarrow$ degree is defined.

정리 1 *(Basic properties of degree)*
(i) $f \simeq g \Rightarrow deg\ f\ = deg\ g$
(ii) $deg\ (f \circ g) \Rightarrow (deg\ f)(deg\ g)$
(iii) $deg\ (id) = 1$
(iv) $deg\ (c) = 0$ *, where c is a constant map.*

증명 (i) $\sim$ (iii) : clear
(iv) c : constant map $\Rightarrow c_* = 0 : \widetilde{H}(X) \to \widetilde{H}(X)$
$(c : X \xrightarrow{\bar{c}} x_0 \overset{i}{\hookrightarrow} X \Rightarrow c_* : \widetilde{H}(X) \xrightarrow{\bar{c_*}} \widetilde{H}(x_0) = 0 \xrightarrow{i_*} \widetilde{H}(X))$ □

Note. (1) $\alpha_n : S^1 \to S^1 \Rightarrow$ deg $(\alpha_n) = n$
$\qquad z \to z^n$

증명

$$\begin{array}{ccc} \pi_1(S^1) & \xrightarrow{(\alpha_n)_\sharp} & \pi_1(S^1) \\ \downarrow{\scriptstyle \chi:\cong} & \curvearrowright & \downarrow{\scriptstyle \chi:\cong} \\ H_1(S^1) & \xrightarrow{(\alpha_n)_*} & H_1(S^1) \end{array}$$

$$\Rightarrow (\alpha_n)_*\{\alpha_1\} = \chi((\alpha_n)_\sharp[\alpha_1]) = \chi([\alpha_n \circ \alpha_1]) = \chi([\alpha_n])$$
$$= \chi(n[\alpha_1]) = n\chi[\alpha_1] = n\{\alpha_1\}$$
□

(2) Brower fixed point theorem on B^n: Prove as follows.

(i) $S^n \xrightarrow{f} S^n \ \Rightarrow$ deg$f = 0$: Applying homology functor to the diagram
$\ \cup \curvearrowright \nearrow \bar{f}$ $\qquad$ and note that $\widetilde{H}(B^n) = 0$
B^{n+1}
(ii) $\nexists$ retraction $r : B^{n+1} \to S^n$:
$id : S^n \xrightarrow{i} B^{n+1} \xrightarrow{r} S^n \overset{(i)}{\Rightarrow}deg(id) = 0$, a contradiction.

(iii) $\forall \phi : B^n \to B^n$ has a fixed point : The proof is the same as before for the case of B^2. (If ϕ has no fixed point, we can construct a retraction:$B^n \to S^{n-1}$.)

정리 2 *Let $S^n \hookrightarrow \mathbb{R}^{n+1}$ be the standard sphere and $f \in O(n+1, \mathbb{R})$ so that $f : S^n \to S^n$. Then deg f = det f(= ± 1)*

증명

보조정리 3 *Let $r : S^n \to S^n$ be the reflection given by $r(x_1, x_2, \cdots, x_{n+1}) = (-x_1, x_2, \cdots, x_{n+1})$. Then deg r = -1 = det r.*

증명 induction on n.
Let $S^0 = \{-1, 1\}$, and x and y be a 0-singular simplex which has image -1 and 1 respectively. $\widetilde{H_0}(S^0) = \{x - y\}$ and $r_* : x - y \mapsto y - x$
$\therefore$ deg $r = -1$.

$n > 0$: MV sequence$\Rightarrow$

$$\begin{array}{ccc} \widetilde{H_n}(S^n) & \xrightarrow{\partial_*:\cong} & \widetilde{H_{n-1}}(S^{n-1}) \\ \downarrow r_* & \circlearrowright & \downarrow r|_* \\ \widetilde{H_n}(S^n) & \xrightarrow{\partial_*:\cong} & \widetilde{H_{n-1}}(S^{n-1}) \end{array}$$

$\Rightarrow$ deg r_* = deg $r|_*$ □

보조정리 4 *$f \in SO(n+1) \Rightarrow f \simeq id$.*

증명 Essentially this is to show $SO(n+1)$ is path-connected.

