I.2 Homology of adjunction space

Assume (X, A) is a collared pair with a collaring B and given $f : A \to Y$, obtain $X \cup_f Y$ which is denoted by Z in this section.

정리 1 *Let $p : X \coprod Y \to Z$ be a quotient map and $\overline{f} := p|_X : (X, A) \to (Z, Y)$ be its restriction. Then $\overline{f}_* : H_q(X, A) \to H_q(Z, Y)$ is an isomorphism $\forall q$.*

증명

$$\begin{array}{ccccc}
H_q(X,A) & \xrightarrow{i_*} & H_q(X,B) & \longleftarrow & H_q(X-A, B-A) \\
\downarrow \overline{f}_* & & \downarrow \overline{f}_* & & \downarrow \overline{f}_* \\
H_q(Z,Y) & \xrightarrow{j_*} & H_q(Z, Y \cup \overline{f}(B)) & \longleftarrow & H_q(Z-Y, \overline{f}(B-A))
\end{array}$$

가로 방향의 map들은 모두 inclusion에 의해 induce된 map들이므로 위 diagram이 commute한다. 오른쪽의 $\overline{f}_*$는 앞절의 정리에 의하여 homeomorphism에 의하여 induce되어 isomorphism이므로 가로 방향의 map들이 모두 isomorphism임을 보이면 증명이 끝난다.
3열에서 2열로 가는 map들은 excision theorem에 의해 isomorphism이고, 1열에서 2열로 가는 map들은 다음 diagram에서 five lemma를 적용하면 isomorphism이다.

$$\begin{array}{ccccccccc}
H_q(A) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X,A) & \longrightarrow & H_q(A) & \longrightarrow & H_q(X) \\
{\scriptstyle\text{deformation retract}}\downarrow\cong & & \downarrow = & & \downarrow & & \downarrow\cong & & \downarrow = \\
H_q(B) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X,B) & \longrightarrow & H_q(B) & \longrightarrow & H_q(X)
\end{array}$$

따라서, $\overline{f}_*$는 isomorphism이다. □

정리 2 (MV-sequence)
We have a long exact sequence

$$\cdots \longrightarrow H_q(A) \longrightarrow H_q(X) \bigoplus H_q(Y) \longrightarrow H_q(Z) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow \cdots$$

$$\alpha \longmapsto (i_*\alpha, f_*\alpha)$$

$$(x, y) \longmapsto j_*y - \overline{f}_*x$$

$$z \longmapsto \partial \overline{f}_*^{-1} k_* z$$

This also holds for reduced homology.

증명

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & H_q(A) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X,A) & \xrightarrow{\partial_*} & H_{q-1}(A) & \longrightarrow & H_{q-1}(X) & \longrightarrow \cdots \\
& \downarrow f_* & & \downarrow \overline{f}_* & & \cong \downarrow \overline{f}_* & & \downarrow f_* & & \downarrow \overline{f}_* & \\
\cdots \longrightarrow & H_q(Y) & \longrightarrow & H_q(Z) & \xrightarrow{k_*} & H_q(Z,Y) & \longrightarrow & H_{q-1}(Y) & \longrightarrow & H_{q-1}(Z) & \longrightarrow \cdots
\end{array}$$

정리 1에 의하여 가운데의 $\overline{f}_*$가 isomorphism이고 다음의 보조정리를 적용하면 원하는 결과를 얻는다. □

보조정리 3 (Barratt-Whitehead Lemma)
Given long exact sequences,

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & C_{i+1} & \longrightarrow & A_i & \xrightarrow{f_i} & B_i & \xrightarrow{g_i} & C_i & \xrightarrow{h_i} & A_{i-1} & \longrightarrow \cdots \\
& \downarrow \gamma_{i+1} & & \downarrow \alpha_i & & \downarrow \beta_i & & \cong \downarrow \gamma_i & & \downarrow \alpha_{i-1} & \\
\cdots \longrightarrow & C'_{i+1} & \longrightarrow & A'_i & \xrightarrow{f'_i} & B'_i & \xrightarrow{g'_i} & C'_i & \xrightarrow{h'_i} & A'_{i-1} & \longrightarrow \cdots
\end{array}$$

If γ_i are all isomorphisms, then we have a long exact sequence,

$$\begin{array}{l}
\cdots \to A_i \xrightarrow{\phi_i} A'_i \bigoplus B_i \xrightarrow{\psi_i} B'_i \xrightarrow{\partial_i} A_{i-1} \longrightarrow \cdots \\
a \longmapsto (\alpha(a), f(a)) \\
(a', b) \longmapsto \beta(b) - f'(a') \\
b' \longmapsto h\gamma^{-1}g'(b')
\end{array}$$

증명 Diagram chasing. □

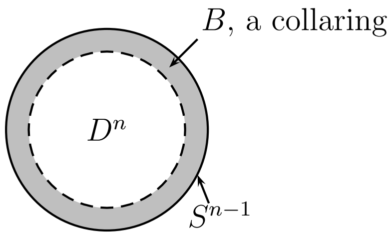
Example 1. Let (X, A) be a collared pair and Y={*}, a point. Then $X \cup_f Y = X/A$ by definition.
정리 1 $\Rightarrow \overline{f}_* : H_q(X,A) \xrightarrow{\cong} H_q(Z,Y) = H_q(X/A, *) = \widetilde{H}_q(X/A)$.
정리 2 $\Rightarrow \cdots \to \widetilde{H}_q(A) \to \widetilde{H}_q(X) \to \widetilde{H}_q(X/A) \to \widetilde{H}_{q-1}(A) \to \cdots$.