Fact. $A \in SO(n) \Rightarrow \exists B$ s.t.

$$BAB^{-1} = \begin{pmatrix} \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} & & \cdots \\ \vdots & \ddots & \\ & & 1 \end{pmatrix} \overset{let}{=} D_\theta$$

where $\theta = (\theta_1, \cdots, \theta_i)$[1]
Using fact, $H_t = B^{-1} D_{t\theta} B : I \simeq A \quad (t = 0$ 일 때 $I, t = 1$ 일 때 $A)$ □

정리 2 증명 계속
Now note that $O(n+1) = SO(n+1) \cup rSO(n+1) \overset{det}{\to} \{-1, 1\} = \mathbb{Z}/2$.(보조정리 3, 4로부터)
$\therefore$ deg = det □

따름정리 5 *$a : S^n \to S^n$ the antipodal map $\Rightarrow$ deg $a = (-1)^{n+1}$.*

정리 6 *$f, g : S^n \to S^n$ with $f(x) \neq g(x) \quad \forall x \in S^n \Rightarrow f \simeq ag$.*
(or equivalently, f and g are never antipodal $\Rightarrow f \simeq g$)

증명 $f(x) \neq g(x) \quad \forall x \Rightarrow f(x)$ and $ag(x)$ are not antipodal.
$\Rightarrow (1-t)f(x) + tag(x) \neq 0$
$F : S^n \times I \Rightarrow S^n$ given by $F(x,t) = \frac{(1-t)f(x)+tag(x)}{|(1-t)f(x)+tag(x)|}$ is the desired homotopy. □

따름정리 7 *(1) f has no fixed point. $\Rightarrow f \simeq a$*
(2) deg $f \neq (-1)^{n+1} \Rightarrow f$ has a fixed point.

증명 (1) $f(x) \neq id(x) \overset{6}{\Rightarrow} f \simeq a$ □

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[1] $<,>$: standard inner product on $\mathbb{R}^n \rightsquigarrow (\ ,\ )$: standard Hermitian inner product on $\mathbb{C}^n$
$A \in O(n) \Rightarrow A$ is an isometry for $(\ ,\ )$
Let $Ax = \lambda x \Rightarrow \lambda\bar{\lambda}(x,x) = (Ax, Ax) = (x,x) \Rightarrow |\lambda| = 1$
A : real $\Rightarrow e^{i\theta}, e^{-i\theta}$: conjugate eigenvlaues in general $\Rightarrow Az = (cos\theta + isin\theta)z \Rightarrow x = \frac{z+\bar{z}}{2}$
$A\bar{z} = (cos\theta - isin\theta)\bar{z} \quad y = \frac{z-\bar{z}}{2}i$
Then A can be written as $\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$
with repect to a basis $\{x, y\}$ where $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2}i$.

exc. Is the converse of (1) true?

숙제 1. 문제 16.11.

정리 8 *S^n has a non-vanishing vector field iff $n =$ odd.*

증명 $(\Leftarrow) n = 2m+1 \Rightarrow$ let $v(x_0, \cdots, x_{2m+1}) = (-x_1, x_0, -x_3, x_2, \cdots, -x_{2m+1}, x_{2m})$

$(\Rightarrow)$ Suppose $\exists$ a non-vanishing vector field v, we may assume $|v(x)| = 1$
$\Rightarrow F(x,t) = (cos\, t\pi)x + (sin\, t\pi)v(x)$ gives a homotopy : $id \simeq a$
$\Rightarrow$ deg $(id) =$ deg $(a) = (-1)^{n+1}$
$\Rightarrow n =$ odd. □

Remark. vector field problem for S^{odd}:
Adams solution : Let $n+1 = (odd)2^{4a+b}, \quad 0 \leqslant b \leqslant 3$. Then the number of independent vector fields on $S^n = 2^b + 8a - 1$.

숙제 2. 문제 16.10.