2. Wedge (or one point union) of (X, x) and (Y, y), where (X, x) and (Y, y) are collared pairs.
$X \vee Y = X \coprod Y/x \sim y$.
$A = \{x\}$, $f : \{x\} \to \{y\} \subset Y$로 두면, 정리 2에 의해 exact sequence,

$$\cdots \to \widetilde{H}_q(x) \to \widetilde{H}_q(X) \bigoplus \widetilde{H}_q(Y) \to \widetilde{H}_q(X \vee Y) \to \cdots$$

를 얻고 $\widetilde{H}_q(x) = 0$이므로 $\widetilde{H}_q(X \vee Y) \cong \widetilde{H}_q(X) \bigoplus \widetilde{H}_q(Y)$이다.

3. $(X, A) = (D^n, S^{n-1})$ is a collared pair.



Then $Z = D^n \cup_f Y$ is a space obtained from Y by attaching an n-cell by $f : S^{n-1} \to Y$. For example, $S^{n+1} = D^n \cup_f \{\text{point}\}$, $T^2 = D^2 \cup_f \{\text{figure eight}\}$, and $\mathbb{P}^n = D^n \cup_f \mathbb{P}^{n-1}$.

Z의 homology를 구해보자. MV-sequence로부터

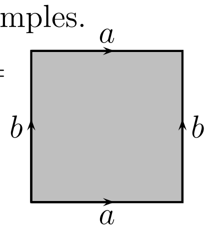
$$\cdots \to \widetilde{H}_q(S^{n-1}) \to \widetilde{H}_q(Y) \to \widetilde{H}_q(Z) \to \widetilde{H}_{q-1}(S^{n-1}) \to \cdots$$

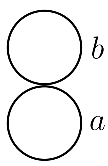
를 얻는다. 따라서, $q \neq n, n-1$의 경우에는 $\widetilde{H}_q(Z) \cong \widetilde{H}_q(Y)$임을 알 수 있다. 또한,

$$0 \to \widetilde{H}_n(Y) \to \widetilde{H}_n(Z) \to \widetilde{H}_{n-1}(S^{n-1}) \xrightarrow{f_*} \widetilde{H}_{n-1}(Y) \to \widetilde{H}_{n-1}(Z) \to 0$$

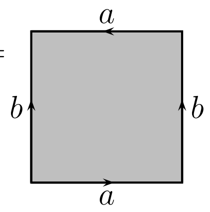
에서 $\widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$는 free abelian group이므로, $\widetilde{H}_n(Z) = \widetilde{H}_n(Y) \bigoplus ker f_*$, $\widetilde{H}_{n-1}(Z) = \widetilde{H}_{n-1}(Y)/im f_*$임을 알 수 있다.

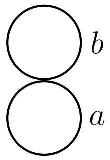
Examples.

$T^2 =$ 

, $Y =$ 

$f_* : \widetilde{H}_1(S^1) \to \widetilde{H}_1(Y)$는 $\widetilde{H}_1(S^1) \cong \mathbb{Z}$의 generator c를 $a + b - a - b = 0$으로 보내므로 zero map이다. 따라서, $H_2(T^2) = H_2(Y) \bigoplus \mathbb{Z} = \mathbb{Z}$, $H_1(T^2) = H_1(Y)/0 = \mathbb{Z} \bigoplus \mathbb{Z}$이다.

$K=$ 

, $Y=$ 

$f_*(c) = a + b + a - b = 2a$이므로 $H_2(K) = H_2(Y) \bigoplus ker f_* = 0$, $H_1(K) = H_1(Y)/im f_* = \mathbb{Z} \bigoplus \mathbb{Z}/2$이다.

마찬가지 방법으로 $\mathbb{P}^2 = D^2 \cup_f S^1$의 경우에는 $f_*(c) = 2a$이므로, $H_2(\mathbb{P}^2) = 0$, $H_1(\mathbb{P}^2) = \mathbb{Z}/2$임을 알 수 있다.

숙제 6. Compute the followings.
(1) $H_*(\Sigma_g)$
(2) $H_*(N_k)$
(3) $H_*(D^2 \cup_f S^1)$, where $f : \partial D^2 = S^1 \to S^1$ is given by $f(z) = z^3$
(4) $H_*(\mathbb{P}^n)$